HIGHER MATHEMATICS
COURSE NOTES

UNIT 1

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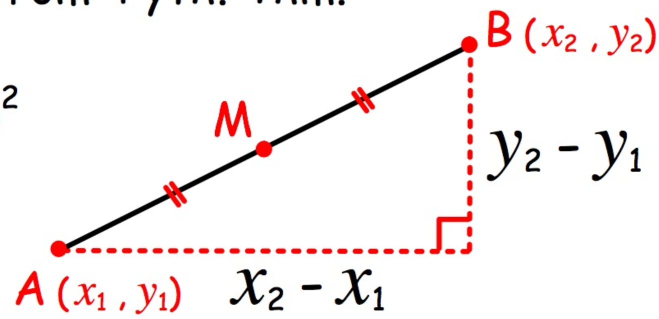
STRAIGHT LINE

DISTANCE FORMULA   from Pyth. Thm.

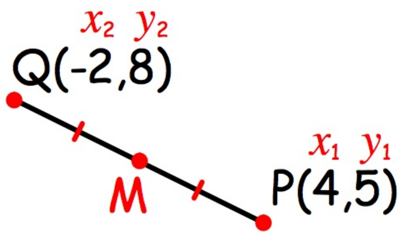
$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$



MID-POINT

$$M_{AB}\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

Points P(4,5) and Q(-2,8).



$$PQ^2 = (\overset{x_2}{-2} - \overset{x_1}{4})^2 + (\overset{y_2}{8} - \overset{y_1}{5})^2$$

$$= (-6)^2 + 3^2$$

$$= 36 + 9$$

$$= 45$$

$$PQ = \sqrt{45}$$

$$= \sqrt{9} \times \sqrt{5}$$

$$PQ = 3\sqrt{5} \text{ units}$$

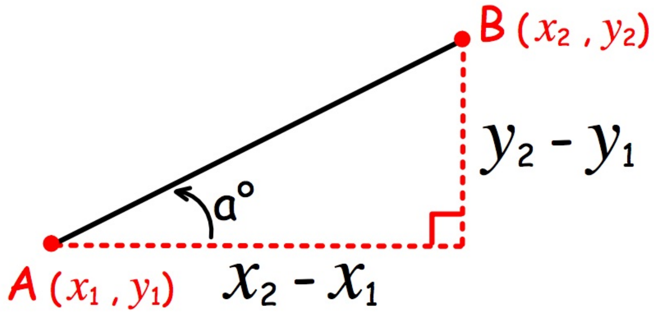
$$\frac{\overset{x_1}{4} + \overset{x_2}{(-2)}}{2} \qquad \frac{\overset{y_1}{5} + \overset{y_2}{8}}{2}$$

$$M_{PQ}\left(1, \frac{13}{2}\right)$$

Note: same result whichever is the first point.

GRADIENT FORMULA

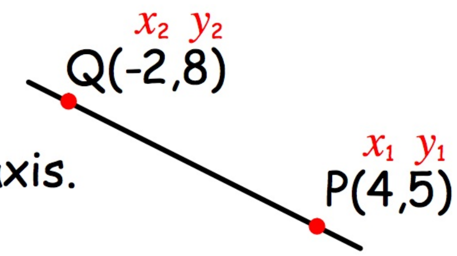
$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} \quad , \; x_2 \neq x_1$$



$$m_{AB} = \tan a° \quad , \; 0 < a < 180$$

a° is the anti-clockwise angle with the positive OX direction

Find the angle the line through P(4,5) and Q(-2,8) makes with the x-axis.



$$m_{PQ} = \frac{\overset{y_2}{8} - \overset{y_1}{5}}{\underset{x_2}{-2} - \underset{x_1}{4}}$$

$$\frac{5 - 8}{4 - (-2)}$$

NOTE: same result whichever is the first point.

$$= \frac{3}{-6}$$

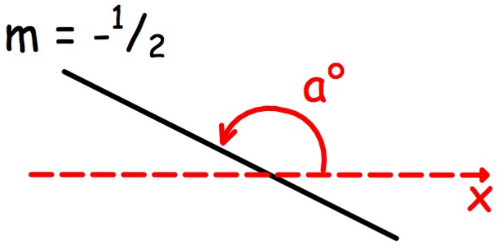
$$= -\frac{1}{2}$$

$$\tan a° = -\frac{1}{2}$$

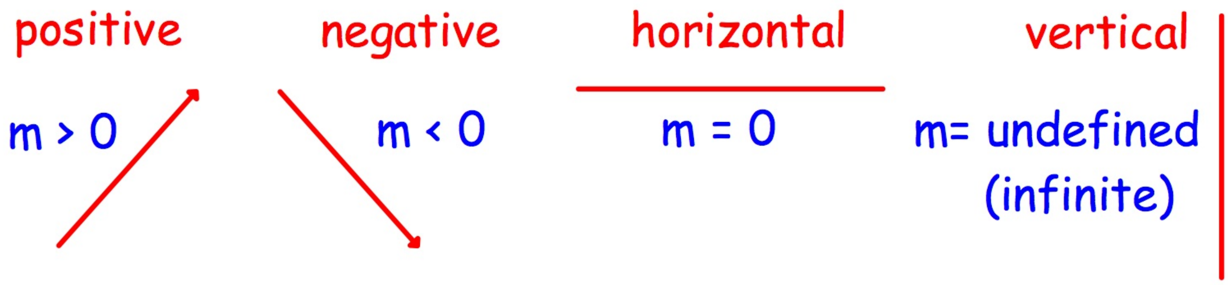
$\tan^{-1}(^1/_2)$

$$a = 180 - 26.565....$$

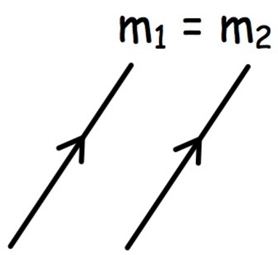
<u>angle 153.4°</u>

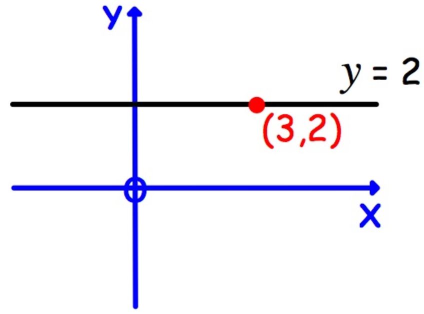


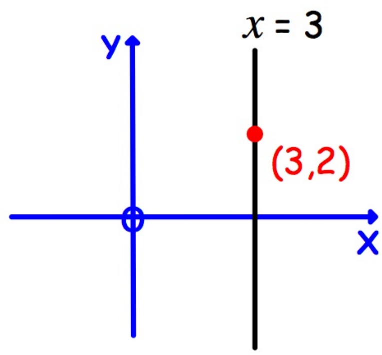
NOTE:



parallel lines: equations:



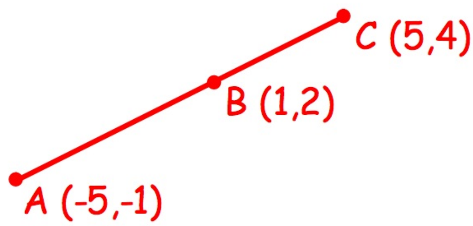




COLLINEARITY

Collinear points lie on the same straight line.

Show that the points A, B and C are collinear.



$$m_{AB} = \frac{2-(-1)}{1-(-5)} = \frac{3}{6} = \frac{1}{2}$$

$$m_{BC} = \frac{4-2}{5-1} = \frac{2}{4} = \frac{1}{2}$$

$m_{AB} = m_{BC}$

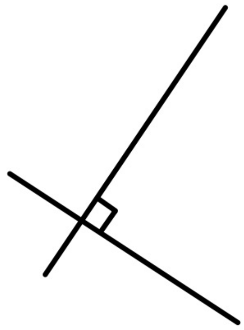
$\Rightarrow$ AB is parallel to BC

and since lines share point B

$\Rightarrow$ points A, B and C are collinear

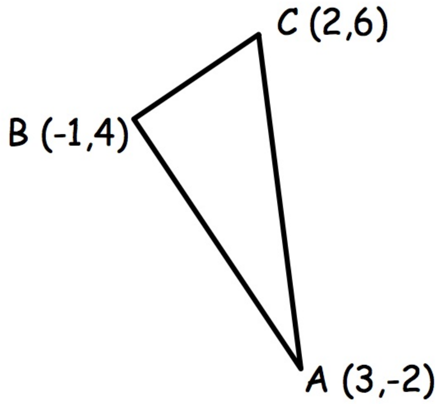
PERPENDICULAR LINES

$$m_1 \times m_2 = -1$$



$$m_1 = \frac{a}{b}$$

$$m_2 = -\frac{b}{a}$$



Show that ΔABC is right-angled at B.

$$m_{AB} = \frac{4-(-2)}{-1-3} = \frac{6}{-4} = -\frac{3}{2}$$

$$m_{BC} = \frac{6-4}{2-(-1)} = \frac{2}{3}$$

$$m_{AB} \times m_{BC} = -\frac{3}{2} \times \frac{2}{3} = -1$$

$$m_{AB} \times m_{BC} = -1$$

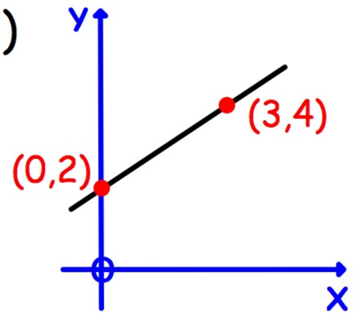
$\Rightarrow$ AB is perpendicular BC

$\Rightarrow \angle ABC = 90°$

EQUATION OF A LINE

Gradient m and Y-intercept C : $y = mx + C$

(1)



$m = \frac{4-2}{3-0} = \frac{2}{3}$

$C = 2$

$y = mx + C$

$y = \frac{2}{3}x + 2$

x3 to remove fraction

$3y = 2x + 6$

$0 = 2x - 3y + 6$

Can also be written in the form Ax + By + C = 0

$2x - 3y + 6 = 0$

(2) Find the gradient of the line $2x - 3y - 9 = 0$.

rearrange to $y = mx + C$

$2x - 9 = 3y$

$\frac{2}{3}x - 3 = y$

$y = \frac{2}{3}x - 3$

$m = \frac{2}{3}$

(3) If point (k,4) lies on line $2x + 3y = 6$, find k.

$x = k$, $y = 4$

$2 \times k + 3 \times 4 = 6$

$2k = -6$

$k = -3$

EQUATION OF A LINE

Gradient m and through point (a,b) : $y - b = m(x - a)$

(1) Equation of the line gradient $-^2/_3$, through P(-1,2).

P($\overset{a}{-1}$, $\overset{b}{2}$)

$m = -^2/_3$

$$y - b = m(x - a)$$

$$y - 2 = \frac{-2}{3}(x - (-1))$$

remove fraction before breaking brackets

$$3(y - 2) = -2(x + 1)$$

multiplied both sides by 3

$$3y - 6 = -2x - 2$$

$$\underline{\underline{3y = -2x + 4}}$$

(2) Equation of the line through P(2,1) and Q(3,-1).

P($\overset{x_2}{2}$, $\overset{y_2}{1}$)

Q($\overset{x_1}{3}$, $\overset{y_1}{-1}$)

$$m_{PQ} = \frac{\overset{y_2}{1} - \overset{y_1}{(-1)}}{\underset{x_2}{2} - \underset{x_1}{3}} = \frac{2}{-1} = -2$$

P($\overset{a}{2}$, $\overset{b}{1}$)

$m = -2$

$$y - b = m(x - a)$$

$$y - 1 = -2(x - 2)$$

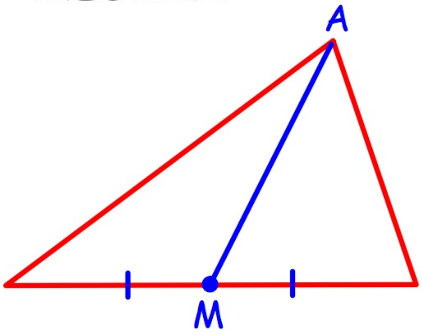
or can use Q(3,-1)

$$y - 1 = -2x + 4$$

$$\underline{\underline{y = -2x + 5}}$$

LINES ASSOCIATED WITH TRIANGLES

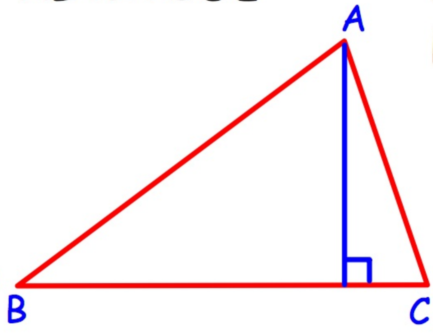
MEDIAN



FIND:
mid-point M

EQUATION:
use points A and M

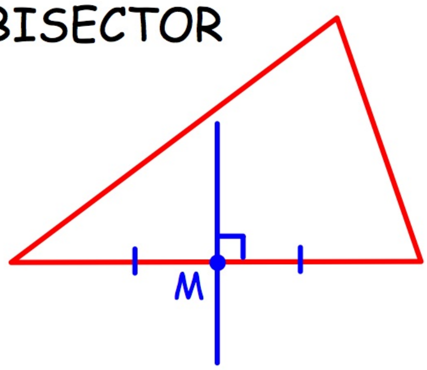
ALTITUDE



gradient BC
ppn. gradient
(by $m_1 \times m_2 = -1$)

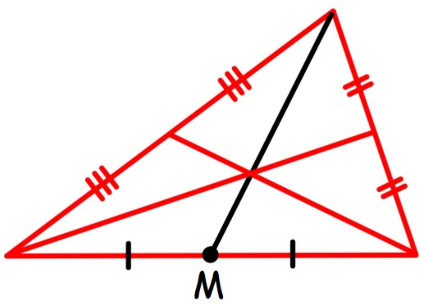
use point A
and ppn. gradient

PERPENDICULAR BISECTOR

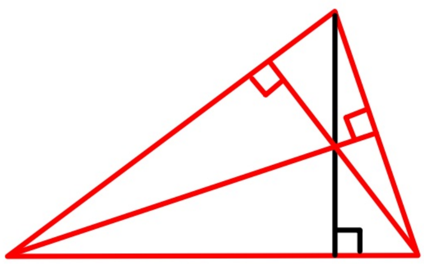


gradient BC
ppn. gradient
mid-point M

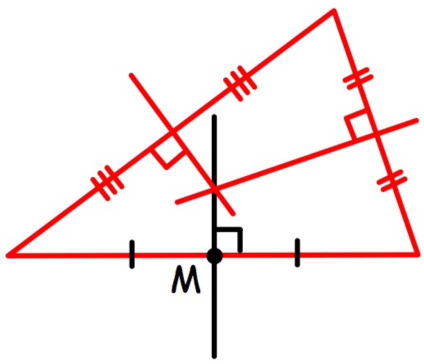
use point M
and ppn. gradient



the 3 medians are concurrent



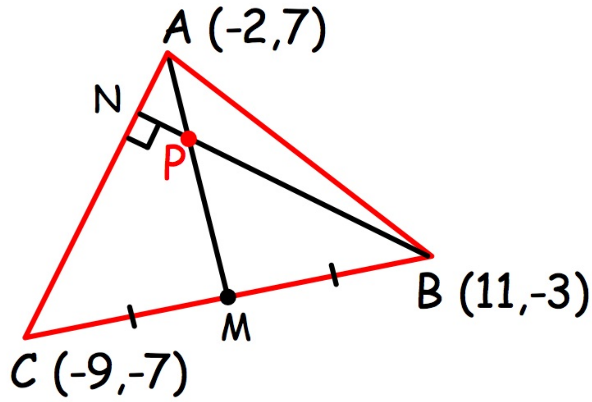
the 3 altitudes are concurrent



the 3 ppn bis are concurrent

Lines are concurrent if they pass through a common point.

Find P, the point of intersection of the median AM and altitude BN.



MEDIAN AM

midpoint: $M(\overset{x_1}{1}, \overset{y_1}{-5})$ $\left(\dfrac{-9+11}{2}, \dfrac{-7+(-3)}{2}\right)$

gradient: $A(\overset{x_2}{-2}, \overset{y_2}{7})$

$$m_{AM} = \frac{\overset{y_2}{7} - \overset{y_1}{(-5)}}{\underset{x_2}{-2} - \underset{x_1}{1}} = \frac{12}{-3} = -4$$

line:

$A(\overset{a}{-2}, \overset{b}{7})$ $m = -4$

$$y - b = m(x - a)$$

$$y - 7 = -4\,(x - (-2))$$

$$y - 7 = -4x - 8$$

$$\underline{\underline{4x + y = -1}}$$

ALTITUDE BN

gradient:

$$A(\overset{x_2}{-2}, \overset{y_2}{7}) \qquad C(\overset{x_1}{-9}, \overset{y_1}{-7})$$

$$m_{AC} = \frac{\overset{y_2}{7} - \overset{y_1}{(-7)}}{\underset{x_2}{-2} - \underset{x_1}{(-9)}} = \frac{14}{7} = 2$$

ppn. gradient: by $m_1 \times m_2 = -1$ $\qquad m_{BN} = -{}^1/_2$

line:

$A(\overset{a}{11}, \overset{b}{-3})$ $\quad m = -{}^1/_2$

$$y - b = m(x - a)$$

$$y - (-3) = \frac{-1}{2}(x - 11)$$

$$2y + 6 = -x + 11$$

$$\underline{\underline{x + 2y = 5}}$$

INTERSECTION

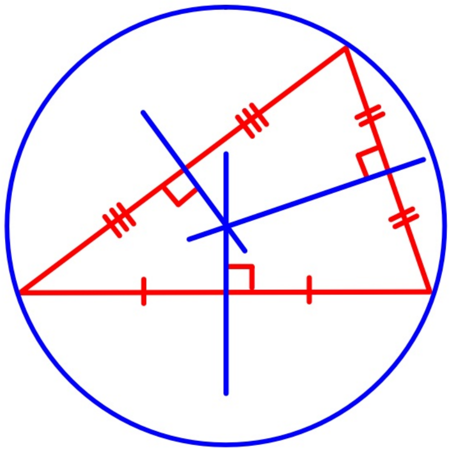
solve the system of equations

$$\begin{array}{lrcl} \text{AM} & 4x + y & = & -1 \quad \times 2 \\ \text{BN} & x + 2y & = & 5 \quad \times (-1) \\ \hline & 8x + 2y & = & -2 \\ & -x - 2y & = & -5 \\ \hline \text{add} & 7x & = & -7 \\ & x & = & -1 \end{array}$$

$$\begin{array}{rcl} 4x + y & = & -1 \\ -4 + y & = & -1 \\ y & = & 3 \end{array}$$

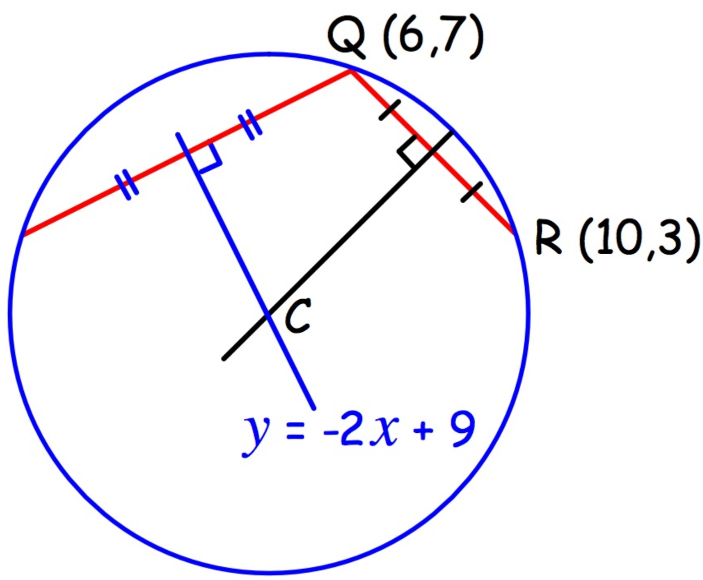
point of intersection $\underline{\underline{P(-1,3)}}$

CIRCLE



The 3 perpendicular bisectors are concurrent at the centre of the CIRCUMCIRCLE.

Find C, the centre of the circle.



PERPENDICULAR BISECTOR CM

gradient: $Q(\overset{x_2}{6}, \overset{y_2}{7})$ $R(\overset{x_1}{10}, \overset{y_1}{3})$

$$m_{QR} = \frac{\overset{y_2}{7} - \overset{y_1}{3}}{\underset{x_2}{6} - \underset{x_1}{10}} = \frac{4}{-4} = -1$$

ppn. gradient: by $m_1 \times m_2 = -1$ $\qquad m_{CM} = 1$

midpoint: $M_{QR}(8, 5)$

line:

$M(\overset{a}{8}, \overset{b}{5})$ $\quad m = 1$

$$y - b = m(x - a)$$
$$y - 5 = 1(x - 8)$$
$$\underline{\underline{y = x - 3}}$$

INTERSECTION

solve the system of equations

$$\begin{aligned} y &= -2x + 9 \\ y &= x - 3 \end{aligned}$$

$$\begin{aligned} -2x + 9 &= x - 3 \\ -3x &= -12 \\ x &= 4 \end{aligned} \qquad \begin{aligned} y &= x - 3 \\ &= 4 - 3 \\ y &= 1 \end{aligned}$$

centre $\underline{\underline{C\,(4,1)}}$

CONCURRENCY

Lines are concurrent if they pass through a common point.

FIND the point of intersection
SHOW this point lies on the other line

solve any pair of equations and show the point 'fits' the equation

Show that the lines with equations
$x + 3y = 10$, $2x - y = -1$ and $3x + y = 6$
are concurrent.

$$\begin{aligned} x + 3y &= 10 \quad \times 1 \\ 2x - y &= -1 \quad \times 3 \end{aligned}$$

$$\begin{aligned} x + 3y &= 10 \\ 6x - 3y &= -3 \end{aligned}$$

add $7x = 7$

$$x = 1$$

$$\begin{aligned} x + 3y &= 10 \\ 1 + 3y &= 10 \\ 3y &= 9 \\ y &= 3 \end{aligned}$$

point of intersection (1,3)

third line $3x + y = 6$

(1,3) , $x = 1$ $\quad 3 \times 1 + y = 6$

$$y = 3$$

so (1,3) lies on the third line

All 3 lines pass through (1,3)

So lines concurrent at (1,3)

FUNCTIONS

Pairs members of one set of numbers with another.
Each DOMAIN element has a unique IMAGE.

This may require RESTRICTIONS on the domain.
If no domain specified - assume largest possible domain.

(1) $f(x) = \dfrac{1}{x+2}$ cannot $\div$ 0

not defined for $x = -2$

$\{\, x : x \in \mathrm{R}\,,\ x \neq -2 \,\}$

cannot $\sqrt{\text{negative}}$

(2) $f(x) = \sqrt{x - 3}$

assume $+\sqrt{\ }$

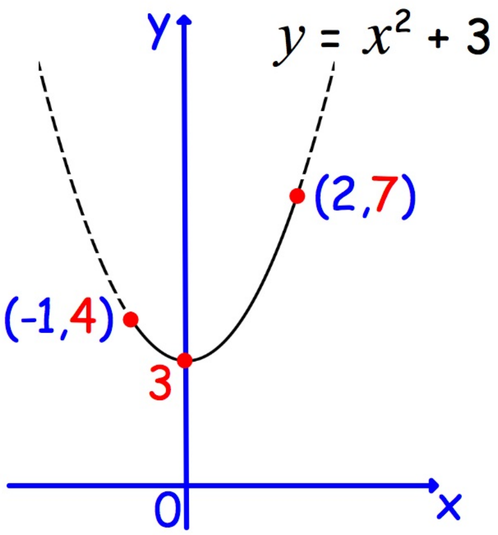
not defined for $x < 3$

$\{\, x : x \in \mathrm{R}\,,\ x \geq 3 \,\}$

The RANGE is the set of images.

Examine the graph and consider the domain specified.

Find the range of $f(x) = x^2 + 3$, $\{\, x : x \in \mathrm{R}\,,\ -1 \leq x \leq 2 \,\}$



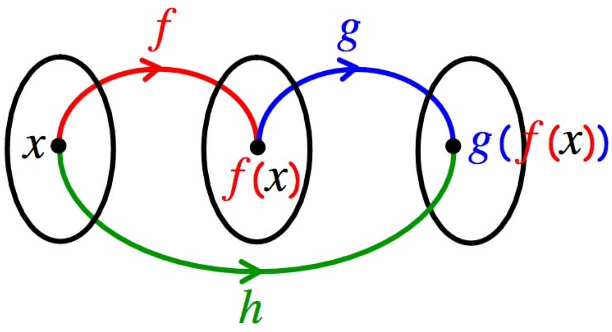
max. value 7

min. value 3

$3 \leq f(x) \leq 7$

Range $\underline{\underline{\{\, y : y \in \mathrm{R}\,,\ 3 \leq y \leq 7 \,\}}}$

COMPOSITE FUNCTIONS



The COMPOSITE function, $h(x) = g(f(x))$

acts second (g) acts first (f)

The order matters: generally $g(f(x)) \neq f(g(x))$

If $f(x) = x^2 - 2x$ and $g(x) = 1 - 2x$

(i) $g(f(x))$

$= g(x^2 - 2x)$

$= 1 - 2(x^2 - 2x)$

$= 1 - 2x^2 + 4x$

$= 1 + 4x - 2x^2$

(ii) $f(g(x))$

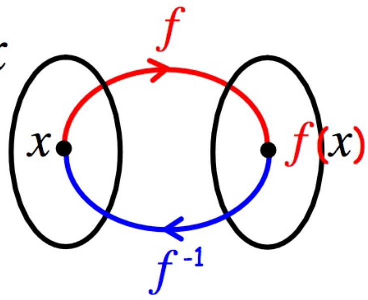
$= f(1 - 2x)$

$= (1 - 2x)^2 - 2(1 - 2x)$

$= 1 - 4x + 4x^2 - 2 + 4x$

$= 4x^2 - 1$

INVERSE FUNCTIONS $f^{-1}(f(x)) = x$



Requires a unique pairing:
one-to-one correspondence

$f(x) = x^2 + 2\ ,\ x \geq 0$ and $g(x) = \sqrt{x-2}\ ,\ x \geq 2$

Find $f(g(x))$ and comment on the result.

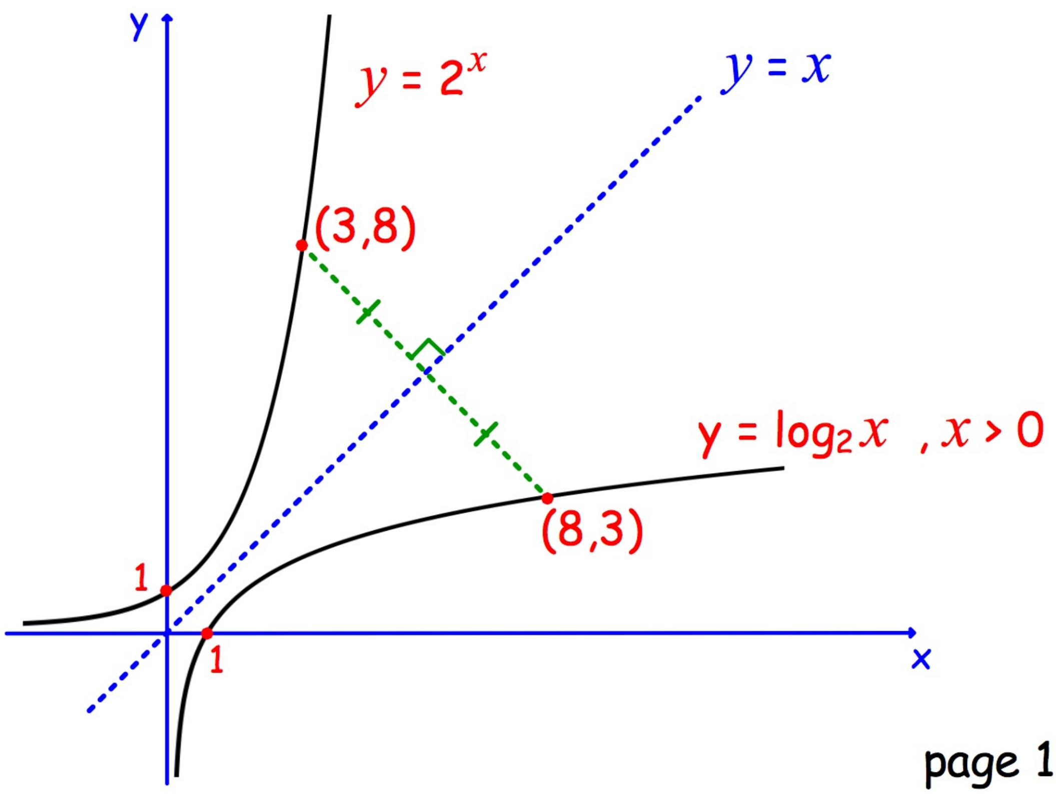
$$\begin{aligned} & f(g(x)) \\ &= f(\sqrt{x-2}\,) \\ &= (\sqrt{x-2}\,)^2 + 2 \\ &= x - 2 + 2 \\ &= x \end{aligned}$$

$f(g(x)) = x$

$\Rightarrow$ f and g are inverse functions

EXPONENTIAL and LOGARITHMIC FUNCTIONS

These are inverse functions.



FUNCTIONS and GRAPHS

COMPLETING THE SQUARE: form $a(x \pm b)^2 + c$

Squaring brackets $(x \pm b)^2 = x^2 \pm 2bx + b^2$

For example, $(x - 3)^2 = x^2 - 6x + 9$

$(x + {}^3/_2)^2 = x^2 + 3x + {}^9/_4$

÷ 2

÷ 2 and square

(1) $x^2 - 6x + 10$

÷ 2 and square

$= x^2 - 6x + 9 \quad - 9 \quad + 10$

$= \quad (x - 3)^2 \quad + \quad 1$

(2) $x^2 + 3x - 1$

÷ 2 and square

$= x^2 + 3x + {}^9/_4 \quad - {}^9/_4 \quad - 1$

$= \quad (x + {}^3/_2)^2 \quad - \quad {}^{13}/_4$

(3) $2x^2 - 12x + 10$

× 2

$= 2(x^2 - 6x) \qquad + 10$

$= 2(x^2 - 6x + 9 - 9) \; + 10$

$= 2(x^2 - 6x + 9) - 18 + 10$

$= 2(x - 3)^2 \; - \; 8$

(4) $3 - 6x - x^2$

× -1

$= -1(x^2 + 6x) \qquad + 3$

$= -1(x^2 + 6x + 9 - 9) \; + 3$

$= -1(x^2 + 6x + 9) + 9 \; + 3$

$= -1(x + 3)^2 \; + 12$

$= 12 - (x + 3)^2$

MAXIMUM and MINIMUM VALUES $a(x + b)^2 + c$

For all x, $(x + b)^2 \geq 0$

expression $(x + b)^2$ has a minimum value 0 when $x = -b$

a > 0 ie. positive

for all x, $(x - 3)^2 \geq 0$

$2(x - 3)^2 - 8 \geq -8$

minimum value -8
when $x = 3$

a < 0 ie. negative

for all x, $-1(x + 3)^2 \leq 0$

$12 - (x + 3)^2 \leq 12$

maximum value 12
when $x = -3$

FRACTIONS

$\dfrac{1}{\text{min.}}$ gives max. value to a fraction

$\dfrac{1}{\text{max.}}$ gives min. value to a fraction

$$\frac{1}{3(x + 2)^2 + 5}$$

for all x, $(x + 2)^2 \geq 0$

$3(x + 2)^2 + 5 \geq 5$

minimum value 5 when $x = -2$

$$\boxed{\frac{1}{3(x + 2)^2 + 5} \quad \text{maximum value } {}^1/_5 \text{ when } x = -2}$$

GRAPHS $y = a(x + b)^2 + c$

(i) TURNING POINT $(-b\,,\,c)$

minimum if $a > 0$ ie. positive

maximum if $a < 0$ ie. negative

(ii) Y-INTERCEPT: substitute $x = 0$

(iii) ZEROS (if any): substitute $y = 0$,

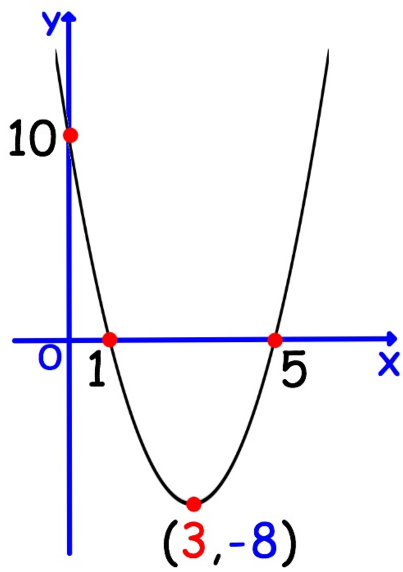
solve $ax^2 + bx + c = 0$

factorise $(\quad)(\quad) = 0$

Sketch $y = 2x^2 - 12x + 10$

(i) $y = 2(x - 3)^2 - 8$

minimum TP: $(3, -8)$



(ii) $y = 2 \times 0^2 - 12 \times 0 + 10 = 10$

(iii)

$$2x^2 - 12x + 10 = 0$$
$$x^2 - 6x + 5 = 0$$
$$(x - 1)(x - 5) = 0$$
$$x = 1 \quad \text{or} \quad x = 5$$

TRANSFORM GRAPHS

Draw the basic shape of the transformed graph.
Annotate with the images of key points.

$y = f(x) + k \qquad (x\,,\ y + k)$

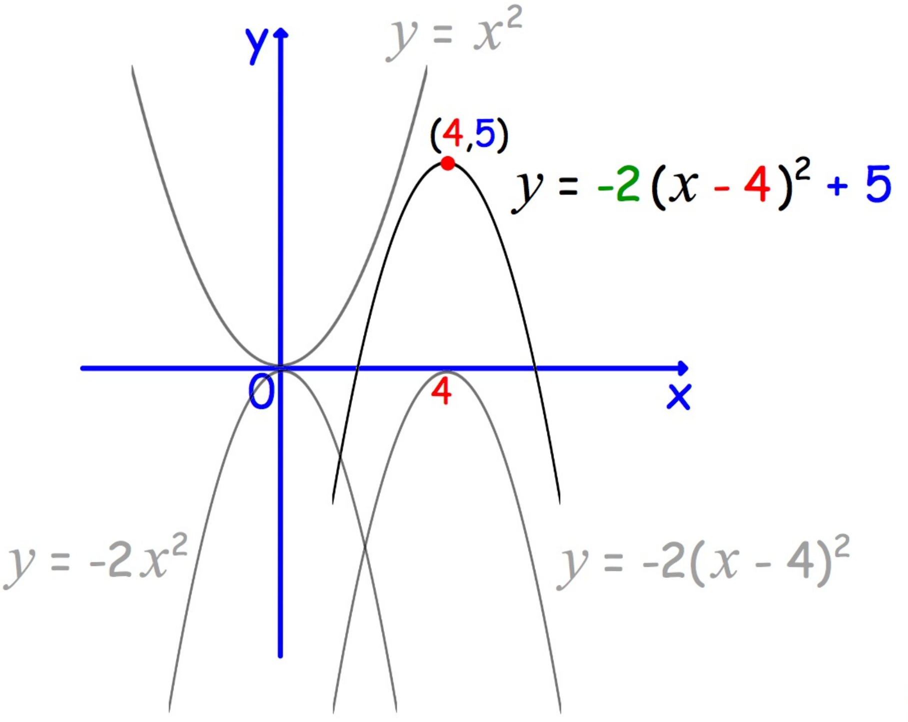
$y = f(x + k) \qquad (x - k\,,\ y)$

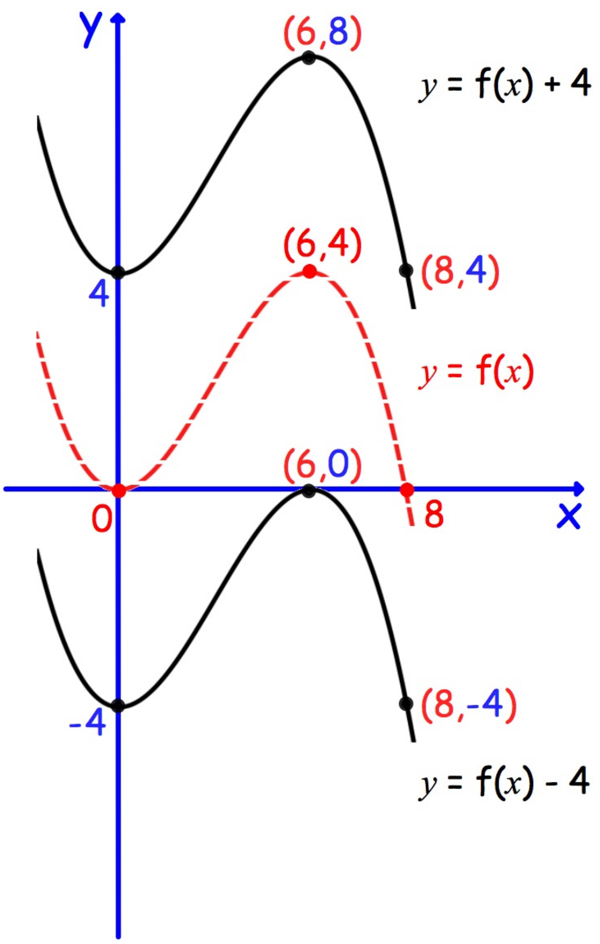
$y = kf(x) \qquad (x\,,\ ky)$

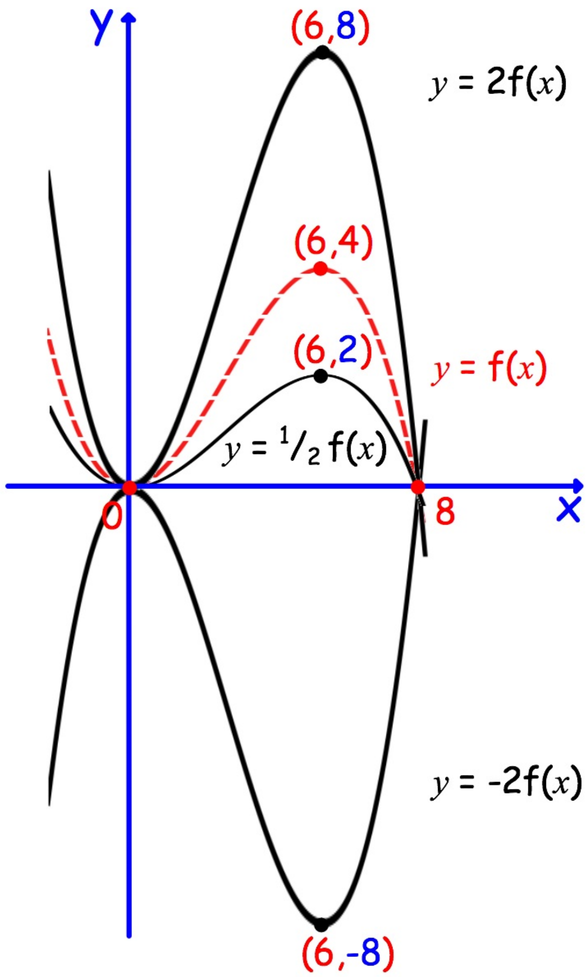
$y = f(kx) \qquad (^1/_k\, x\,,\ y)$

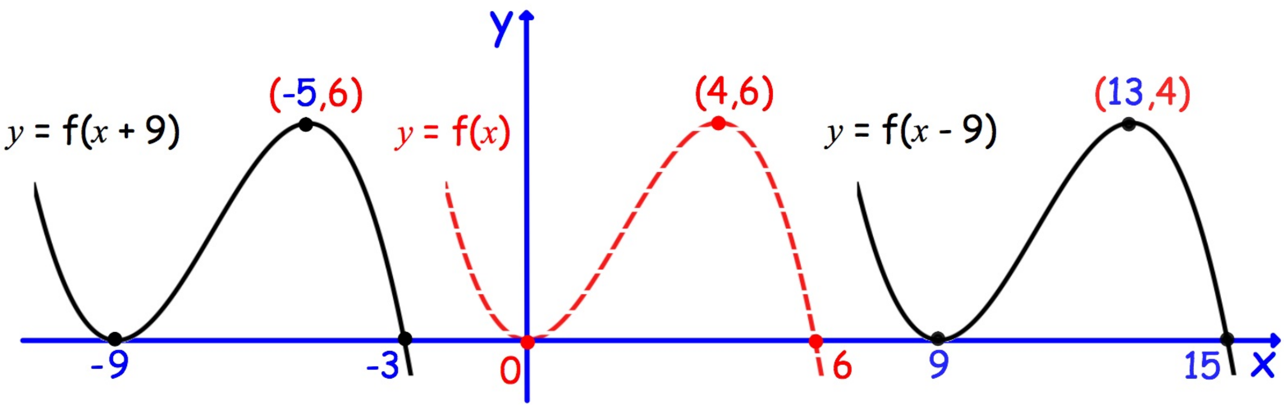
COMPLETED SQUARE

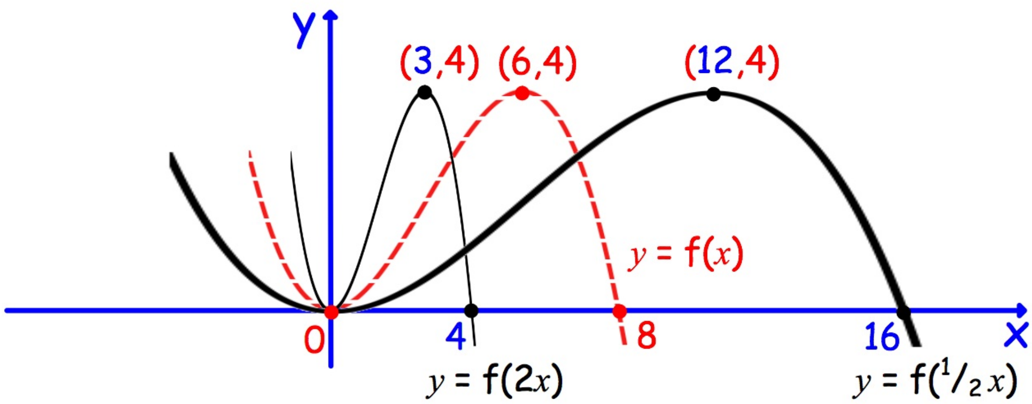
$y = 5 - 2(x - 4)^2$




y
(6,8)
y = f(x) + 4
(6,4)
4
(8,4)
y = f(x)
(6,0)
0
8
x
-4
(8,-4)
y = f(x) - 4

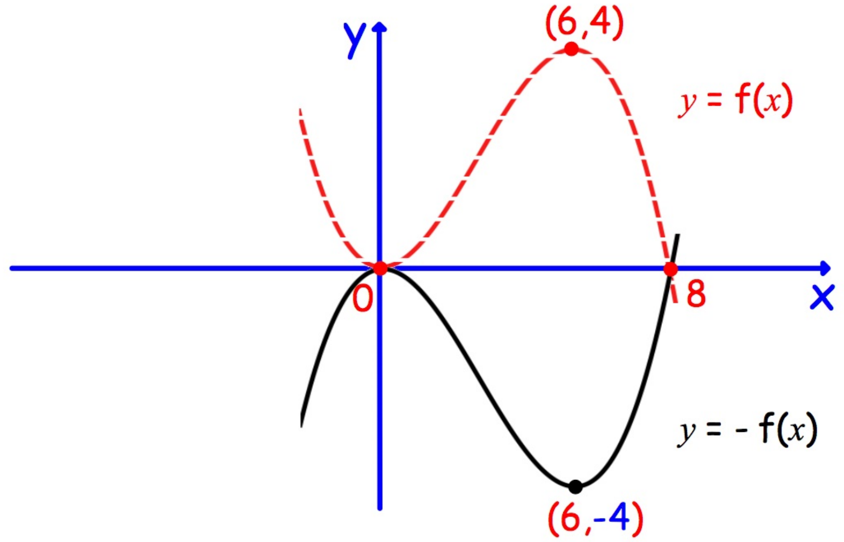

y
(6,8)
y = 2f(x)
(6,4)
(6,2)
y = f(x)
y = 1/2 f(x)
0
8
x
y = -2f(x)
(6,-8)

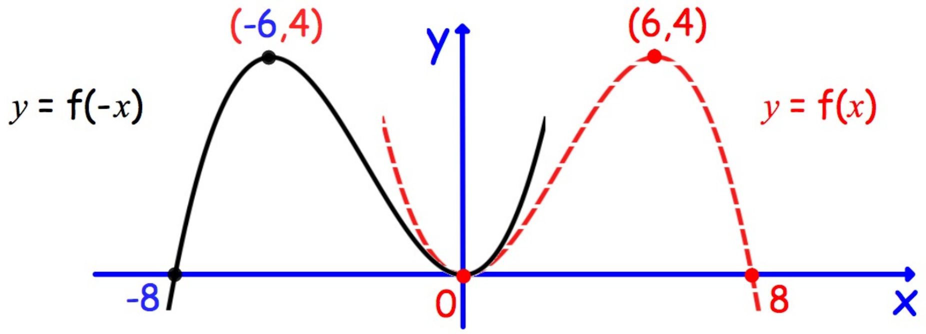

y
(-5,6)
y = f(x + 9)
(4,6)
y = f(x)
y = f(x - 9)
(13,4)
-9
-3
0
6
9
15
x

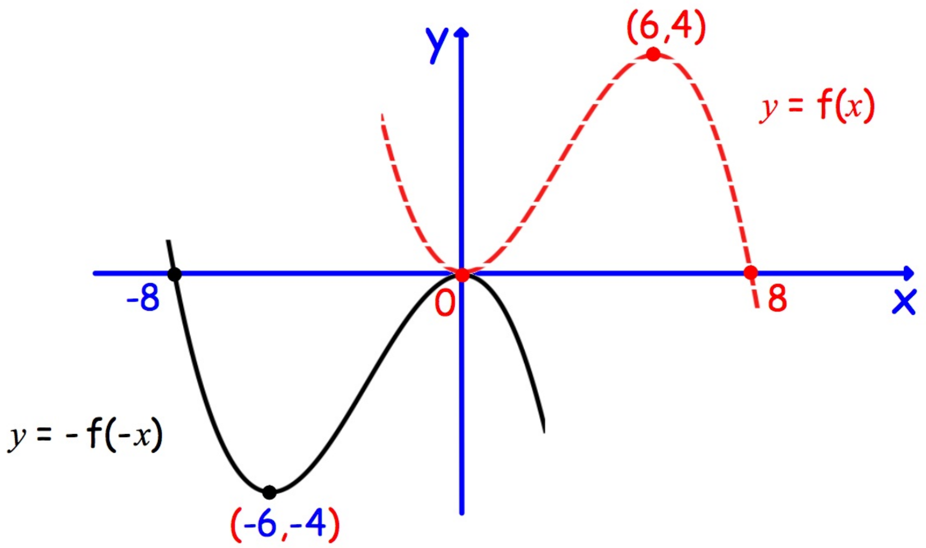

y
(3,4)
(6,4)
(12,4)
y = f(x)
0
4
8
16
x
y = f(2x)
y = f(1/2 x)

SPECIAL CASES:

REFLECT in X- axis	$y = -f(x)$	$(x, -y)$
REFLECT in Y- axis	$y = f(-x)$	$(-x, y)$
HALF-TURN about O (or REFLECT in O)	$y = -f(-x)$	$(-x, -y)$



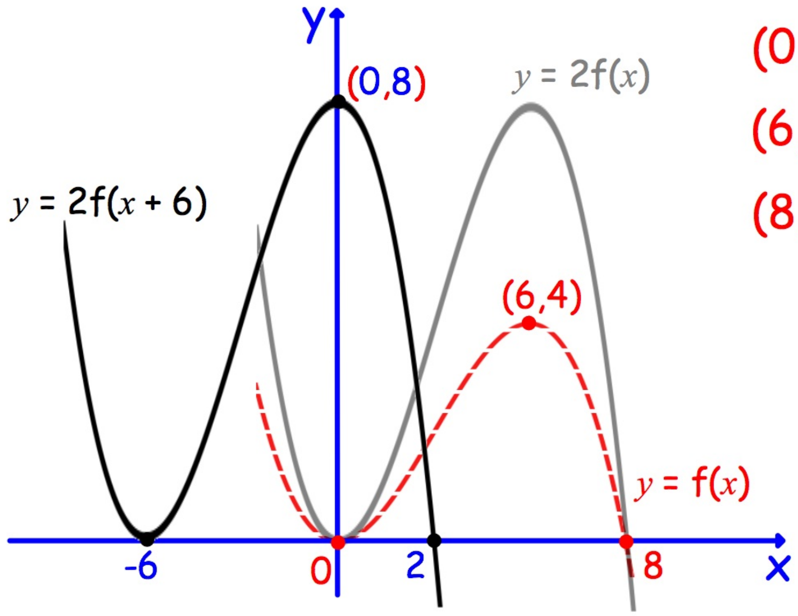




COMBINING TRANSFORMATIONS

Transform coordinates; also show intermediate graphs.

(1) $y = 2f(x + 6)$ $\qquad (x, y) \longrightarrow (x - 6, 2y)$



$(0,0) \longrightarrow (-6, 0)$

$(6,4) \longrightarrow (0, 8)$

$(8,0) \longrightarrow (2, 0)$

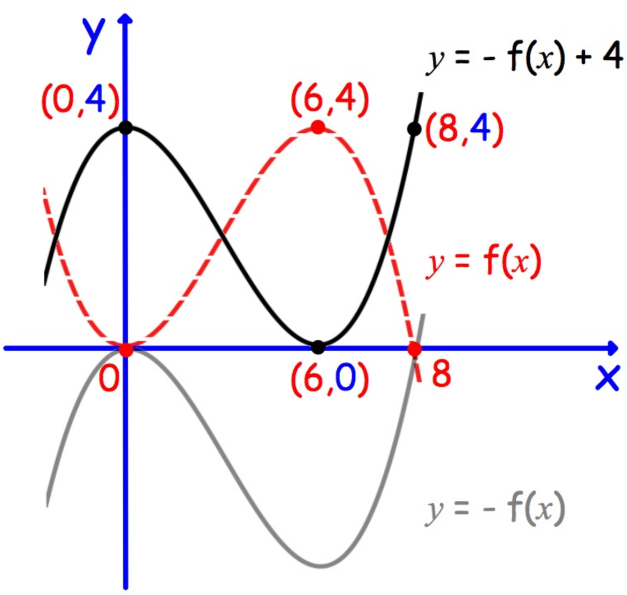
(2) $y = 4 - f(x)$

or $y = -f(x) + 4$ $\qquad (x, y) \longrightarrow (x, -y + 4)$

$(0,0) \longrightarrow (0, 4)$

$(6,4) \longrightarrow (6, 0)$

$(8,0) \longrightarrow (8, 4)$

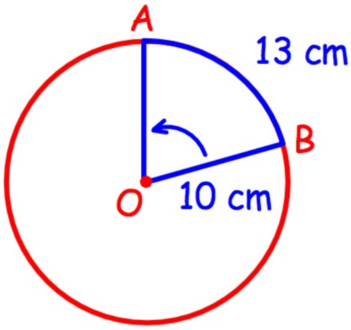


TRIGONOMETRIC FUNCTIONS

RADIAN MEASURE: given by the ratio $\dfrac{\text{arc } AB}{r}$

$$\frac{\text{arc } AB}{r} = \frac{13}{10} = 1.3$$

$$\angle AOB = 1.3 \text{ rads}$$



The ratio for a complete turn is 2π, so

$$180° = \pi \text{ radians}$$

$$\text{RADS} \underset{\div 180 \ \times \pi}{\overset{\div \pi \ \times 180}{\rightleftarrows}} \text{DEGREES}$$

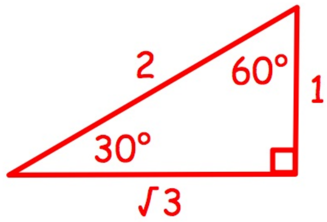
(1) $1.3 \text{ rads} = \dfrac{1.3}{\pi} \times 180° = 74.484\ldots \approx 74.5°$

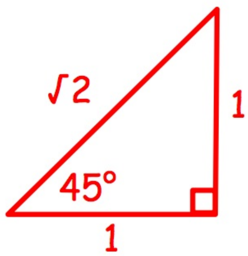
(2) $140° = \dfrac{140}{180} \times \pi = 2.4434\ldots \approx 2.44 \text{ rads}$

as multiples of π

(3) $260°$ $\quad \dfrac{260}{180}\pi = \dfrac{13}{9}\pi = \dfrac{13\pi}{9}$

EXACT VALUES:





30°	45°	60°	90°
$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$

multiples:

(1) $5\pi/3$ $\quad 5 \times \pi/3 = 5 \times 60° = 300°$

(2) $210°$ $\quad 7 \times 30° = 7 \times \pi/6 = 7\pi/6$

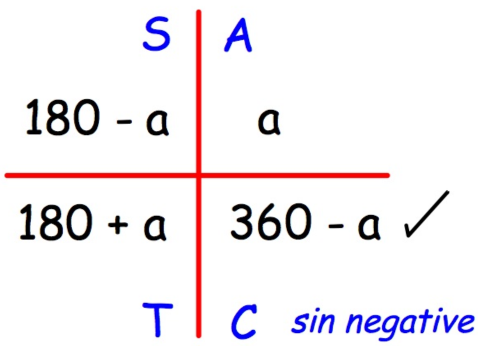
beyond acute angles:

(1) $\sin 300°$

$= \sin (360 - 60)°$

$= -\sin 60°$

$= -\sqrt{3}/2$

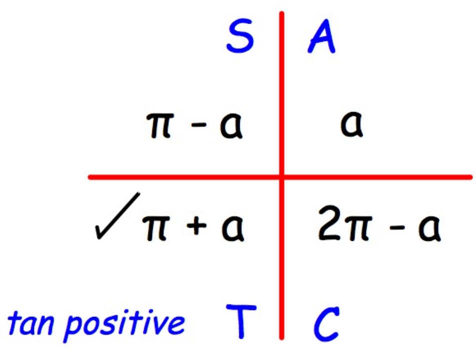


(2) $\tan 7\pi/6$

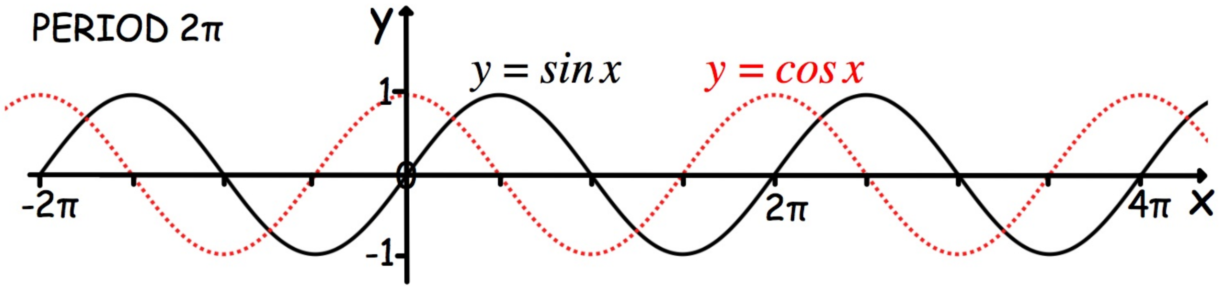
$= \tan (\pi + \pi/6)$

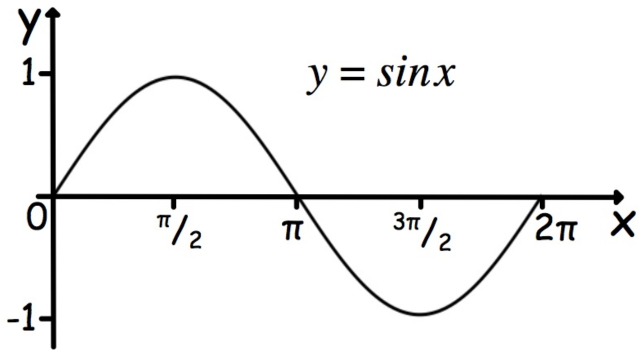
$= +\tan \pi/6$

$= 1/\sqrt{3}$

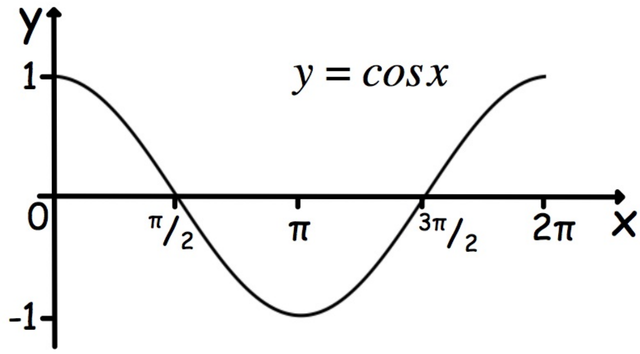


TRIGONOMETRIC GRAPHS





Max TP $(\pi/2, 1)$
Min TP $(3\pi/2, -1)$

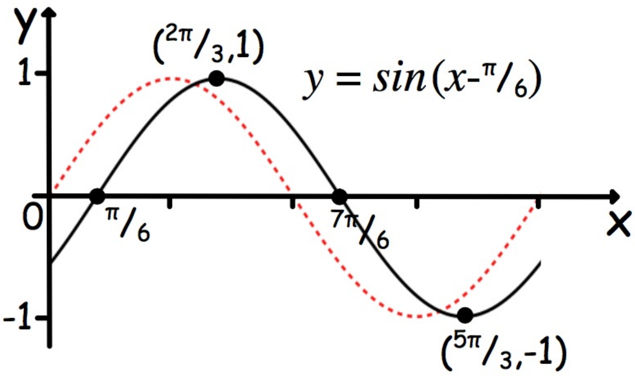


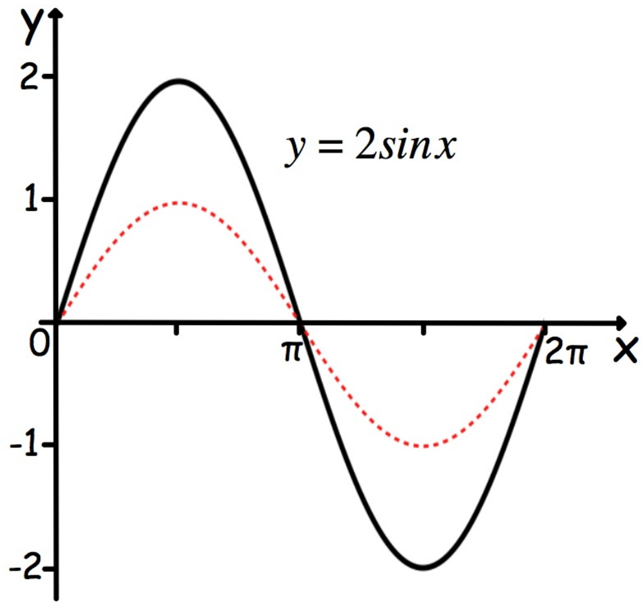
Max TP $(0,1)$
Min TP $(\pi,-1)$

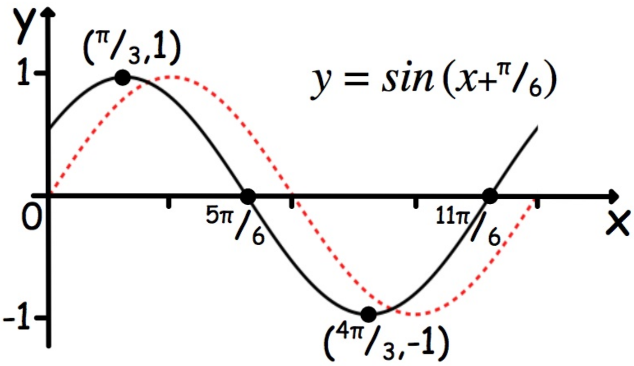
TRANSFORMATIONS

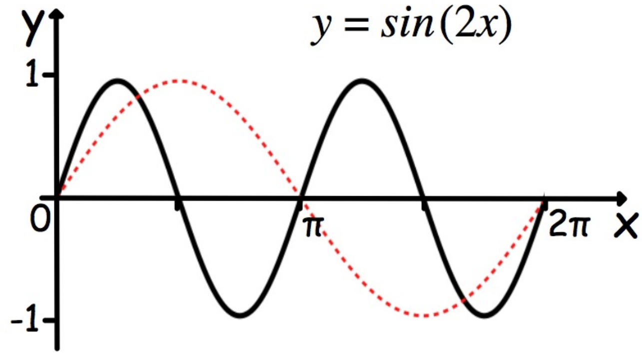
$y = a\sin x$	stretch a units vertically
$y = \sin(bx)$	period $2\pi \div b$
$y = \sin(x + c)$	shift -c rads horizontally
$y = \sin x + d$	shift d units vertically

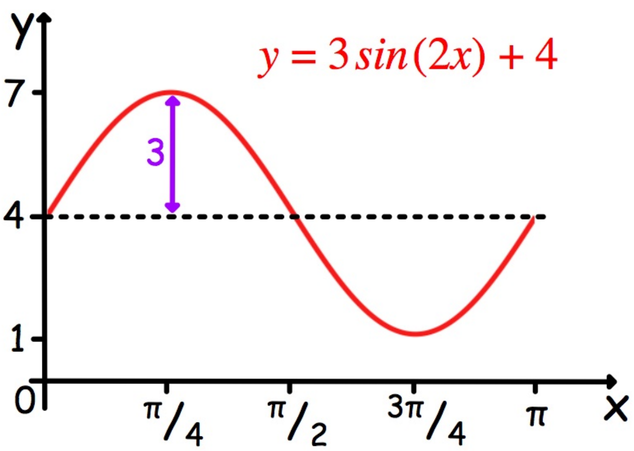
TRANSFORMATIONS:

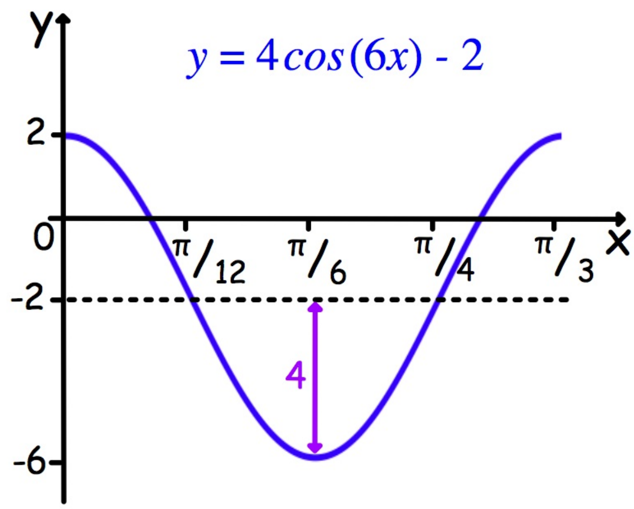






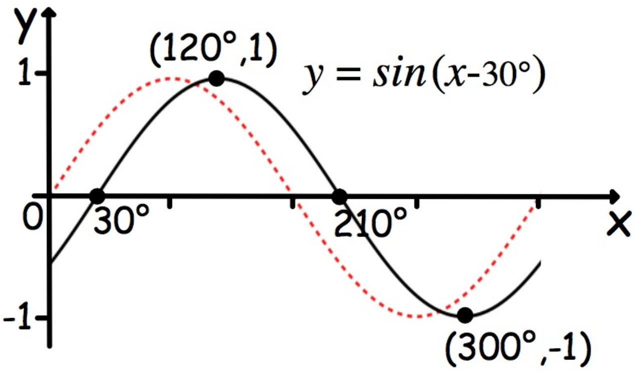


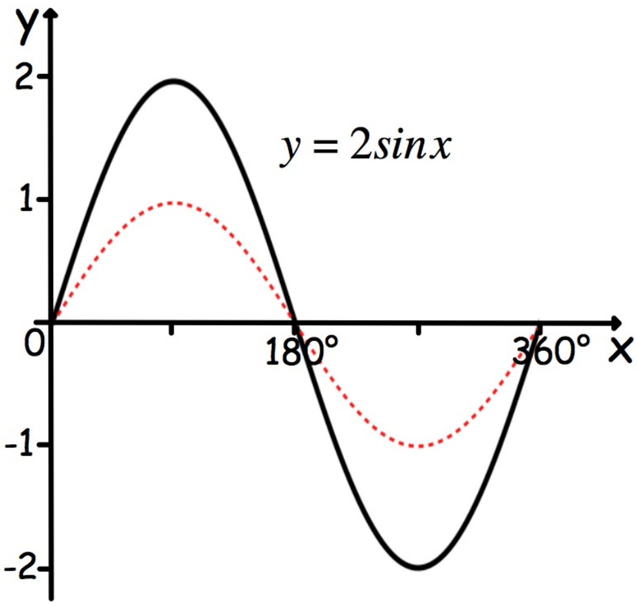


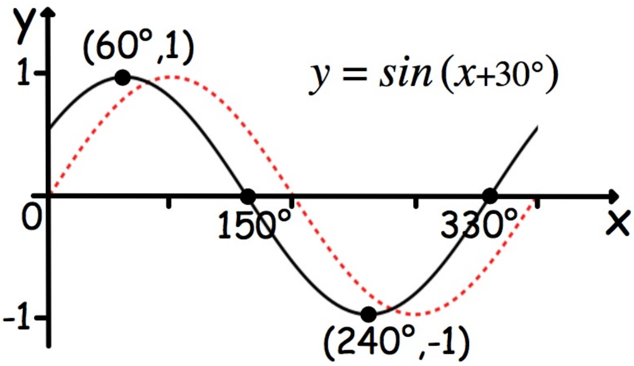


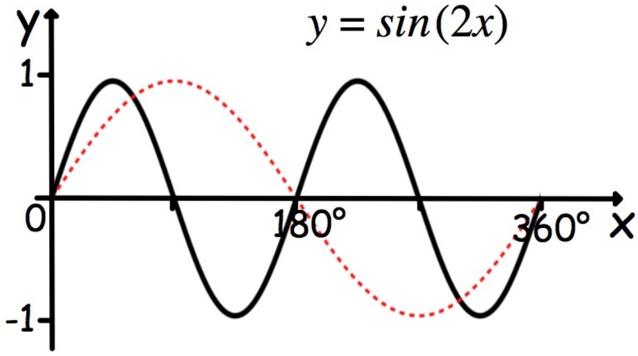
$({}^{\pi}/_{2}, 1)$	$({}^{3\pi}/_{2}, -1)$	$(0, 1)$	$(\pi, -1)$
$\div 2$ $\quad \times 3 + 4$	$\div 2$ $\quad \times 3 + 4$	$\div 6$ $\quad \times 4 - 2$	$\div 6$ $\quad \times 4 - 2$
$({}^{\pi}/_{4}, 7)$	$({}^{3\pi}/_{4}, 1)$	$(0, 2)$	$({}^{\pi}/_{6}, -6)$
MAX. TP	MIN. TP	MAX. TP	MIN. TP

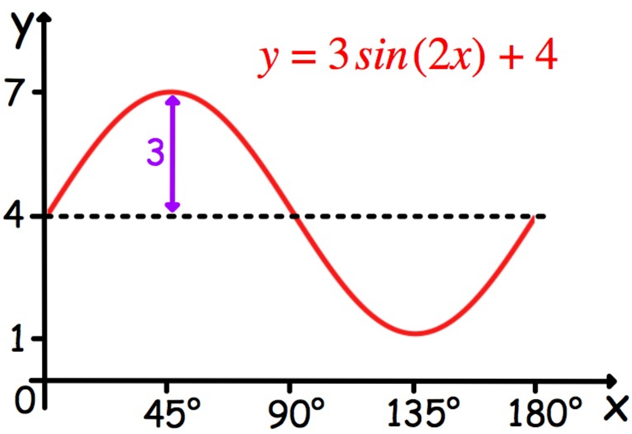
TRANSFORMATIONS:

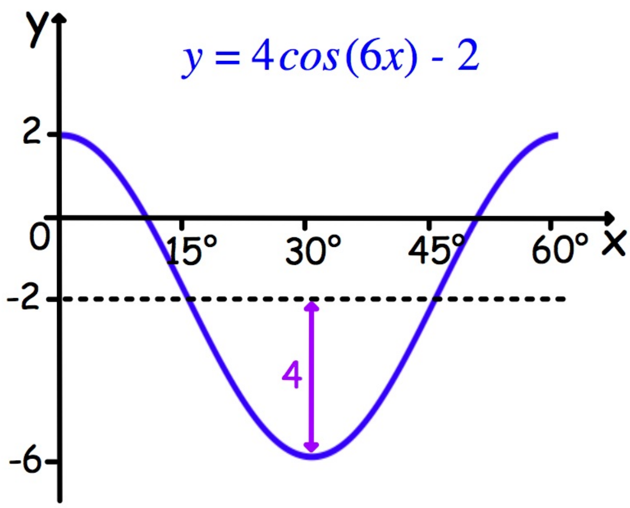










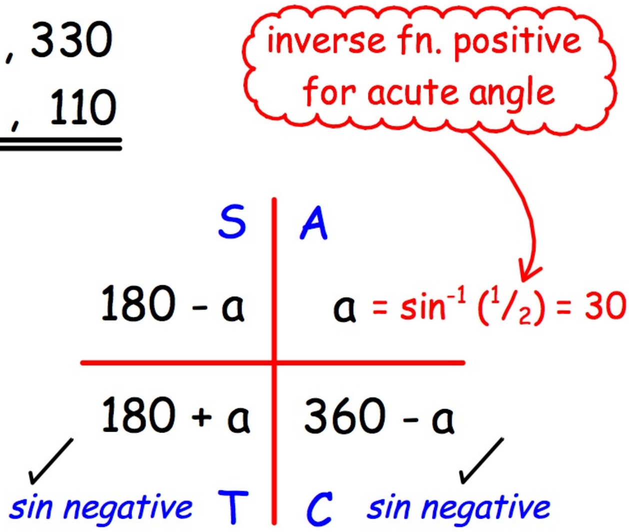


(90°, 1)	(270°, -1)	(0°, 1)	(180°, -1)
÷2 ×3+4	÷2 ×3+4	÷6 ×4-2	÷6 ×4-2
(45°, 7)	(135°, 1)	(0°, 2)	(30°, -6)
MAX. TP	MIN. TP	MAX. TP	MIN. TP

TRIGONOMETRIC EQUATIONS

(1) $2\sin 3x^\circ + 6 = 5 \quad , \quad 0 \leq x \leq 120$

$$2\sin 3x^\circ = -1$$
$$\sin 3x^\circ = -\tfrac{1}{2}$$
$$3x = 210\ ,\ 330$$
$$\underline{\underline{x = 70\ ,\ 110}}$$



(2) $\sin 3x = -\tfrac{1}{2} \quad , \quad 0 \leq x \leq \tfrac{2\pi}{3}$

$$3x = \tfrac{7\pi}{6}\ ,\ \tfrac{11\pi}{6}$$
$$\underline{\underline{x = \tfrac{7\pi}{18}\ ,\ \tfrac{11\pi}{18}}}$$

MULTIPLE ANGLES: solving over more than one period

(3) $\sin 3x^\circ = -\tfrac{1}{2} \quad , \quad 0 \leq x \leq 360$

$$3x = 210\ ,\ 330$$
$$x = 70\ ,\ 110$$

add multiples of period 120°

$$\underline{\underline{x = 70\ ,\ 110\ ,\ 190\ ,\ 230\ ,\ 310\ ,\ 350}}$$

USING GRAPHS: for $0°$, $90°$, $180°$, $270°$, $360°$

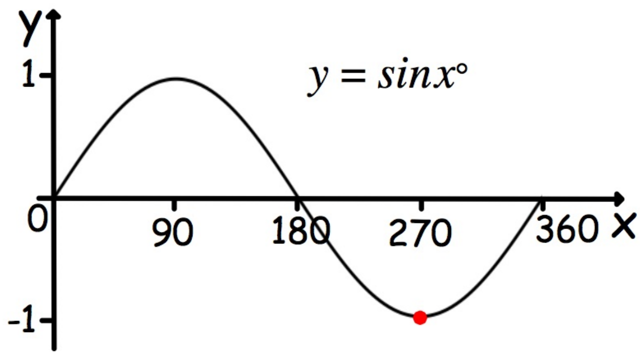
(4) $\sin 2x° = -1$, $0 \leq x \leq 360$

$2x = 270$

$x = 135$

add multiples of period $180°$

$x = 135 , 315$



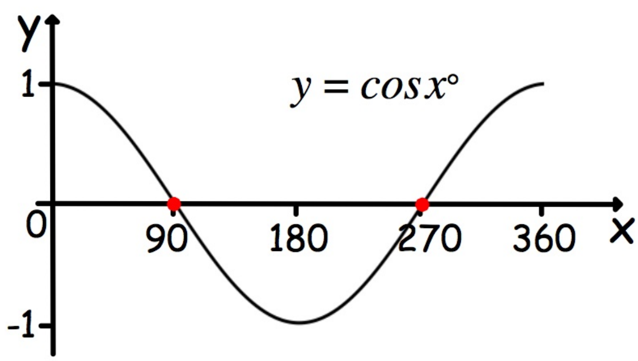
(5) $\cos 2x° = 0$, $0 \leq x \leq 360$

$2x = 90 , 270$

$x = 45 , 135$

add multiples of period $180°$

$x = 45 , 135 , 225 , 315$



SEQUENCES

Sequences can be generated by rules:

(i) n^{th} term formula

relates terms to the natural numbers
the formula can generate any term required

(ii) recurrence relation

relates one term to the next
requires an initial term
work through sequence to generate term required

$1, 2, 4, 8, 16 \ldots$

n^{th} term formula $\quad U_n = 2^{n-1}$

5th term $\quad U_5 = 2^{5-1} = 2^4 = 16$

recurrence relation $\quad U_{n+1} = 2U_n \quad , \quad U_1 = 1$

$U_2 = 2U_1 = 2\times1 = 2$

$U_3 = 2U_2 = 2\times2 = 4$

$U_4 = 2U_3 = 2\times4 = 8$

$U_5 = 2U_4 = 2\times8 = 16$

LINEAR RECURRENCE RELATIONS

$$U_{n+1} = aU_n + b$$

initial term often U_0 , "U nought"

NOTE: $U_n = aU_{n-1} + b$ is the same rule

A park has 500 of a pest species which increases by 20% each year, so 200 are removed each year.

How many years to eliminate the pest?

$U_{n+1} = 1.2\,U_n - 200$, $U_0 = 500$

Growth Factor
100% + 20% = 120%

$U_0 = 500$

$U_1 = 1.2 \times 500 - 200 = 400$

$U_2 = 1.2 \times 400 - 200 = 280$

$U_3 = 1.2 \times 280 - 200 = 136$

$U_4 = 1.2 \times 136 - 200 = -36.8 < 0$

4 years required

A recurrence relation is of the form $U_{n+1} = aU_n + b$ where $U_{10} = 2$, $U_{11} = 5$ and $U_{12} = 17$.

Find (i) a and b (ii) U_{13} (iii) U_9

(i) $U_{n+1} = aU_n + b$

$U_{12} = aU_{11} + b$

$17 = a \times 5 + b$

$U_{11} = aU_{10} + b$

$5 = a \times 2 + b$

solve simultaneous equations

$5a + b = 17$

$2a + b = 5$

subtract

$3a = 12$

$a = 4$

$2a + b = 5$

$8 + b = 5$

$b = -3$

$U_{n+1} = aU_n + b$

$U_{n+1} = 4U_n - 3$

(ii) $U_{n+1} = 4U_n - 3$

$U_{13} = 4U_{12} - 3$

$U_{13} = 4 \times 17 - 3$

$U_{13} = 65$

(iii) $U_{n+1} = 4U_n - 3$

$U_{10} = 4U_9 - 3$

$2 = 4U_9 - 3$

$5 = 4U_9$

$U_9 = {}^5/_4$

LIMITS

Some sequences converge to a particular value.

n^{th} term formula

$1, 2, 4, 8, 16 \ldots$ for $U_n = 2^{n-1}$

heading to infinity (∞) as $n \to \infty$, $U_n \to \infty$

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots$ for $U_n = (\frac{1}{2})^{n-1}$

heading to the limit zero as $n \to \infty$, $U_n \to 0$

A sequence is given by the formula $U_n = \dfrac{3n + 1}{n}$

(a) Find the first 3 terms of the sequence.

(b) Find the limit of the sequence.

(a) $U_1 = \dfrac{3 \times 1 + 1}{1} = 4$

$U_2 = \dfrac{3 \times 2 + 1}{2} = 3\frac{1}{2}$

$U_3 = \dfrac{3 \times 3 + 1}{3} = 3\frac{1}{3}$

(b) choose very large values:

$U_{1000} = \dfrac{3 \times 1000 + 1}{1000} = 3.001$

$U_{10000} = \dfrac{3 \times 10000 + 1}{10000} = 3.0001$

as $n \to \infty$, $U_n \to 3$

<u>LIMIT is 3</u>

recurrence relation $U_{n+1} = aU_n + b$

has a limit if $-1 < a < 1$ ie. a fraction

for limit L , $L = aL + b$

The sequence converges on the limit regardless of U_o

$U_{n+1} = 0.2U_n + 4$

$U_o = 4$; 4 , 4.8 , 4.96 , 4.992 , 4.994 , 4.99968 ...

$U_o = 6$; 6 , 5.2 , 5.04 , 5.008 , 5.0016 , 5.00032 ...

$U_o = 5$; 5 , 5 , 5 , 5 , 5 , 5 ...

start with the limit 5 and next term is the limit.

for $U_{n+1} = 0.2U_n + 4$

as $n \longrightarrow \infty$, $U_n \longrightarrow 5$

<u>LIMIT is 5</u>

can generalise the result:

$$L = aL + b$$

$$L - aL = b$$

$$L(1 - a) = b$$

$$L = \frac{b}{1 - a}$$

(1) In a park, each day 80% of litter is cleared but 4 kg of new litter is dropped. Find the amount of litter in the long-term.

80% removed, so 20% remains

$U_{n+1} = 0.2U_n + 4$

has a limit since $-1 < 0.2 < 1$

$$L = 0.2L + 4$$
$$0.8L = 4$$
$$L = 4 \div 0.8$$
$$L = 5$$

check:

$0.2 \times 5 = 1$ Min. level

$1 + 4 = 5$ Max. level

In the long-term the litter settles at 5 kg.

(2) A car leaks 25% of its oil each day so 1 litre of oil is added daily. The oil level must never fall below 4 litres. Is it safe in the long-term?

25% removed, so 75% remains

$U_{n+1} = 0.75U_n + 1$

has a limit since $-1 < 0.75 < 1$

$$L = 0.75L + 1$$
$$0.25L = 1$$
$$L = 1 \div 0.25$$
$$L = 4$$

check:

$0.75 \times 4 = 3$ Min. level

$3 + 1 = 4$ Max. level

NOT safe in the long-term as the level falls to 3 litres before topping-up.

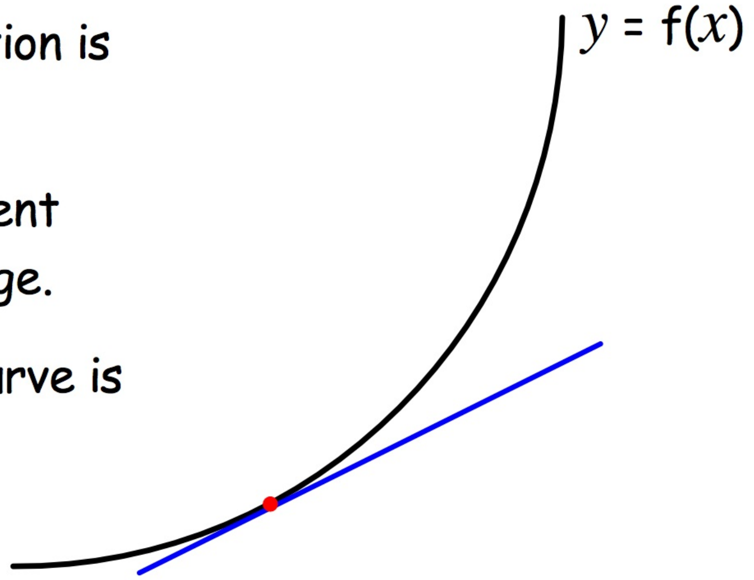
DIFFERENTIAL CALCULUS

Examines the effect of change on a function.
A change in the domain produces a change in the range.

The value of the function is continuously changing.

At any point the gradient gives the rate of change.

The gradient of the curve is given by the tangent.



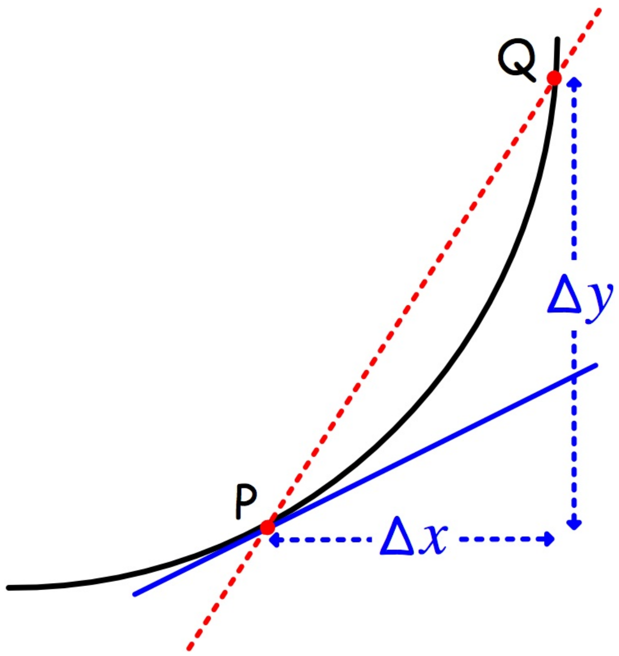
The gradient of line PQ is ${}^{\Delta y}/_{\Delta x}$

and as Q moves closer to P , $\Delta x \rightarrow 0$

The gradient at P is ,

$\dfrac{\Delta y}{\Delta x}$ as $\Delta x \rightarrow 0$

written $\dfrac{dy}{dx}$



RULE: variety of notations

$y = x^n$ $\qquad$ $f(x) = x^n$

$\dfrac{dy}{dx} = n x^{n-1}$ $\qquad$ $f'(x) = n x^{n-1}$ $\qquad$ $\dfrac{d}{dx}\left(x^n\right) = n x^{n-1}$

TERMINOLOGY:

gradient of curve/tangent

rate of change of a function
(speed , acceleration)

derived function
derivative

differential
differentiation

OTHER RULES:

$F(x) = k\,f(x)$

$F'(x) = k\,f'(x)$

$F(x) = f(x) + g(x)$

$F'(x) = f'(x) + g'(x)$

(1) $f(x) = x^3 - x^2 + x + 5$

$f'(x) = 3x^2 - 2x + 1$

differentiates to 0

(2) $g(u) = 2u^5 - 6u^3 + 7u - 3$

$g'(u) = 10u^4 - 18u^2 + 7$

(3) Find the gradient of the curve $y = x^2 - 3x + 2$ at the point where $x = 4$.

$f(x) = x^2 - 3x + 2$

$f'(x) = 2x - 3$

$f'(4) = 2 \times 4 - 3 = 5$

gradient 5

INDICES

Use indices rules to express terms in the form x^n.

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$\frac{1}{a^p} = a^{-p}$

$\sqrt[n]{a^m} = a^{\frac{m}{n}}$

(1) $f(x) = \frac{3}{x^2}$

$= 3x^{-2}$

$f'(x) = -6x^{-3}$

$= -\frac{6}{x^3}$

(2) $g(x) = \frac{1}{3x}$

$= \frac{1}{3}x^{-1}$

$g'(x) = -\frac{1}{3}x^{-2}$

$= -\frac{1}{3x^2}$

(3) $h(x) = \sqrt{x^3}$, find $h'(9)$

$h(x) = x^{\frac{3}{2}}$

$h'(x) = \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{3}{2}\sqrt{x}$

$h'(9) = \frac{3}{2}\sqrt{9}$

$= \frac{9}{2}$

(4) $k(x) = \frac{1}{\sqrt{x}}$, find $k'(4)$

$k(x) = x^{-\frac{1}{2}}$

$k'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$

$= -\frac{1}{2\sqrt{x^3}}$

$k'(4) = -\frac{1}{2\sqrt{4^3}}$

$\sqrt{4}$ and cube

$= -\frac{1}{16}$

BRACKETS AND QUOTIENTS

Differentiate sums and differences of terms x^n,
so 'break' brackets and 'split' quotients.

(1) $f(x) = \dfrac{(x+1)^2}{x}$

$$\frac{x^2 + 2x + 1}{x}$$

$$= \frac{x^2}{x} + \frac{2x}{x} + \frac{1}{x}$$

$$= x + 2 + x^{-1}$$

$f(x) = x + 2 + x^{-1}$

$f'(x) = 1 + 0 - x^{-2}$

$= 1 - \dfrac{1}{x^2}$

(2) $f(x) = \dfrac{x^2 + 1}{\sqrt{x}}$

$$\frac{x^2}{x^{1/2}} + \frac{1}{x^{1/2}}$$

$$= x^{3/2} + x^{-1/2}$$

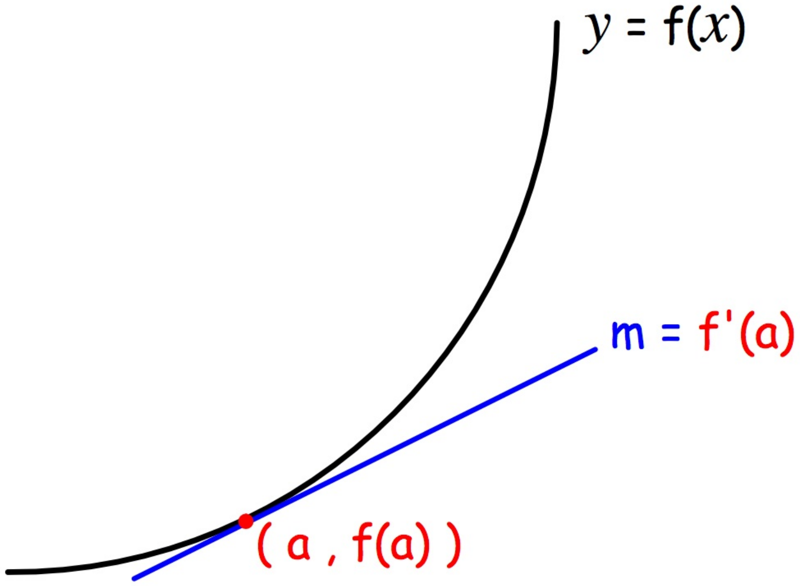
$f(x) = x^{3/2} + x^{-1/2}$

$f'(x) = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-3/2}$

$= \frac{3}{2}\sqrt{x} - \dfrac{1}{2\sqrt{x^3}}$

EQUATION OF A TANGENT

The gradient of the curve at some point is given by the tangent to the curve at that point.



Find the equation of the tangent to the curve $y = x^2 - x$ at the point where $x = 3$.

$f(x) = x^2 - x$

$f(3) = 3^2 - 3 = 6$

point $(3,6)$

$f(x) = x^2 - x$

$f'(x) = 2x - 1$

$f'(3) = 2\times3 - 1 = 5$

gradient 5

$\begin{matrix} a & b \\ (3, & 6) \end{matrix}$

$m = 5$

$y - b = m(x - a)$

$y - 6 = 5(x - 3)$

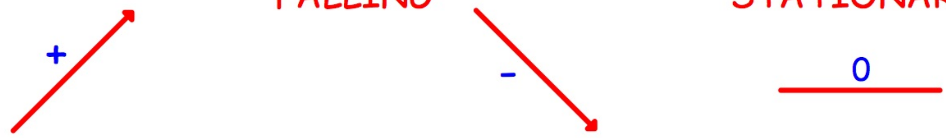
$y - 6 = 5x - 15$

$$\underline{\underline{y = 5x - 9}}$$

GRAPH OF THE DERIVED FUNCTION , $y = f'(x)$

The graph of the gradient.

Graph RISING | FALLING | STATIONARY



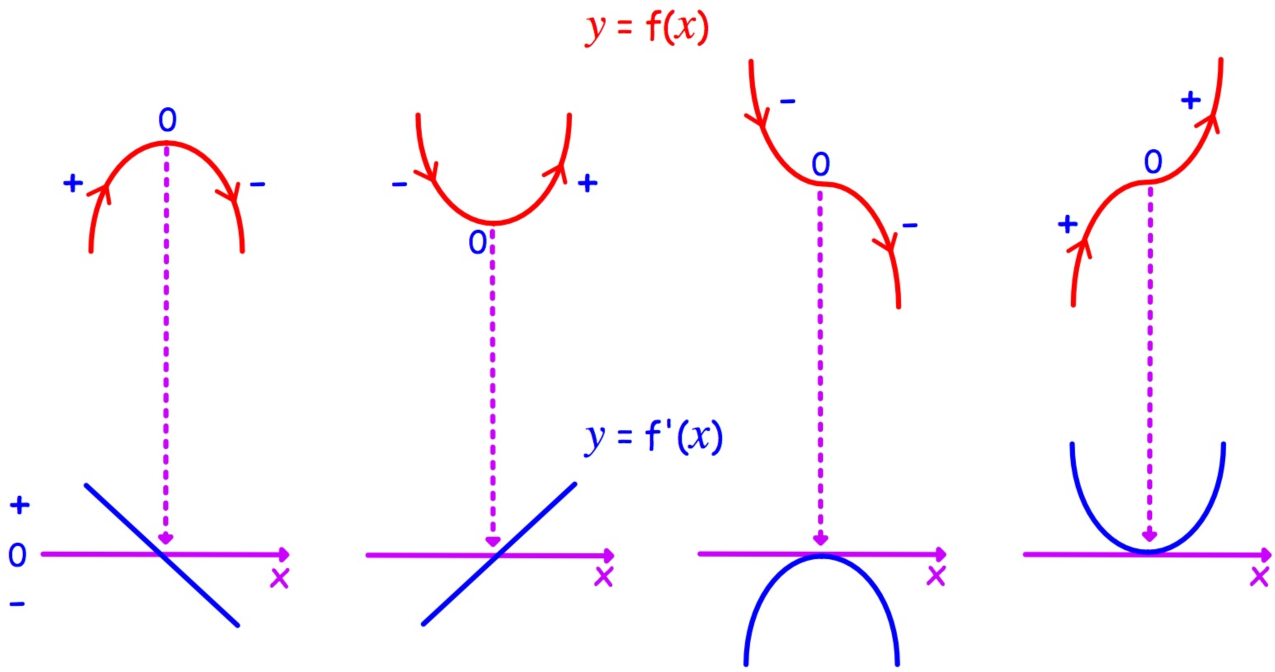
$f'(x) > 0$ | $f'(x) < 0$ | $f'(x) = 0$

STATIONARY POINTS

Main features of a graph occur at stationary points.

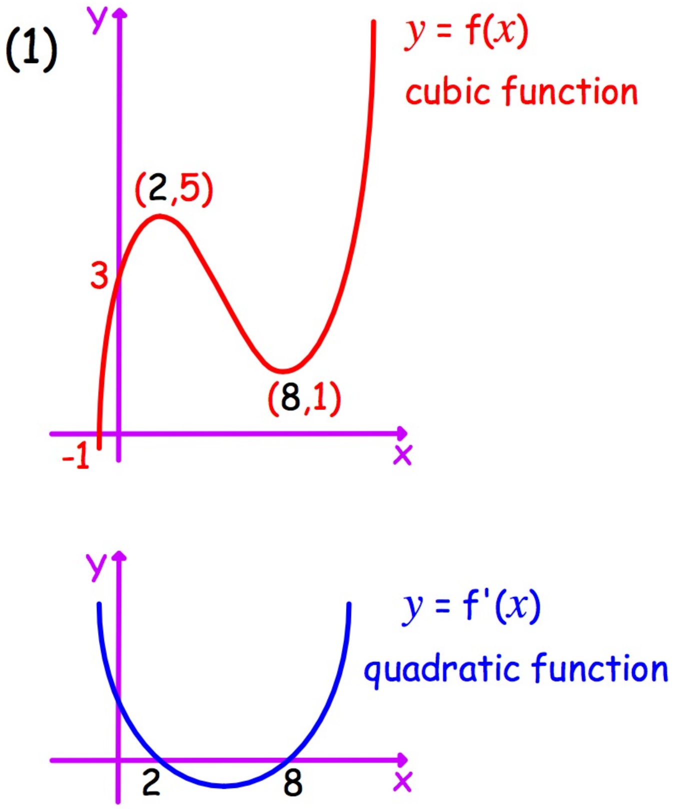
By examining the gradient around stationary points the graph of the gradient can be sketched.

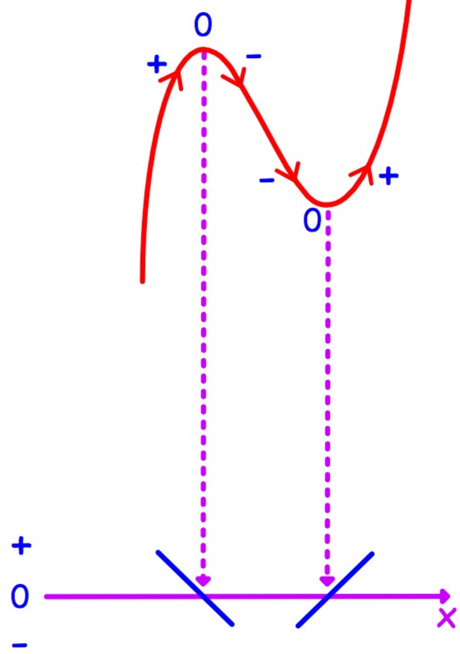
turning points (maximum or minimum) | points of inflexion (falling or rising)



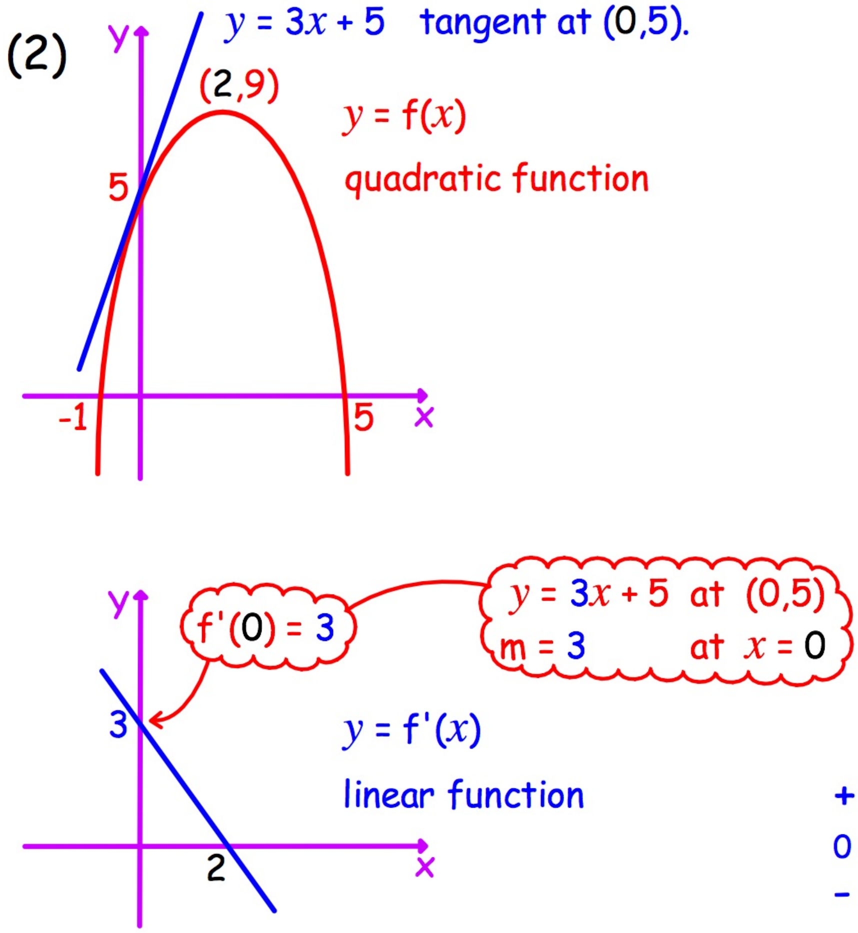
Sketch the graph of $y = f'(x)$

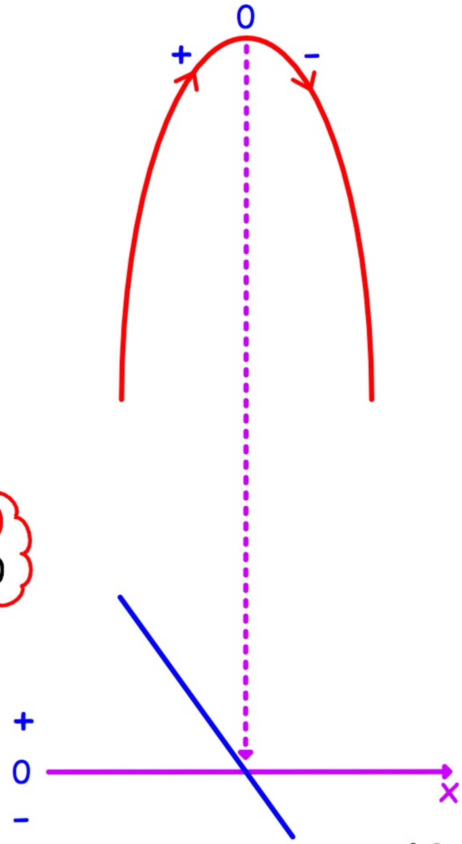
(1)





(2)



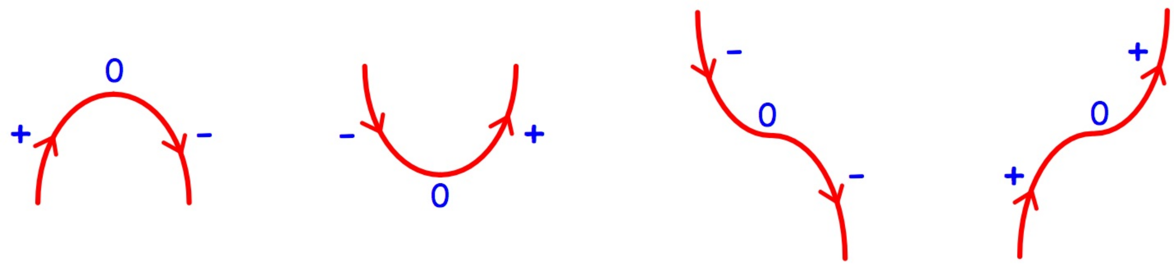


CURVE SKETCHING

The key features of the graph occur at stationary points where $f'(x) = 0$.

turning points (maximum or minimum)

points of inflexion (falling or rising)



A NATURE TABLE gives the shape of the graph.

Determine:

(i) the stationary points and their nature

(ii) where the graph meets the axes

(iii) the behaviour of the graph for very large positive and negative values of x.

$y = x^3(x - 4)$

STATIONARY POINTS:

$f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

function stationary where $f'(x) = 0$

$$4x^2(x - 3) = 0$$

$$4x^2 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = 0 \quad \text{or} \quad x = 3$$

$f(x) = x^3(x - 4)$

$f(0) = 0^3(0 - 4) = 0$ point (0,0)

$f(3) = 3^3(3 - 4) = -27$ point (3,-27)

NATURE TABLE:

x	$\rightarrow$	0	$\rightarrow$	$\rightarrow$	3	$\rightarrow$
$4x^2$	+	0	+	+	+	+
$x - 3$	-	-	-	-	0	+
$f'(x)$	-	0	-	-	0	+
shape	$\searrow$	$\rightarrow$	$\searrow$	$\searrow$	$\rightarrow$	$\nearrow$
nature		point of inflexion (0,0)			minimum TP (3,-27)	

AXES:

y - axis , $x = 0$

$f(0) = 0^3(0 - 4) = 0$

point $(0,0)$

x - axis , $y = 0$

$x^3(x - 4) = 0$

$x = 0$ or $x = 4$

points $(0,0)$, $(4,0)$

BEHAVIOUR FOR VERY LARGE x :

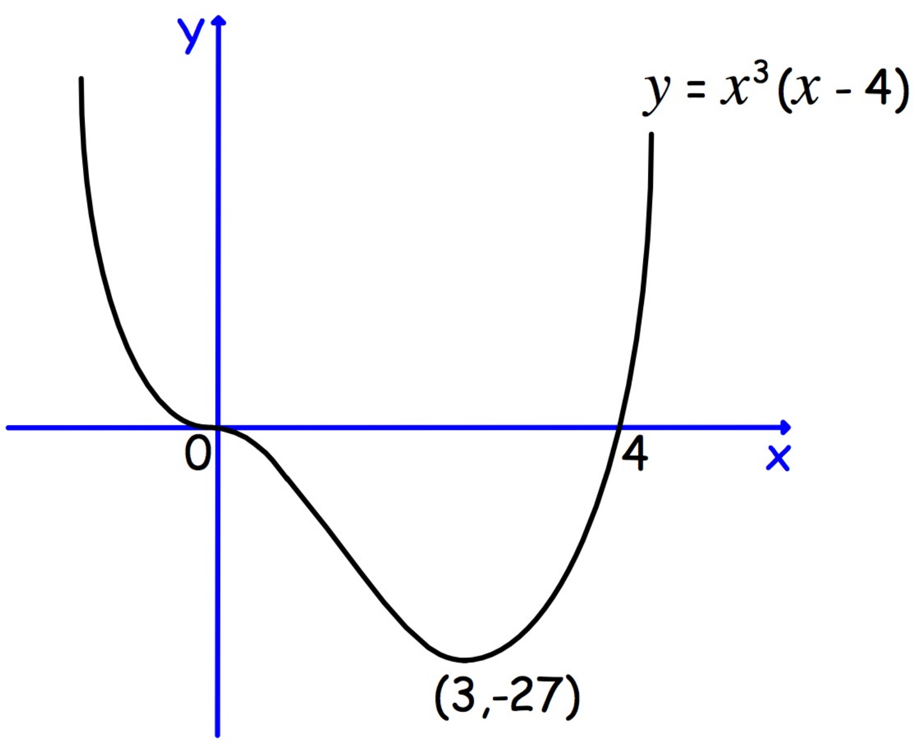
$$y = x^4 - 4x^3$$

as $x \rightarrow \infty$ $\quad y \rightarrow +x^4$, highest power dominates

as $x \rightarrow -\infty$ $\quad y \rightarrow +\infty$

as $x \rightarrow +\infty$ $\quad y \rightarrow +\infty$

SKETCH:



FUNCTION INCREASING AND DECREASING

FUNCTION INCREASING $\quad f'(x) > 0$ $\quad +$

FUNCTION DECREASING $\quad f'(x) < 0$ $\quad -$

From nature table of $f(x) = x^3(x - 4)$ earlier:

x	$\rightarrow$	0	$\rightarrow$	$\rightarrow$	3	$\rightarrow$
$f'(x)$	-	0	-	-	0	+

function increasing where $f'(x) > 0$ $\quad \underline{\underline{x > 3}}$

function decreasing where $f'(x) < 0$ $\quad \underline{\underline{x < 0 \text{ or } 0 < x < 3}}$

Show that the function
$f(x) = x^3 - 6x^2 + 12x - 5$ is never decreasing.

$$\begin{aligned} f'(x) &= 3x^2 - 12x + 12 \\ &= 3(x^2 - 4x + 4) \\ &= 3(x - 2)^2 \end{aligned}$$

function decreasing where $f'(x) < 0$

for all values of x $\quad 3(x - 2)^2 \geq 0$

so the function is never decreasing.

MAXIMUM and MINIMUM VALUES

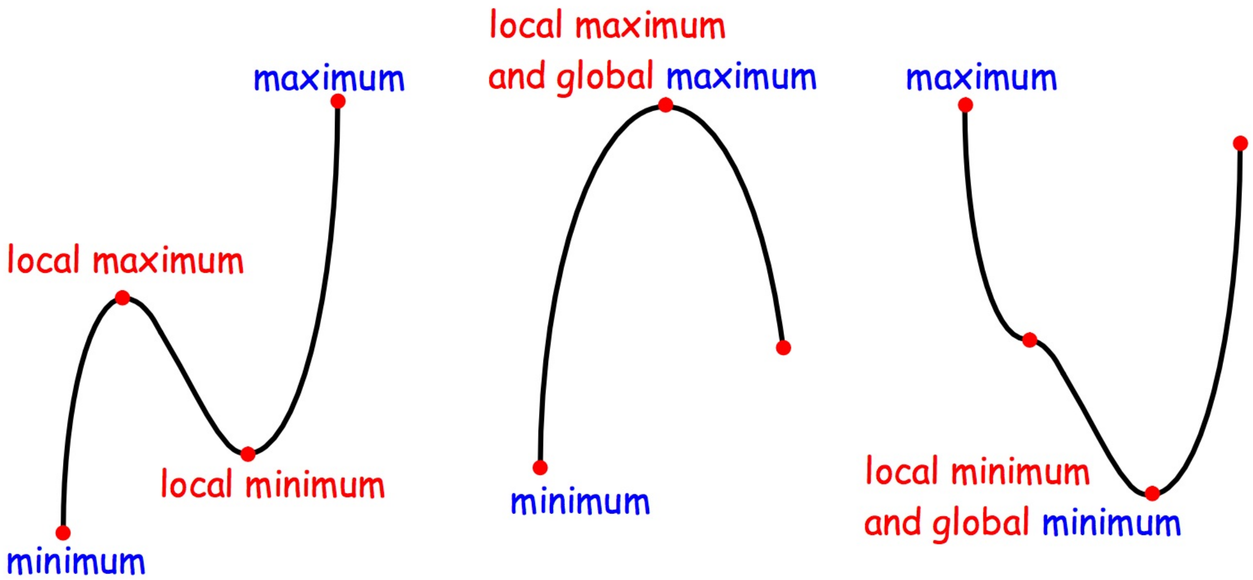
For the maximum and minimum value of a function in a closed interval, examine the values at:

(i) stationary points within the interval

(ii) the end points of the interval

LOCAL max/min values occur at stationary points

GLOBAL max/min values may be at stationary points or at the end points of the interval.



Find the maximum and minimum values of the function
$f(x) = x^2(x^2 - 8)$, $[-1,3]$

$$f(x) = x^4 - 8x^2$$

$$f'(x) = 4x^3 - 16x \quad = 4x(x^2 - 4) \quad = 4x(x + 2)(x - 2)$$

function stationary where $f'(x) = 0$

$$4x(x + 2)(x - 2) = 0$$

$$x = 0 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 2$$

for the interval $-1 \le x \le 3$, $\quad x = 0$ or $x = 2$

$$f(x) = x^4 - 8x^2$$

STATIONARY POINTS:

$$f(0) = 0^4 - 8 \times 0^2 \quad = 0$$

$$f(2) = 2^4 - 8 \times 2^2 \quad = -16$$

END POINTS OF INTERVAL:

$$f(-1) = (-1)^4 - 8(-1)^2 \quad = -7$$

$$f(3) = 3^4 - 8 \times 3^2 \quad = 9$$

maximum value 9 , minimum value -16

OPTIMISATION

Problems involving maxima and minima.

The problem is modelled by a function.
Examine stationary points and end points of interval.

(1) Two numbers have product 16. Find the minimum sum.

product: $xy = 16$

$$y = \frac{16}{x}$$

sum: $x + y$

$$= x + \frac{16}{x}$$

model: $S(x) = x + \dfrac{16}{x} \,, \; x \neq 0$

Stationary points: $S(x) = x + 16x^{-1}$

$$S'(x) = 1 - 16x^{-2} = 1 - \frac{16}{x^2}$$

function stationary where $S'(x) = 0$

$$1 - \frac{16}{x^2} = 0$$

$$x^2 = 16$$

$$x = \pm 4$$

x	$\rightarrow$ -4 $\rightarrow$	$\rightarrow$ 4 $\rightarrow$
$S'(x)$	+ 0 -	- 0 +
shape	↗ — ↘	↘ — ↗
nature	max. value at $x = -4$	min. value at $x = 4$

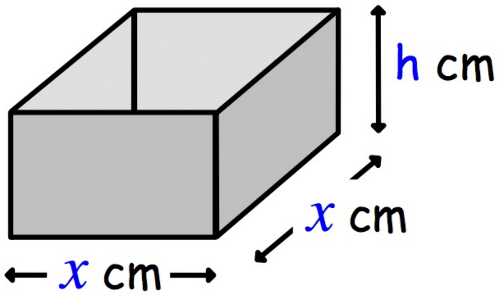
$$S(x) = x + \frac{16}{x}$$

$$S(4) = 4 + \frac{16}{4} = 8$$

minimum sum 8

(2) A square-based open-top cuboid has volume 4000cm^3.

Find the dimensions for the minimum surface area.



$$V = lbh$$

area: $A = x^2 + 4xh$ $\qquad\qquad 4000 = x^2 h$

$$= x^2 + 4x\frac{4000}{x^2} \qquad\qquad h = \frac{4000}{x^2}$$

$$A(x) = x^2 + \frac{16000}{x}, \quad x > 0$$

$$A(x) = x^2 + 1600x^{-1}$$

$$A'(x) = 2x - 1600x^{-2} = 2x - \frac{16000}{x^2}$$

function stationary where $A'(x) = 0$ $\qquad 2x - \dfrac{16000}{x^2} = 0$

x	$\rightarrow 20 \rightarrow$
$A'(x)$	- 0 +
shape	↘ — ↗
nature	min. value when $x = 20$

$$h = \frac{4000}{x^2} \qquad\qquad 2x = \frac{16000}{x^2}$$

$$= \frac{4000}{20^2} \qquad\qquad x^3 = 8000$$

$$= 10 \qquad\qquad x = 20$$

DIMENSIONS: $\underline{\underline{20 \times 20 \times 10 \text{ cm}}}$