Higher : Differentiation 1 Revision

2008 P1

21. A function f is defined on the set of real numbers by $f(x) = x^3 - 3x + 2$.

(*a*) Find the coordinates of the stationary points on the curve $y = f(x)$ and determine their nature. 6

(*b*) (i) Show that $(x - 1)$ is a factor of $x^3 - 3x + 2$.

(ii) Hence or otherwise factorise $x^3 - 3x + 2$ fully. 5

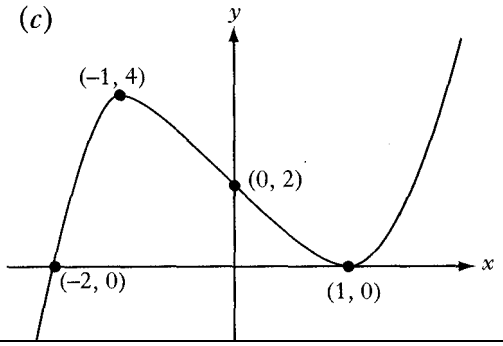
(*c*) State the coordinates of the points where the curve with equation $y = f(x)$ meets both the axes and hence sketch the curve. 4

Ans

(*a*) (–1, 4) maximum
(1, 0) minimum

(*b*) (i) $x = 1$, $f(x) = 0$
so $(x - 1)$ is a factor
(ii) $(x - 1)(x - 1)(x + 2)$

(*c*)

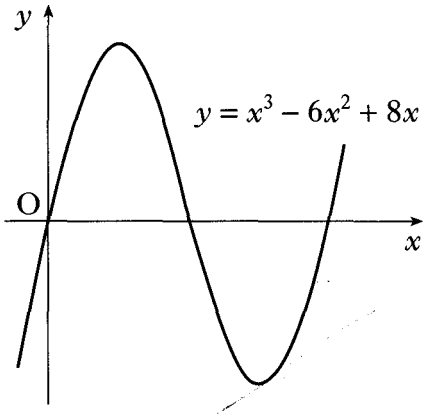


2008 P1

22. The diagram shows a sketch of the curve with equation $y = x^3 - 6x^2 + 8x$.

(*a*) Find the coordinates of the points on the curve where the gradient of the tangent is –1. 5

(*b*) The line $y = 4 - x$ is a tangent to this curve at a point A. Find the coordinates of A. 2



Ans (a) (1,3), (3,-3) (b) (1,3)

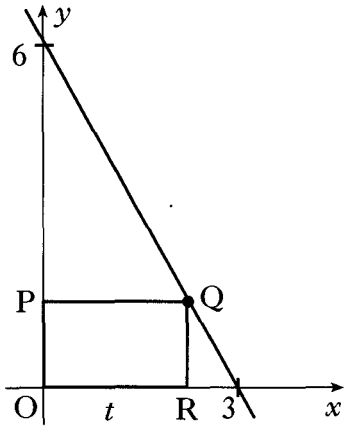
2008 P2

6. In the diagram, Q lies on the line joining (0, 6) and (3, 0).

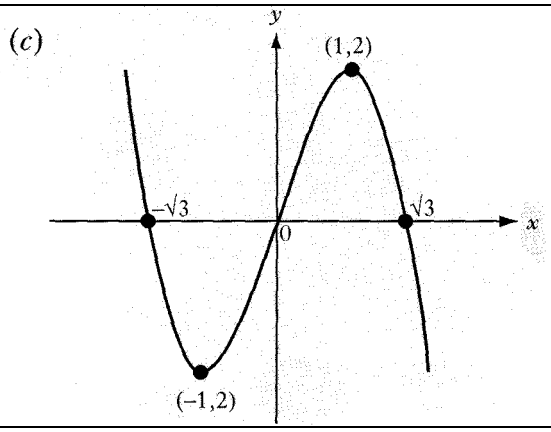
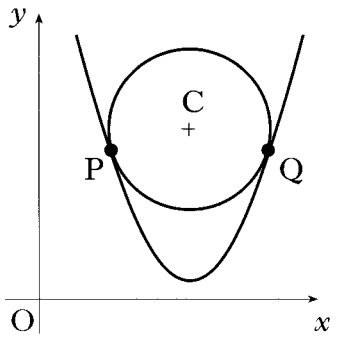
OPQR is a rectangle, where P and R lie on the axes and OR = t.

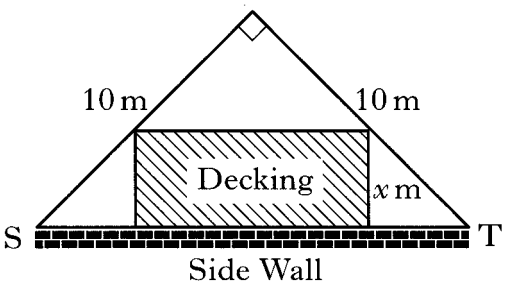
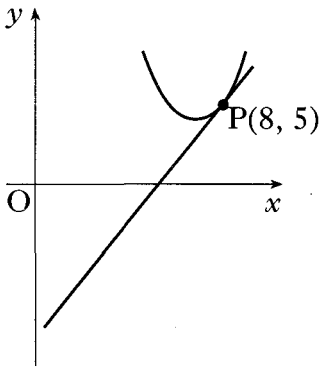
(*a*) Show that QR = $6 - 2t$. 3

(*b*) Find the coordinates of Q for which the rectangle has a maximum area. 6



Ans (a) proof (b) (1.5,3)

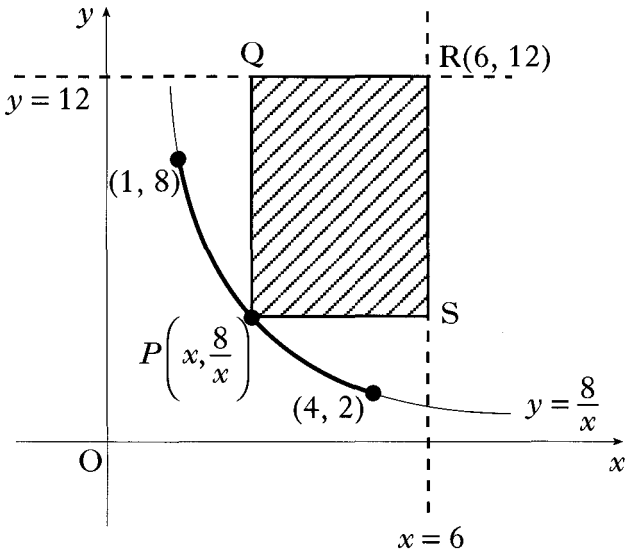
2007 P1	**9.** A function f is defined by the formula $f(x) = 3x - x^3$. (*a*) Find the exact values where the graph of $y = f(x)$ meets the x- and y-axes. (*b*) Find the coordinates of the stationary points of the function and determine their nature. (*c*) Sketch the graph of $y = f(x)$.	2 7 1
Ans	(*a*) $(-\sqrt{3}, 0)$, $(0,0)$, $(\sqrt{3},0)$ (*b*) (1,2): maximum (−1,−2): minimum (*c*) 	
2007 P2	**5.** A circle centre C is situated so that it touches the parabola with equation $y = \frac{1}{2}x^2 - 8x + 34$ at P and Q. (*a*) The gradient of the tangent to the parabola at Q is 4. Find the coordinates of Q. (*b*) Find the coordinates of P. (*c*) Find the coordinates of C, the centre of the circle. 	5 2 2
Ans	(*a*) Q = (12,10) (*b*) P = (4,10) (*c*) C = (8,11)	

2007 P2	**6.** A householder has a garden in the shape of a right-angled isosceles triangle. It is intended to put down a section of rectangular wooden decking at the side of the house, as shown in the diagram.  (*a*) (i) Find the exact value of ST. (ii) Given that the breadth of the decking is x metres, show that the area of the decking, A square metres, is given by $A = \left(10\sqrt{2}\right)x - 2x^2.$ (*b*) Find the dimensions of the decking which maximises its area.	3 5
Ans	(*a*) (i) ST = $\sqrt{200}$ (ii) Length of decking = $\sqrt{200} - 2x$ So A = $x(\sqrt{200} - 2x)$ = $(10\sqrt{2}) - 2x^2$ (*b*) $x = \frac{10\sqrt{2}}{4}$ length = $5\sqrt{2}$	
2006 P2	**3.** The parabola with equation $y = x^2 - 14x + 53$ has a tangent at the point P(8, 5). (*a*) Find the equation of this tangent.  	4
Ans	(*a*) $y - 5 = 2(x - 8)$	

2006 P2

12. PQRS is a rectangle formed according to the following conditions:

- it is bounded by the lines $x = 6$ and $y = 12$
- P lies on the curve with equation $y = \frac{8}{x}$ between (1, 8) and (4, 2)
- R is the point (6, 12).



(*a*) (i) Express the lengths of PS and RS in terms of x, the x-coordinate of P.

(ii) Hence show that the area, A square units, of PQRS is given by

$$A = 80 - 12x - \frac{48}{x}.$$

3

(*b*) Find the greatest and least possible values of A and the corresponding values of x for which they occur.

8

Ans

(*a*) (i) PS = $6 - x$

RS = $12 - \frac{8}{x}$

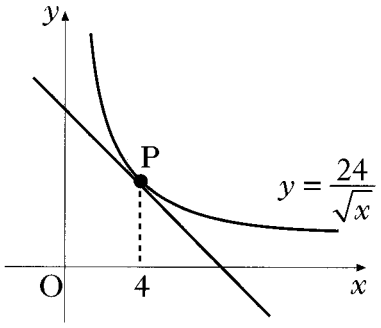
(ii) Area = $(6 - x)\left(12 - \frac{8}{x}\right)$ and complete

(*b*) max.A = 32 at $x = 2$ **and**
min.A = 20 at $x = 1$ **or** $x = 4$

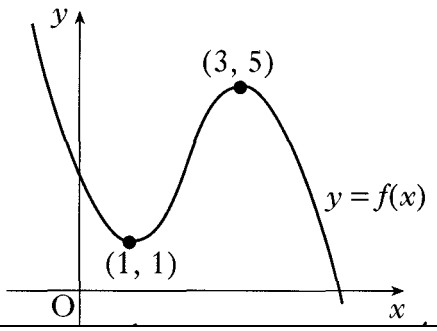
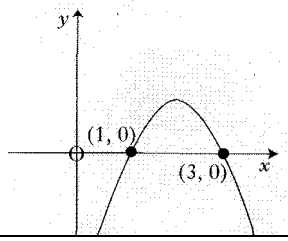
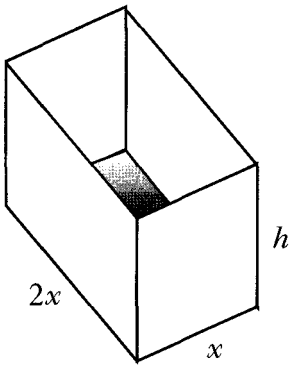
2005 P2

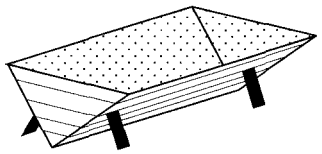
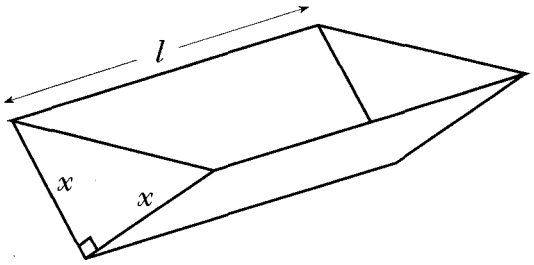
6. The diagram shows the graph of $y = \frac{24}{\sqrt{x}}$, $x > 0$.

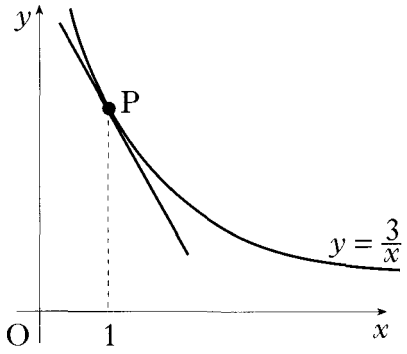
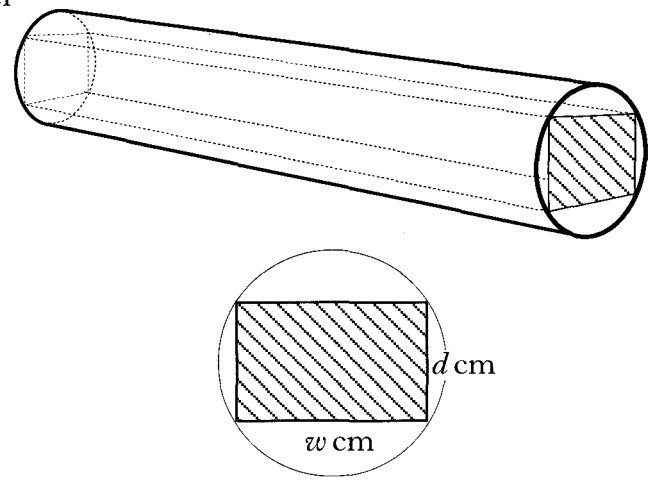
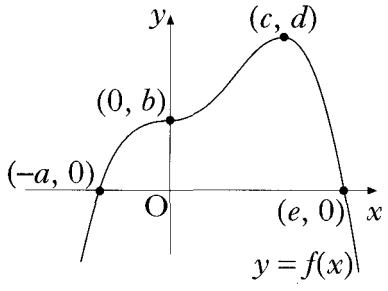
Find the equation of the tangent at P, where $x = 4$.



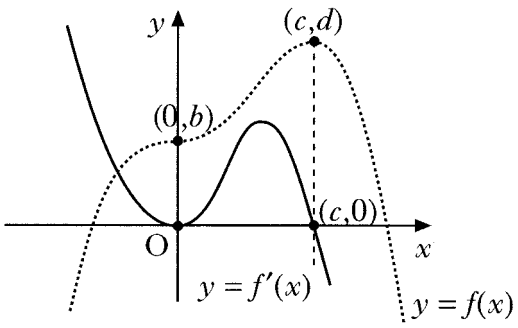
6

Ans	$y - 12 = -\frac{3}{2}(x - 4)$	
2004 P2	**5.** The point P(x, y) lies on the curve with equation $y = 6x^2 - x^3$. (*a*) Find the value of x for which the gradient of the tangent at P is 12. (*b*) Hence find the equation of the tangent at P.	5 2
Ans	(*a*) $x = 2$ (*b*) $y = 12x - 8$	
2004 P2	**7.** The graph of the cubic function $y = f(x)$ is shown in the diagram. There are turning points at (1, 1) and (3, 5). Sketch the graph of $y = f'(x)$. 	3
Ans		
2004 P2	**9.** An open cuboid measures internally x units by $2x$ units by h units and has an inner surface area of 12 units^2.  (*a*) Show that the volume, V units^3, of the cuboid is given by $V(x) = \frac{2}{3}x(6 - x^2)$. (*b*) Find the exact value of x for which this volume is a maximum.	3 5
Ans	(*a*) $A = 2x^2 + 2xh + 4xh = 12$ $V = 2x \times x \times h$ $V = 2x \times \frac{12 - 2x^2}{6}$ $V = \frac{2}{3}x(6 - x^2)$ (*b*) $x = \sqrt{2}$	

2003 P1	**5.** Given that $f(x) = \sqrt{x} + \frac{2}{x^2}$, find $f'(4)$.	5
Ans	$\frac{3}{16}$	
2003 P2	**4.** (*a*) Find the equation of the tangent to the curve with equation $y = x^3 + 2x^2 - 3x + 2$ at the point where $x = 1$.	5
Ans	(*a*) $y = 4x - 2$	
2003 P2	**8.** An open water tank, in the shape of a triangular prism, has a capacity of 108 litres. The tank is to be lined on the inside in order to make it watertight.  The triangular cross-section of the tank is right-angled and isosceles, with equal sides of length x cm. The tank has a length of l cm.  (*a*) Show that the surface area to be lined, A cm^2, is given by $A(x) = x^2 + \frac{432000}{x}$. (*b*) Find the value of x which minimises this surface area.	3 5
Ans	(*a*) length $= \frac{108000}{\frac{1}{2}x^2}$ $SA = 2 \times \frac{1}{2}x^2 + 2x \times \text{length}$ $SA = x^2 + \frac{432000}{x}$ $\frac{dA}{dx} = 2x - \frac{432000}{x^2}$ $\frac{dA}{dx} = 0$ $x = 60$ Justify minimum using, e.g. nature table (*b*) 60	

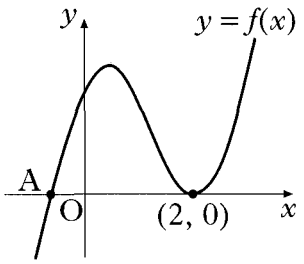
2002W P1	**3.** Find the equation of the tangent to the curve with equation $y = \frac{3}{x}$ at the point P where $x = 1$.  	5
Ans	$y + 3x = 6$	
2002W P2	**7.** A rectangular beam is to be cut from a cylindrical log of diameter 20 cm.  The diagram shows a cross-section of the log and beam where the beam has a breadth of w cm and a depth of d cm. The strength S of the beam is given by $S = 1{\cdot}7\ w\ (400 - w^2)$. Find the dimensions of the beam for maximum strength. 	5
Ans	Solve $\frac{dS}{dw} = 0$ and test for max/min $d = 20\sqrt{\frac{2}{3}}$	
2002 P1	**4.** Find the coordinates of the point on the curve $y = 2x^2 - 7x + 10$ where the tangent to the curve makes an angle of 45° with the positive direction of the x-axis.	4
Ans	(2, 4)	
2002 P1	**6.** The graph of a function f intersects the x-axis at $(-a, 0)$ and $(e, 0)$ as shown. There is a point of inflexion at $(0, b)$ and a maximum turning point at (c, d). Sketch the graph of the derived function f'.  	3

Ans



2002 P2

3. The diagram shows part of the graph of the curve with equation $y = 2x^3 - 7x^2 + 4x + 4$.



(*a*) Find the x-coordinate of the maximum turning point. 5

(*b*) Factorise $2x^3 - 7x^2 + 4x + 4$. 3

(*c*) State the coordinates of the point A and hence find the values of x for which $2x^3 - 7x^2 + 4x + 4 < 0$. 2

Ans

(*a*) $f'(x) = 6x^2 - 14x + 4 = 0$

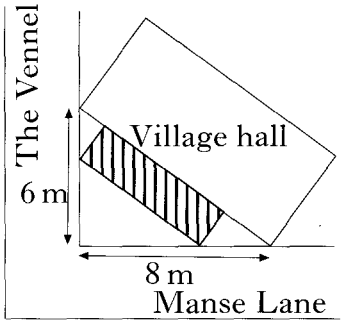
$x = \frac{1}{3}$

(*b*) $(x-2)(2x+1)(x-2)$

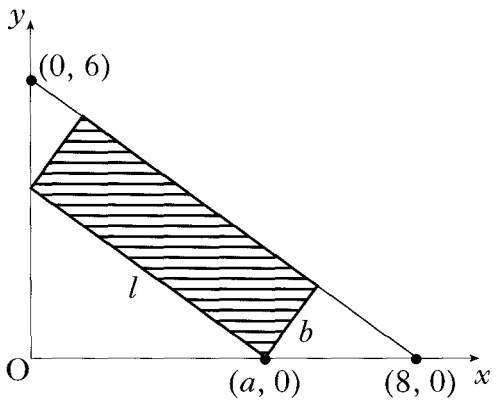
(*c*) $A(-\frac{1}{2}, 0)$, $x < -\frac{1}{2}$

2002 P2

10. The shaded rectangle on this map represents the planned extension to the village hall. It is hoped to provide the largest possible area for the extension.



The coordinate diagram represents the right angled triangle of ground behind the hall. The extension has length l metres and breadth b metres, as shown. One corner of the extension is at the point $(a, 0)$.



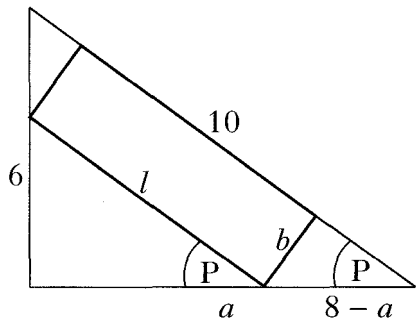
(a) (i) Show that $l = \frac{5}{4}a$.

(ii) Express b in terms of a and hence deduce that the area, A m^2, of the extension is given by $A = \frac{3}{4}a(8-a)$. 3

(b) Find the value of a which produces the largest area of the extension. 4

Ans

(a) proof



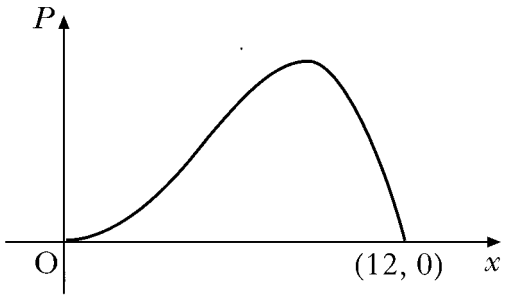
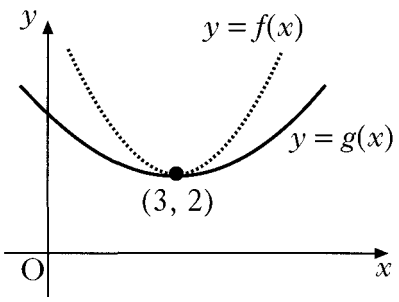
$\cos P = \frac{8}{10} = \frac{a}{1} \Rightarrow l = \frac{10}{8}a = \frac{5}{4}a$

$\sin P = \frac{6}{10} = \frac{b}{8-a} \Rightarrow b = \frac{6}{10}(8-a)$

Area $= lb = \frac{5}{4}a \times \frac{6}{10}(8-a) = \frac{3}{4}a(8-a)$

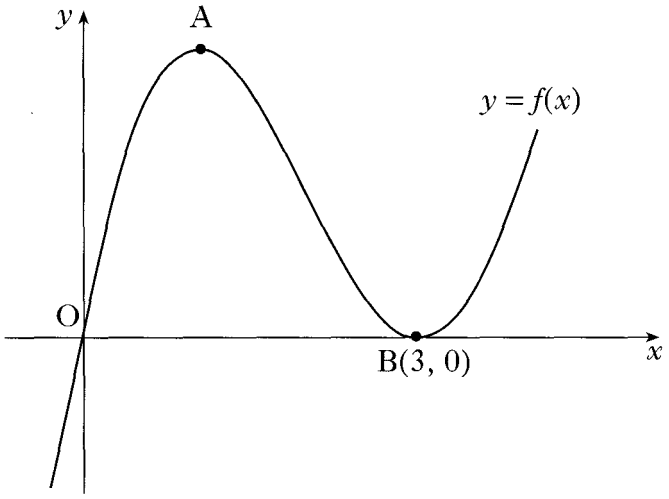
(b) Solve $\frac{dA}{da} = 0$ and test for max/min

$a = 4$

2001 P1	**6.** A company spends x thousand pounds a year on advertising and this results in a profit of P thousand pounds. A mathematical model, illustrated in the diagram, suggests that P and x are related by $P = 12x^3 - x^4$ for $0 \le x \le 12$. Find the value of x which gives the maximum profit. 	5
Ans	$x = 9$	
2001 P1	**9.** The diagram shows the graphs of two quadratic functions $y = f(x)$ and $y = g(x)$. Both graphs have a minimum turning point at (3, 2). Sketch the graph of $y = f'(x)$ and on the same diagram sketch the graph of $y = g'(x)$. 	2
Ans	straight line for f' through (3,0), $m_{f'} > 0$ straight line for g' through (3,0), $m_{f'} > m_{g'} > 0$	
2001 P2	**2.** A curve has equation $y = x - \frac{16}{\sqrt{x}}$, $x > 0$. Find the equation of the tangent at the point where $x = 4$.	6
Ans	$y = 2x - 12$	

2000 P1

2. A sketch of the graph of $y = f(x)$ where $f(x) = x^3 - 6x^2 + 9x$ is shown below. The graph has a maximum at A and a minimum at B(3, 0).

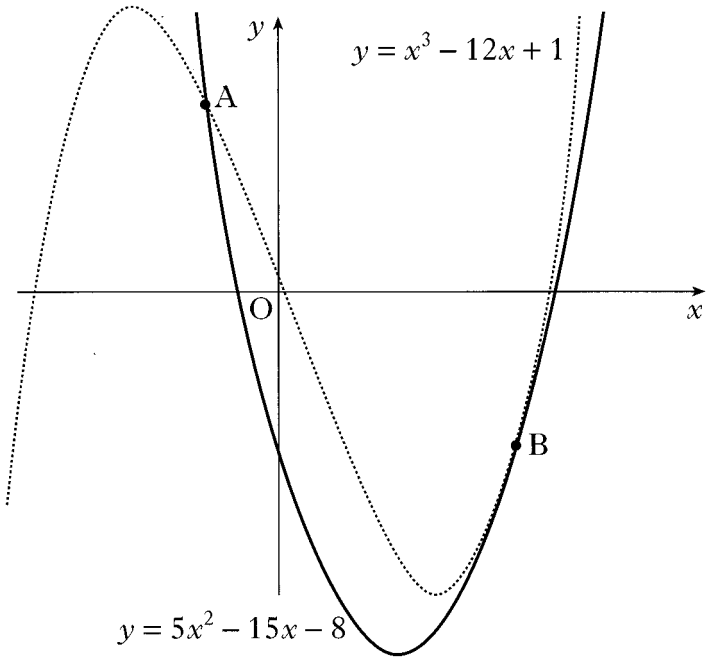


(*a*) Find the coordinates of the turning point at A. 4

Ans (*a*) A = (1,4)

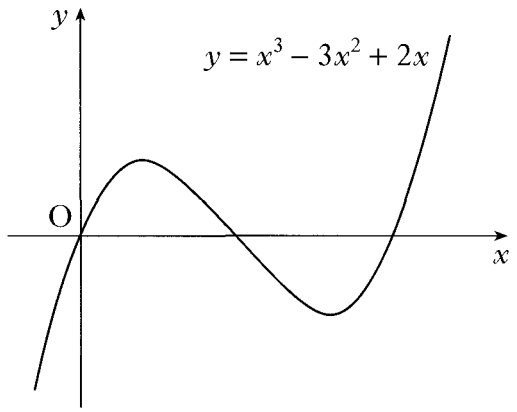
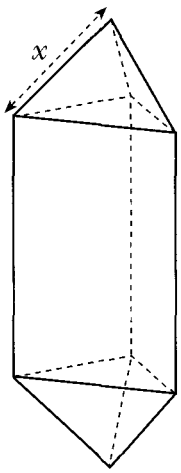
2000 P1

4. The diagram shows a sketch of the graphs of $y = 5x^2 - 15x - 8$ and $y = x^3 - 12x + 1$. The two curves intersect at A and touch at B, ie at B the curves have a common tangent.



(*a*) (i) Find the x-coordinates of the points on the curves where the gradients are equal. 4

(ii) By considering the corresponding y-coordinates, or otherwise, distinguish geometrically between the two cases found in part (i). 1

Ans	(*a*) (i) $x = \frac{1}{3}$ and $x = 3$ (ii) parallel and coincident	
2000 P2	**1.** The diagram shows a sketch of the graph of $y = x^3 - 3x^2 + 2x$. (*a*) Find the equation of the tangent to this curve at the point where $x = 1$. (*b*) The tangent at the point (2, 0) has equation $y = 2x - 4$. Find the coordinates of the point where this tangent meets the curve again. 	5 5
Ans	(*a*) $x + y = 1$ (*b*) $(-1, -6)$	
2000 P2	**6.** A goldsmith has built up a solid which consists of a triangular prism of fixed volume with a regular tetrahedron at each end. The surface area, A, of the solid is given by $$A(x) = \frac{3\sqrt{3}}{2}\left(x^2 + \frac{16}{x}\right)$$ where x is the length of each edge of the tetrahedron. Find the value of x which the goldsmith should use to minimise the amount of gold plating required to cover the solid. 	6
Ans	$x = 2$	
Specimen 2 P1	**3.** The point P(–1, 7) lies on the curve with equation $y = 5x^2 + 2$. Find the equation of the tangent to the curve at P.	3
Ans	$y - 7 = -10(x - (-1))$	
Specimen 2 P2	**9.** A curve has equation $y = 2x^3 + 3x^2 + 4x - 5$. Prove that this curve has no stationary points.	5

Ans

$\frac{dy}{dx} = 6x^2 + 6x + 4$

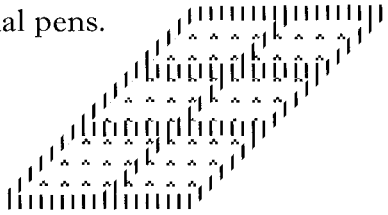
$b^2 - 4ac = -60$

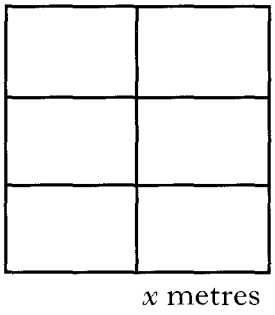
$6x^2 + 6x + 4$ has no roots

$\frac{dy}{dx} = 0$ has no solutions so curve has no stationary points

Specimen 2 P1

10. A zookeeper wants to fence off six individual animal pens.





Each pen is a rectangle measuring x metres by y metres, as shown in the diagram.

(*a*) (i) Express the total length of fencing in terms of x and y.

(ii) Given that the total length of fencing is 360 m, show that the total area, A m^2, of the six pens is given by $A(x) = 240x - \frac{16}{3}x^2$. 4

(*b*) Find the values of x and y which give the maximum area and write down this maximum area. 6

Ans

10. (*a*) length $= 9y + 8x = 360$

$A = 3y \times 2x = 2x.3.\frac{1}{9}(360 - 8x)240x - \frac{16}{3}x^2$

(*b*) $A'(x) = 240 - \frac{32}{3}x$

$A'(x) = 0 \quad \Rightarrow x = 22\frac{1}{2}, \quad y = 20$

x	$22\frac{1}{2}^-$	$22\frac{1}{2}$	$22\frac{1}{2}^+$
$A'(x)$	+	0	−
		maximum	

$A_{max} = 2700$

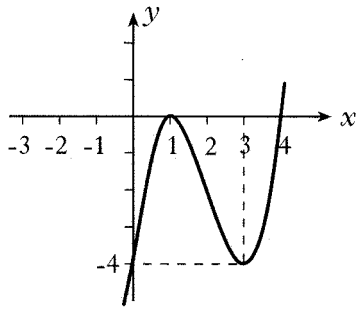
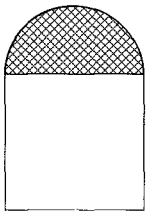
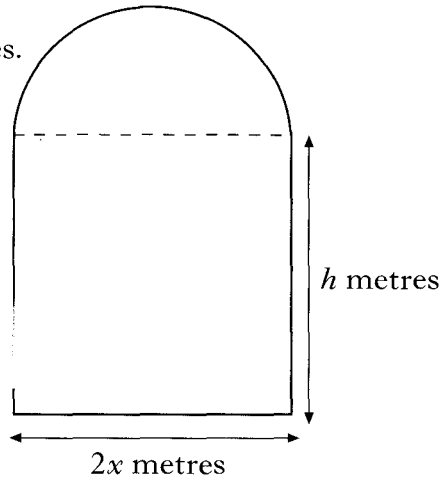
Specimen 1 P1

3. (*a*) Show that $(x - 1)$ is a factor of $f(x) = x^3 - 6x^2 + 9x - 4$ and find the other factors. 3

(*b*) Write down the coordinates of the points at which the graph of $y = f(x)$ meets the axes. 1

(*c*) Find the stationary points of $y = f(x)$ and determine the nature of each. 5

(*d*) Sketch the graph of $y = f(x)$. 1

Ans	(*d*) (*a*) $f(1) = 0$, $(x-4)$, $(x-1)$ (*b*) (1,0), (4,0), (0, −4) (*c*) max at (1,0), min at (3, −4) 	
Specimen 1 P2	**4.** In the diagram below, a winding river has been modelled by the curve $y = x^3 - x^2 - 6x - 2$ and a road has been modelled by the straight line AB. The road is a tangent to the river at the point A(1, −8). (*a*) Find the equation of the tangent at A.	3
Ans	(*a*) $y = -5x - 3$	
Specimen 1 P2	**9.** A window in the shape of a rectangle surmounted by a semicircle is being designed to let in the maximum amount of light. The glass to be used for the semicircular part is stained glass which lets in one unit of light per square metre; the rectangular part uses clear glass which lets in 2 units of light per square metre.  The rectangle measures $2x$ metres by h metres.  (*a*) (i) If the perimeter of the whole window is 10 metres, express h in terms of x. (ii) Hence show that the amount of light, L, let in by the window is given by $L = 20x - 4x^2 - \frac{3}{2}\pi x^2$. (*b*) Find the values of x and h that must be used to allow this design to let in the maximum amount of light.	2 2 5

Ans	(*a*) (i) $h = \frac{1}{2}(10 - \pi x - 2x)$ (ii) $L = 2 \times 2xh + \frac{1}{2}\pi x^2$ $= 4x \times \frac{1}{2}(10 - \pi x - 2x) + \frac{1}{2}\pi x^2$ $= 20x - 2\pi^2 - 4x^2 + \frac{1}{2}\pi x^2$ (*b*) $x = \frac{20}{3\pi + 8}$, $h = \frac{5(\pi + 4)}{3\pi + 8}$