St Peter the Apostle High

Mathematics Dept.

Higher Prelim Revision 2

Paper I - Non~calculator

Time allowed - 1 hour 10 minutes

FORMULAE LIST

Circle:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

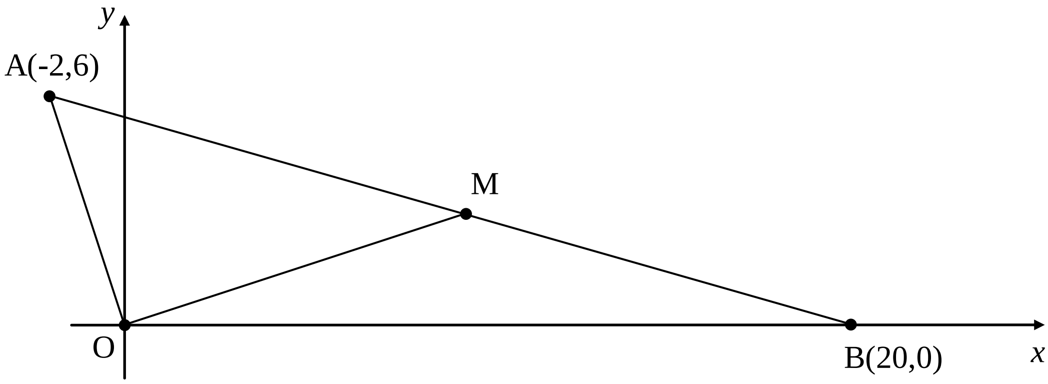
The equation $(x-a)^2 + (y-b)^2 = r^2$ represents a circle centre (a, b) and radius r.

Trigonometric formulae:

$$\begin{aligned}
\sin(A \pm B) &= \sin A\cos B \pm \cos A\sin B \\
\cos(A \pm B) &= \cos A\cos B \mp \sin A\sin B \\
\sin 2A &= 2\sin A\cos A \\
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A
\end{aligned}$$

All questions should be attempted

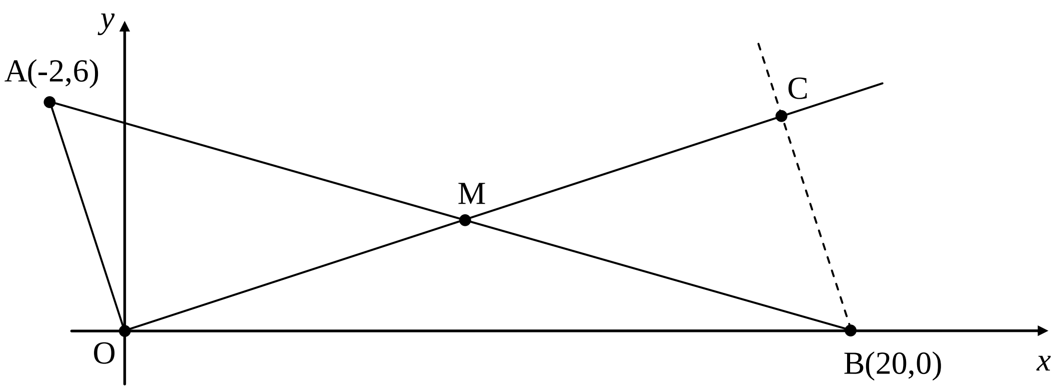
1. The diagram shows triangle OAB with M being the mid-point of AB.
The coordinates of A and B are (-2,6) and (20,0) respectively.



(a) Establish the coordinates of M. **1**

(b) Hence find the equation of the median OM. **2**

(c) A line through B, perpendicular to OM meets OM produced at C.



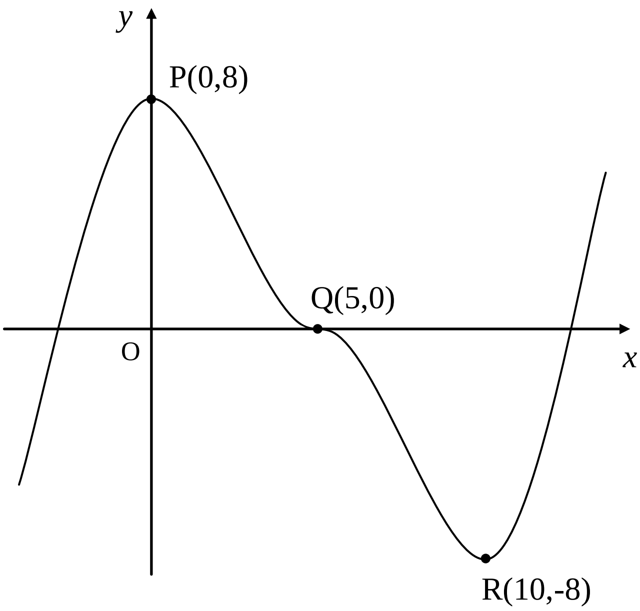
(i) Find the equation of the line BC and hence establish the coordinates of C. **4**

(ii) What can you say about triangles OAM and BMC? Explain your answer. **2**

2. A curve has as its equation $y = \dfrac{x^2 - 4x}{\sqrt{x}}$, where $x \in R$ and $x > 0$.

Find the gradient of the tangent to this curve at the point where $x = 4$. **6**

3. The diagram shows part of the graph of $y = f(x)$.



The function has stationary points at P(0,8), Q(5,0) and R(10,-8) as shown.

Sketch a possible graph for $y = f'(x)$, where $f'(x)$ is the derivative of $f(x)$. **4**

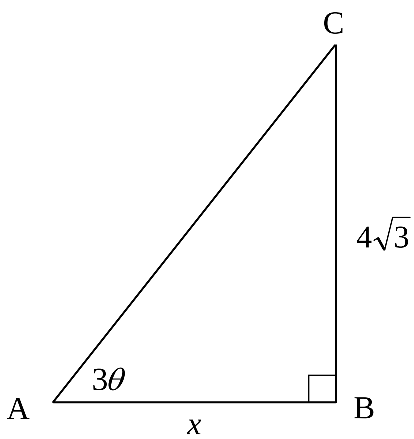
4. Two functions, defined on suitable domains, are given as

$$g(x) = x^2 - 3x \qquad \text{and} \qquad h(x) = 2x + 1.$$

Show that the composite function $g\big(h(x)\big)$ can be written in the form $a(ax+b)(x-b)$, where a and b are constants, and state the value(s) of a and b. **4**

5. Consider the triangle opposite.

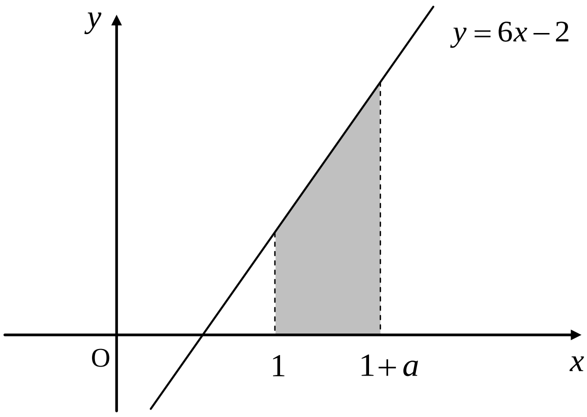
AB is x units long, BC $= 4\sqrt{3}$ units long and angle BAC = 3θ radians.



(a) Given that the exact area of the triangle is $8\sqrt{3}$ units^2, **show clearly** that $x = 4$. **3**

(b) Hence find the value of θ, **in radians**, given that 3θ is acute. **3**

6. The diagram below, which is not to scale, shows part of the graph of the line with equation $y = 6x - 2$. Also shown are ordinates at $x = 1$ and at $x = 1 + a$.



Find a given that the shaded part of the diagram has an area of 4 square units. **7**

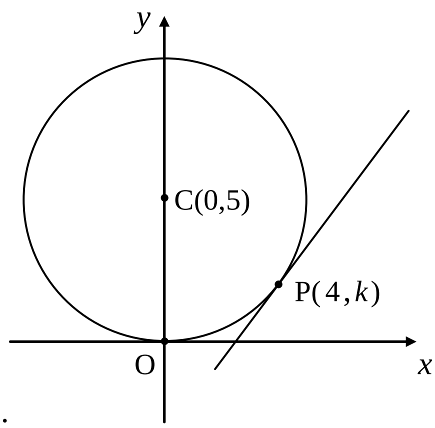
7. Two sequences are defined by the following recurrence relationships

$$U_{n+1} = 0 \cdot 6U_n + 20 \quad \text{and} \quad U_{n+1} = 0 \cdot 9U_n + b \text{ , where } b \text{ is a constant.}$$

(a) Explain why both sequences have a limit as $n \to \infty$. **1**

(b) Find the value of b if both these sequences have the same limit. **4**

8. A circle passes through the origin and has the point C(0,5) as its centre.



(a) Establish the equation of this circle giving your answer in **expanded form**. **4**

(b) The point P$(4, k)$ lies on the circumference of this circle as shown.
Find **algebraically** the value of k. **5**

(c) Find the equation of the tangent to the circle at P. **3**

9. A curve has as its equation $y = (p+1)x^3 - 3px^2 + 4x + 1$, where p **is a positive integer**.

(a) Find $\dfrac{dy}{dx}$. **2**

(b) Hence establish the value of p given that this curve has only **one stationary point**. **5**

[END OF QUESTION PAPER]

St Peter the Apostle High

Mathematics Dept.

Higher Prelim Revision 2

Paper 2 - Calculator

Time allowed - 1 hour 30 minutes

FORMULAE LIST

Circle:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

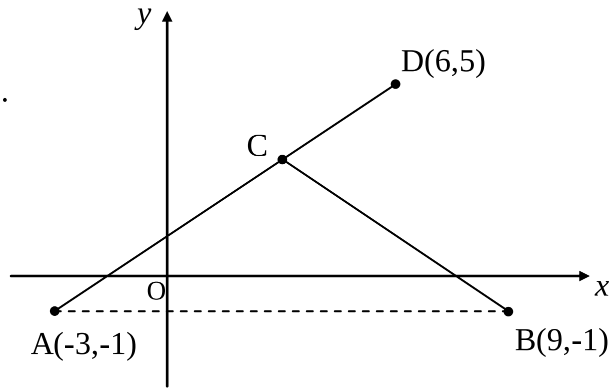
The equation $(x-a)^2 + (y-b)^2 = r^2$ represents a circle centre (a, b) and radius r.

Trigonometric formulae:

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\sin 2A &= 2 \sin A \cos A \\
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A
\end{aligned}$$

All questions should be attempted

1. The diagram shows a line joining the points A(-3,-1) and D(6,5).
B has coordinates (9,-1) and C is a point on AD.

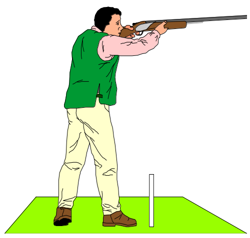


(a) Find the equation of the line AD. **2**

(b) Hence establish the coordinates of C given that triangle ABC is isosceles. **3**

(c) Use gradient theory to calculate the size of angle BCD, giving your answer correct to the nearest degree. **3**

2. A lead shot is discharged from a gun at a clay pigeon.



The height, h feet, of the shot after t seconds is given by the function

$$h(t) = 288t - 48t^2 .$$

(a) What is the maximum height the shot can reach ? **4**

(b) For the shot to actually break the clay pigeon it must strike the pigeon at a speed greater than <u>or</u> equal to 48 feet per second.

The speed, s , of the shot after t seconds can be found from $s = h'(t)$, where $0 < t \leq 3$.

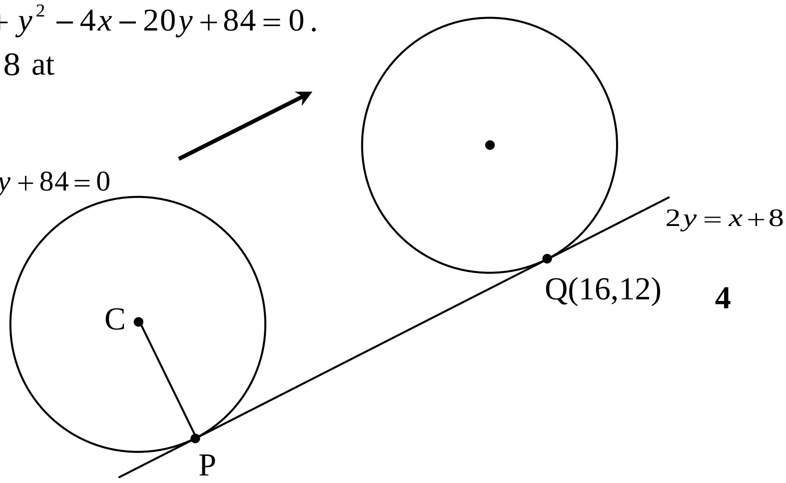
Will the shot break the clay pigeon after a flight of $2 \cdot 7$ seconds ? Explain. **2**

(c) Calculate the maximum **height** the shot can reach **and** still break the clay pigeon. **3**

3. Solve algebraically the equation $9\sin x^\circ + 4 = 2\cos 2x^\circ$ where $0 \leq x < 360$ **6**

4. A circle, centre C, has as its equation $x^2 + y^2 - 4x - 20y + 84 = 0$.
It touches the line with equation $2y = x + 8$ at point P, as shown.

$x^2 + y^2 - 4x - 20y + 84 = 0$



(a) Find **algebraically** the coordinates of P. **4**

(b) The circle is rolled up the line until Q(16,12) becomes the new point of tangency.

Establish the equation of the circle in this new position. **5**

5. A sequence is defined by the recurrence relation $U_{n+1} = aU_n + b$, where a and b are constants.

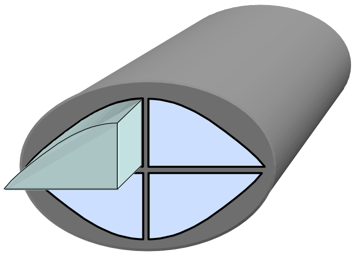
(*a*) Given that $U_0 = a - 2$ and $b = 1$, show clearly that $U_1 = a^2 - 2a + 1$. **2**

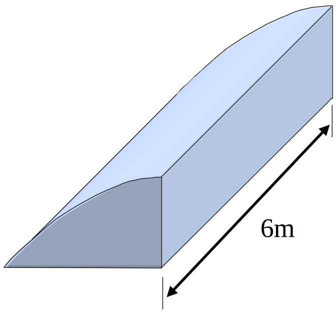
(*b*) Hence find an expression for U_2 in terms of a. **2**

(c) Given now that $U_2 = 37$, form an equation and solve it to find a.

Explain why there is only one possible answer for a. **4**

6. A titanium rod from a nuclear reactor is a solid prism which slots into an elliptical chamber along with three other identical rods. It has a cross-sectional shape made up of two straight lines and a curved edge.



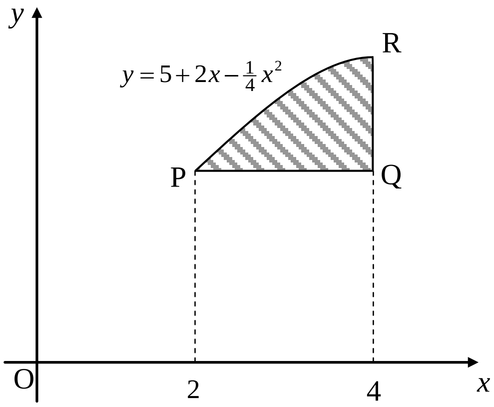


Each rod has a depth of 6 metres.

The cross section of a rod is shown geometrically in the coordinate diagram below where the **units are in metres**. The diagram is not drawn to scale.

The curved section is part of the graph of the curve with equation $y = 5 + 2x - \frac{1}{4}x^2$.

PQ is horizontal and QR is vertical.



(a) Calculate the shaded area in square metres. **7**

(b) Hence calculate the **total volume** of titanium contained in **all four rods**. **2**

7. The angle θ is such that $\tan\theta = \frac{2}{\sqrt{2}}$ where $0 < \theta < \frac{\pi}{2}$.

(a) Find the exact values of $\sin\theta$ and $\cos\theta$. **3**

(b) Hence show clearly that the exact value of $\sin(\theta + \frac{\pi}{3})$ can be expressed as

$$\sin(\theta + \tfrac{\pi}{3}) = \tfrac{1}{6}(\sqrt{6} + 3).$$ **5**

8. Three functions are defined on suitable domains as

$$f(x) = x - 1, \quad g(x) = 3x^2 - 3 \quad \text{and} \quad h(x) = x^3 - 6x.$$

(a) Given that $y = g(f(x)) - h(x)$, find a formula for y in its simplest form. **3**

(b) Hence find the coordinates of the maximum turning point of the graph of $y = g(f(x)) - h(x)$, **justifying your answer**. **4**

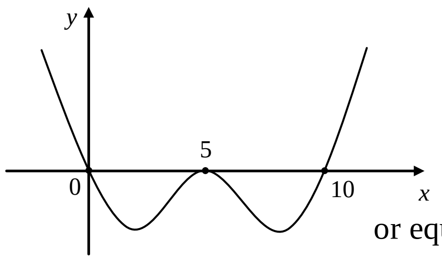
9. An equation is given as $ax(x-1) = c(x-1)$, where $a \neq 0$, $c \neq 0$, and a and c are constants.

(a) Show clearly that this equation can be written in the form

$$ax^2 - (a+c)x + c = 0.$$ **2**

(b) What condition needs to be met for this quadratic equation to have equal roots? **4**

[END OF QUESTION PAPER]

	Give 1 mark for each •	**Illustration(s) for awarding each mark**
1.	**(a) ans:** M(9,3) **1 mark** •1 answer **(b) ans:** $y=\frac{1}{3}x$ **2 marks** •1 for gradient •2 for strategy **(c) ans:** i) $y=-3x+60$, C(18,6) ii) congruent **6 marks** i) •1 for gradient of BC •2 for sub. to equ. of line •3 knowing to solve system •4 coordinates of C ii) •5 answer •6 explanation	(a) •1 M(9,3) (b) •1 $m_{AC}=\frac{3-0}{9+0}=\frac{1}{3}$ •2 $y=mx$ $y=\frac{1}{3}x$ (c) i) •1 $m=-3$ •2 $y-0=-3(x-20)$ •3 $\frac{1}{3}x=-3x+60$ •4 $x=18 \Rightarrow \therefore y=6$ ii) •5 congruent (or equiv.) •6 explanation of parallel lines (or any suitable explanation)
2.	**ans:** $m=2$ **6 marks** •1 for dealing with denominator •2 for simplifying •3 diff. first term •4 diff. second term •5 substituting •6 answer	•1 $y=x^{-\frac{1}{2}}(x^2-4x)$ •2 $y=x^{\frac{3}{2}}-4x^{\frac{1}{2}}$ •3 $\frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}}$ .... •4 $\frac{dy}{dx}=$ .... $-\ 2x^{-\frac{1}{2}}$ •5 $m=\frac{3}{2}\sqrt{4}-\frac{2}{\sqrt{4}}$ (or equiv.) •6 $m=2$
3.	**ans:** see sketch **4 marks** •1 for stat. points as roots •2 for basic shape … left side •3 basic shape … right side •4 annotation	 or equivalent sketch
4.	**ans:** $a=2$, $b=1$ **4 marks** •1 sub. for the composite function •2 expanding and simplifying •3 factorising •4 answers	•1 $g(h(x))=(2x+1)^2-3(2x+1)$ •2 $g(h(x))=4x^2-2x-2$ •3 $\qquad =2(2x+1)(x-1)$ •4 $a=2$, $b=1$

	Give 1 mark for each •	Illustration(s) for awarding each mark
5.	**(a) ans:** proof **3 marks** •1 for area strategy •2 for substitution •3 for answer **(b) ans:** $\theta = \frac{\pi}{9}$ **3 marks** •1 for strategy and writing .. $\tan 3\theta =$ •2 for knowing exact value •3 calculating answer	(a) •1 $A = \frac{1}{2}bh$ •2 $A = \frac{1}{2}bh = \frac{1}{2} \times x \times 4\sqrt{3}$ •3 $8\sqrt{3} = x \times 2\sqrt{3} \quad \therefore\ x = 4$ (b) •1 $\tan 3\theta = \dfrac{4\sqrt{3}}{4} = \sqrt{3}$ •2 *If* $\tan 3\theta = \sqrt{3}$ *then* $3\theta = \frac{\pi}{3}$ •3 $\therefore\ \theta = \frac{\pi}{9}$
6.	**ans:** $a = \frac{2}{3}$ **7 marks** •1 for setting up integral •2 integrating correctly •3 making integral equal 4 •4 substituting •5 simplifying to quadratic equ. •6 factorising •7 solving to answer	•1 $A = \int_1^{1+a} (6x-2)\ dx$ •2 $= \left[\ 3x^2 - 2x\ \right]_1^{1+a}$ •3 $\left[\ 3x^2 - 2x\ \right]_1^{1+a} = 4$ •4 $(3(1+a)^2 - 2(1+a)) - (1) = 4$ •5 $3a^2 + 4a - 4 = 0$ •6 $(3a-2)(a+2) = 0$ •7 $\therefore\ a = \frac{2}{3}$ (note: -2 is a discard)
7.	**(a) ans:** since for both $-1 < a < 1$ **1 mark** •1 for statement **(b) ans:** $b = 5$ **4 marks** •1 know how to find a limit •2 substitute •3 equate both limits •4 solve for *b*	(a) •1 since for both $-1 < a < 1$ (or equiv.) (b) •1 $L = \dfrac{b}{1-a}$ …. or equivalent •2 $L_1 = \dfrac{20}{1-0\cdot 6}$, $L_2 = \dfrac{b}{1-0\cdot 9}$ •3 $\dfrac{20}{1-0\cdot 6} = \dfrac{b}{1-0\cdot 9}$ •4 $b = \dfrac{20 \times 0\cdot 1}{0\cdot 4} = 5$

	Give 1 mark for each •	Illustration(s) for awarding each mark
8.	**(a) ans:** $x^2 + y^2 - 10y = 0$ **4 marks** •1 for radius (5 units) •2 for strategy •3 for substituting in formula •4 for expanding	(a) •1 $r = 5$ •2 $(x-a)^2 + (y-b)^2 = r^2$ •3 $(x-0)^2 + (y-5)^2 = 25$ •4 $x^2 + y^2 - 10y + 25 - 25 = 0$ ....
	(b) ans: $k = 2$ **5 marks** •1 knowing to substitute point in equ. •2 simplifying to quadratic •3 solving to answers •4 discarding $k = 8$ •5 answer	(b) •1 $4^2 + k^2 - 10k = 0$ •2 $k^2 - 10k + 16 = 0$ •3 $(k-8)(k-2) = 0$ •4 $\cancel{\therefore\ k = 8}$ •5 $k = 2$
	(c) ans: $3y = 4x - 10$ **3 marks** •1 for gradient of radius •2 for gradient of tangent •3 sub. to answer	(c) •1 $m_r = \frac{2-5}{4-0} = -\frac{3}{4}$ •2 $m_{\tan} = \frac{4}{3}$ •3 $y - 2 = \frac{4}{3}(x-4)$
9.	**(a) ans:** $\frac{dy}{dx} = 3(p+1)x^2 - 6px + 4$ **2 marks** •1 differentiating first term •2 differentiating remainder	(a) •1 $\frac{dy}{dx} = 3(p+1)x^2$ .... (or equiv.) •2 $\frac{dy}{dx} =$ .... $-6px + 4$
	(b) ans: $p = 2$ **5 marks** •1 realising strategy i.e. equal roots •2 for a, b and c •3 for substitution •4 for simplifying + factorising •5 choosing correct answer	(b) •1 $b^2 - 4ac = 0$ (stated <u>or</u> implied) •2 $a = 3p + 3,\ b = -6p,\ c = 4$ •3 $(-6p)^2 - 16(3p+3) = 0$ •4 $36p^2 - 48p - 48 = 0$ $12(3p+2)(p-2) = 0$ •5 $\therefore\ \cancel{p = -\frac{2}{3}}\ ,\ p = 2$

Total 60 marks

	Give 1 mark for each •	Illustration(s) for awarding each mark
1.	**(a) ans:** $3y = 2x + 3$ **2 marks** •1 for gradient •2 for sub. to answer **(b) ans:** C(3,3) **3 marks** •1 realising mid-point gives $x = 3$ •2 knowing to sub. in equation •3 calculating *y* correctly then answer **(c) ans:** 67^o **3 marks** •1 for knowing to use $\tan\theta = m$ •2 equating and calculating an angle •3 working towards and finding angle	(a) •1 $m = \frac{5+1}{6+3} = \frac{2}{3}$ •2 $y - 5 = \frac{2}{3}(x-6)$ (b) •1 $mid_{AB} = \frac{-3+9}{2} = 3$ •2 $\therefore 3y = 2(3) + 3$ •3 $3y = 9 \quad \therefore \ y = 3 \Rightarrow$ C(3,3) (c) •1 $\tan\theta = m$ •2 $\tan D\hat{A}B = \frac{2}{3} \quad \therefore \angle DAB \approx 33\cdot7^o$ •3 working through isosceles triangle then $\angle BCD \approx 67^o$
2.	**(a) ans:** 432 feet **4 marks** •1 knowing to differentiate •2 differentiating and solving to zero •3 finding *t* for max. height •4 substituting to find height **(b) ans:** No since $28\cdot8 < 48$ ft/s **2 marks** •1 evaluating value of derivative •2 answer + viable explanation **(c) ans:** 420 feet **3 marks** •1 for knowing to solve derivative to 48 •2 calculating *t* •3 substituting *t* to answer	(a) •1 max height when $h'(t) = 0$ •2 $288 - 96t = 0$ •3 $\therefore \ t = 3$ •4 $h(3) = 288(3) - 48(3^2) = 432$ ft (pupils may use mid-point of roots to find max. height) (b) •1 $h'(2\cdot7) = 288 - 96(2\cdot7) = 28\cdot8$ ft/s •2 No since $28\cdot8 < 48$ ft/s (c) •1 $288 - 96t = 48$ •2 $t = 2\cdot5$ sec. •3 $h(2\cdot5) = 720 - 300 = 420$ ft
3.	**ans:** $\{194\cdot5^o \ , \ 345\cdot5^o\}$ **6 marks** •1 double angle substitution •2 simplifying to standard quad. form •3 factorising and solving •4 discarding -2 solution •5 for 1st angle •6 for 2nd angle	•1 $9\sin x + 4 = 2(1 - 2\sin^2 x)$ •2 $4\sin^2 x + 9\sin x + 2 = 0$ •3 $(4\sin x + 1)(\sin x + 2) = 0$ $\therefore \ \sin x = -\frac{1}{4} \ \ or \ \ \cancel{\sin x = -2}$ •4 $\sin x = -\frac{1}{4}$ … stated or implied •5 $x = 180 + 14\cdot5 = 194\cdot5$ •6 $x = 360 - 14\cdot5 = 345\cdot5$

	Give 1 mark for each •	Illustration(s) for awarding each mark
4.	**(a) ans:** P(4,6) **4 marks** •1 startegy + substituting •2 simplifying to quadratic equation •3 factorising + first coordinate •4 second coordinate **(b) ans:** $(x-14)^2+(y-16)^2=20$ **5 marks** •1 stepping out strategy •2 finding original centre •3 establishing the new centre •4 calculating radius *(may use pyth.)* •5 substituting in general equ. to answer	(a) •1 $(2y-8)^2+y^2-4(2y-8)-20y+84=0$ •2 $5y^2-60y+180=0$ •3 $5(y-6)(y-6)=0 \quad \therefore \quad y=6$ •4 $x=2(6)-8=4$ (b) •1 From P to Q …. 12 along , 6 up (or equivalent strategy) •2 $C_1(2,10)$ •3 $C_2(2+12,10+6)$ = $C_2(14,16)$ •4 $r=\sqrt{(-4)^2+(-10)^2-84}=\sqrt{20}$ •5 $(x-14)^2+(y-16)^2=20$
5.	**(a) ans:** proof **2 marks** •1 for knowing to substitute •2 for simplifying to answer **(b) ans:** $U_2=a^3-2a^2+a+1$ **2 marks** •1 knowing to sub (a) into U_2 = … •2 answer **(c) ans:** $a=4$, quotient has no roots (or equivalent) **4 marks** •1 strategy (synthetic division) •2 finding answer for *a* •3 checking for further roots •4 explanation for no further roots	(a) •1 $U_1=a(a-2)+1$ •2 $U_1=a^2-2a+1$ (b) •1 $U_2=aU_1+b$ $U_2=a(a^2-2a+1)+1$ •2 $U_2=a^3-2a^2+a+1$ (c) •1 $a \,\vert\, 1 \;\; -2 \;\; 1 \;\; -36$ •2 $\begin{array}{r\|rrrr} 4 & 1 & -2 & 1 & -36 \\ & & 4 & 8 & 36 \\ \hline & 1 & 2 & 9 & 0 \end{array}$ $\therefore a=4$ •3 for x^2+2x+9...... $b^2-4ac=-32$ •4 since $b^2-4ac<0$, no further roots
6.	**(a) ans:** Area = $1\frac{1}{3}$ m^2 **7 marks** •1 for setting up integral •2 integrating •3 substituting in limits •4 calculating area •5 finding *y* coordinate at $x=2$ •6 calculating area of rectangle •7 subtracting to work out shaded area **(b) ans:** 32 m^3 **2 marks** •1 for knowing how to calculate volume •2 for calculations to answer	(a) •1 $A=\int_2^4 (5+2x-\frac{1}{4}x^2)\,dx$ •2 $=\left[5x+x^2-\frac{1}{12}x^3\right]_2^4$ •3 $=(20+16-5\frac{1}{3})-(10+4-\frac{2}{3})$ •4 $=17\frac{1}{3}$ square metres •5 $y=5+2(2)-\frac{1}{4}(2^2)=8$ •6 $A_{rec}=8\times2=16$ square metres •7 $A_{sh}=17\frac{1}{3}-16=1\frac{1}{3}$ sq. m (b) •1 $V=$ *face area* $\times$ *depth* •2 $V=1\frac{1}{3}\times6=8$ .... $V_{tot}=8\times4=32$ m^3

	Give 1 mark for each •	Illustration(s) for awarding each mark
7.	**(a) ans:** $\sin\theta = \frac{2}{\sqrt{6}}$, $\cos\theta = \frac{\sqrt{2}}{\sqrt{6}}$ **3 marks** •1 drawing a R.A. triangle •2 calculating hypotenuse •3 lifting answers **(b) ans:** proof **5 marks** •1 expanding •2 putting in all exact values •3 simplifying •4 rationalising the denominator •5 taking out common factor to answer	(a) •1 drawing triangle •2 $h^2 = 2 + 4 = 6 \therefore h = \sqrt{6}$ •3 $\sin\theta = \frac{2}{\sqrt{6}}$, $\cos\theta = \frac{\sqrt{2}}{\sqrt{6}}$ (b) •1 $\sin(\theta + \frac{\pi}{3}) = \sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3}$ •2 $= \frac{2}{\sqrt{6}}(\frac{1}{2}) + \frac{\sqrt{2}}{\sqrt{6}}(\frac{\sqrt{3}}{2})$ •3 $= \frac{1}{\sqrt{6}} + \frac{1}{2}$ •4 $= \frac{\sqrt{6}}{6} + \frac{1}{2}$ •5 $\sin(\theta + \frac{\pi}{3}) = \frac{1}{6}(\sqrt{6} + 3)$
8.	**(a) ans:** $y = 3x^2 - x^3$ **3 marks** •1 dealing with the composite function •2 simplifying the composite function •3 subtracting $h(x)$ to answer **(b) ans:** (2,4) **4 marks** •1 knowing to differentiate and solve to 0 •2 finding the two x values •3 finding corresponding y values •4 justifying $x = 2$ gives max. $\therefore$ (2,4)	(a) •1 $g(f(x)) = 3(x-1)^2 - 3$ •2 $g(f(x)) = 3x^2 - 6x$ •3 $y = 3x^2 - 6x - (x^3 - 6x) = 3x^2 - x^3$ (b) •1 $\frac{dy}{dx} = 6x - 3x^2 = 0$ •2 $3x(2-x) = 0 \quad \therefore \quad x = 0 \; or \; x = 2$ •3 (0,0) , $y = 3(2^2) - 2^3 = 4 \therefore$ (2,4) •4 justification table (or 2^{nd} deriv.)
9.	**(a) ans:** proof **2 marks** •1 expanding and taking to one side •2 removing common factor to required ans. **(b) ans:** a must equal c ($a = c$) **4 marks** •1 condition for equal roots stated or implied •2 drawing out a,b & c and sub. in discrim. •3 simplifying to perfect square •4 conclusion	(a) •1 $ax^2 - ax - cx + c = 0$ •2 $ax^2 - (a+c)x + c = 0$ (b) •1 for equal roots … $b^2 - 4ac = 0$ •2 $a = a$, $b = -(a+c)$, $c = c$ $(a+c)^2 - 4ac = 0$ •3 $a^2 - 2ac - c^2 = (a-c)^2 = 0$ •4 For $(a-c)^2 = 0$ then $a = c$

Total 70 marks