HIGHER MATHEMATICS

COURSE NOTES

UNIT 2

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POLYNOMIALS

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots\ldots\ldots + a_2 x^2 + a_1 x^1 + a_0 x^0$$

The COEFFICIENTS a_0 , a_1 , a_2 ... a_n are constants, $a_n \neq 0$
The DEGREE is the highest power.
The CONSTANT is the term independent of x. (a_0)
Usually written as descending powers of x.

$(x - 2)(x + 2)(x^3 + 3)$	degree 5
$= x^5 - 4x^3 + 3x^2 - 12$	coefficient of x^5 is 1
	coefficient of x^4 is 0
	constant is -12

VALUE

$f(x) = 2x^3 + 4x^2 - 10$, find $f(-3)$.

$f(-3) = 2 \times (-3)^3 + 4 \times (-3)^2 - 10 = \underline{\underline{-28}}$

NESTED FORM: uses detached coefficients.
include ALL descending powers of x.

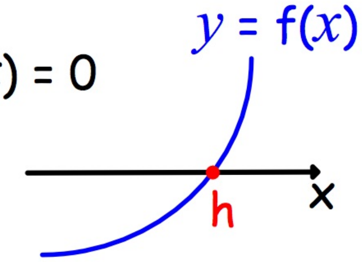
$$2x^3 \quad + 4x^2 \quad + 0x^1 \quad - 10x^0$$

$$\begin{array}{r|rrrrl}
-3 & 2 & 4 & 0 & -10 & \\
 & & & & & \text{add} \\
 & 0 & -6 & 6 & -18 & \\
\hline
 & \times -3 & \times -3 & \times -3 & & \\
 & 2 & -2 & 6 & \underline{\underline{-28}} & \underline{\underline{f(-3) = -28}}
\end{array}$$

FACTOR THEOREM:

$f(h) = 0 \iff x - h$ is a factor

$\iff h$ is a root of equation $f(x) = 0$



(1) Show 3 is a root of $x^3 - 3x^2 - x + 3 = 0$

3	1	-3	-1	3	
					add
	0	3	0	-3	
	×3	×3	×3		
	1	0	-1	$\underline{\underline{0}}$	

$f(3) = 0$

$\iff \underline{\underline{3 \text{ is a root}}}$

(2) Show $x - 3$ is a factor of $x^3 - 3x^2 - x + 3$

as (1) but conclude $f(3) = 0$

$\iff \underline{\underline{x - 3 \text{ is a factor}}}$

(3) Show $x + 2$ is a factor of $x^3 + 2x^2 - 9x - 18$

-2	1	2	-9	-18	
					add
	0	-2	0	18	
	×-2	×-2	×-2		
	1	0	-9	$\underline{\underline{0}}$	

$f(-2) = 0$

$\iff \underline{\underline{x + 2 \text{ is a factor}}}$

REMAINDER THEOREM:

$$f(x) = \underbrace{(\,x - h\,)}_{\text{divisor}}\ \underbrace{Q(x)}_{\text{quotient}} \; + \; \underbrace{f(h)}_{\text{remainder}}$$

POLYNOMIAL = DIVISOR × QUOTIENT + REMAINDER

$Q(x)$ is a polynomial of degree one lower than $f(x)$

SYNTHETIC DIVISION

When $f(x)$ is divided by $x - h$ the remainder is $f(h)$.

(1) Find the quotient and remainder on dividing
$2x^3 + 5x^2 - 28x - 9$ by $x - 3$

$$\begin{array}{c|cccc l}
3 & 2 & 5 & -28 & -9 & \\
 & & & & & \text{add} \\
 & 0 & 6 & 33 & 15 & \\
\hline
 & 2 & 11 & 5 & \underline{\underline{6}} &
\end{array}$$

$f(3) = 6$

$\Leftrightarrow$ remainder 6

$$2x^3 + 5x^2 - 28x - 9 = (x - 3)(2x^2 + 11x + 5) + 6$$

$$\underline{\underline{\text{quotient } 2x^2 + 11x + 5 \text{ , remainder } 6}}$$

(2) Find the quotient and remainder on dividing

$4x^3 - 3x + 5$ by $2x + 1$

root $2x + 1 = 0$

$2x = -1$

$x = -{}^1/_2$

$$\begin{array}{c|cccc} -{}^1/_2 & 4 & 0 & -3 & 5 \\ & 0 & -2 & 1 & 1 \\ \hline & 4 & -2 & -2 & \underline{\underline{6}} \end{array}$$

add

$\times -{}^1/_2$ $\quad \times -{}^1/_2$ $\quad \times -{}^1/_2$

$f(-{}^1/_2) = 0$

$\Leftrightarrow x + {}^1/_2$ is a factor

$$4x^3 - 3x + 5 = (x + {}^1/_2)(4x^2 - 2x - 2) + 6$$

$$= (x + {}^1/_2)\,2(2x^2 - x - 1) + 6$$

$$= (2x + 1)(2x^2 - x - 1) + 6$$

quotient $2x^2 - x - 1$, remainder 6

UNKNOWNS

(1) If $2x^3 + x^2 - bx - 1$ divided by $x + 2$ has a remainder of 3 , find the value of b.

-2	2	1	- b	-1
				add
	0	- 4	6	2b-12
	$\times$-2	$\times$-2	$\times$-2	
	2	-3	-b+6	2b-13

remainder 3

$\Leftrightarrow f(-2) = 3$

$2b - 13 = 3$

$2b = 16$

$b = 8$

(2) If $x^3 + ax^2 - 5x - 6$ has a factor $x - 2$, find the value of a and fully factorise.

2	1	a	-5	-6
				add
	0	2	2a+4	4a-2
	$\times$2	$\times$2	$\times$2	
	1	a+2	2a-1	4a-8

factor $x - 2$

$\Leftrightarrow f(2) = 0$

$4a - 8 = 0$

$a = 2$

a = 2

1 4 3

$$x^3 + 2x^2 - 5x - 6 = (x - 2)(1x^2 + 4x + 3)$$

$$= (x - 2)(x + 1)(x + 3)$$

FACTORISATION

Find a factor h of the CONSTANT such that f(h) = 0

(1) Fully factorise $2x^3 + 5x^2 - 28x - 15$

factors of -15 : ± 1 , ± 3 , ± 5 , ± 15

$f(1) = 2\times 1^3 + 5\times 1^2 - 28\times 1 - 15 \neq 0$

$f(-1) = 2\times(-1)^3 + 5\times(-1)^2 - 28\times(-1) - 15 \neq 0$

$f(3) = 2\times 3^3 + 5\times 3^2 - 28\times 3 - 15 = 0$ ✓

3 is a root

3	2	5	-28	-15	
					add
	0	6	33	15	
	×3	×3	×3		
	2	11	5	$\underline{\underline{0}}$	

$f(3) = 0$

$\Longleftrightarrow x - 3$ is a factor

coefficients of a poly. of degree one lower

$$2x^3 + 5x^2 - 28x - 15 = (x - 3)(2x^2 + 11x + 5)$$

$$= \underline{\underline{(x - 3)(2x + 1)(x + 5)}}$$

(2) Given $2x + 1$ is a factor,

fully factorise $2x^3 + 5x^2 - 28x - 15$

and so solve $2x^3 + 5x^2 - 28x - 15 = 0$

root
$$2x + 1 = 0$$
$$2x = -1$$
$$x = -\tfrac{1}{2}$$

$$\begin{array}{r|rrrr}
-\tfrac{1}{2} & 2 & 5 & -28 & -15 \\
 & 0 & -1 & -2 & 15 \\
\hline
 & 2 & 4 & -30 & \underline{\underline{0}}
\end{array}$$

add

$\times -\tfrac{1}{2}$ $\times -\tfrac{1}{2}$ $\times -\tfrac{1}{2}$

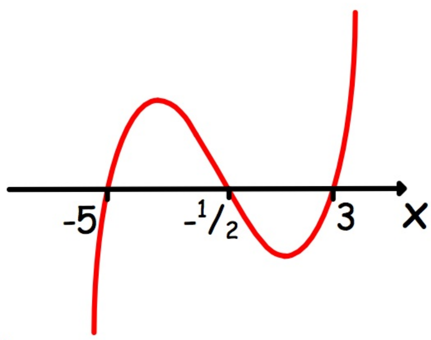
$f(-\tfrac{1}{2}) = 0$

$\Leftrightarrow x + \tfrac{1}{2}$ is a factor

$$\begin{aligned}
2x^3 + 5x^2 - 28x - 15 &= (x + \tfrac{1}{2})(2x^2 + 4x - 30) \\
&= (x + \tfrac{1}{2})\,2(x^2 + 2x - 15) \\
&= (2x + 1)(x^2 + 2x - 15) \\
&= \underline{\underline{(2x + 1)(x + 5)(x - 3)}}
\end{aligned}$$

$$2x^3 + 5x^2 - 28x - 15 = 0$$
$$(2x + 1)(x + 5)(x - 3) = 0$$
$$x = -\tfrac{1}{2} \quad \text{or} \quad x = -5 \quad \text{or} \quad x = 3$$

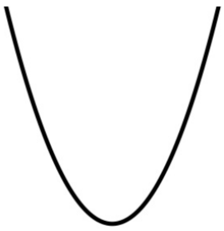
<u>ROOTS are -5, $-\tfrac{1}{2}$ and 3</u>



$y = 2x^3 + 5x^2 - 28x - 15$

EQUATION FROM THE GRAPH

Quadratics: 'U'

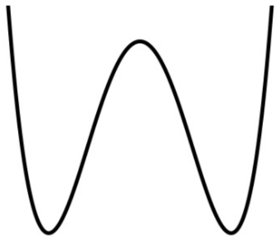


two roots

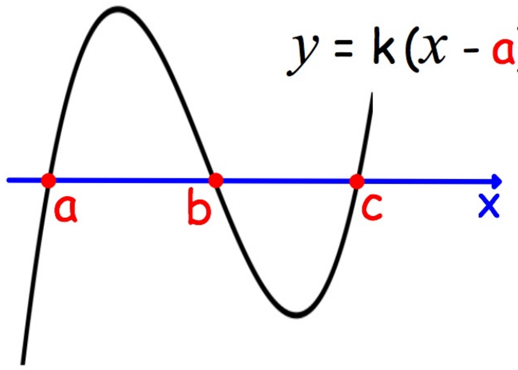
Cubics: 'S'

three roots

Quartics: 'W'

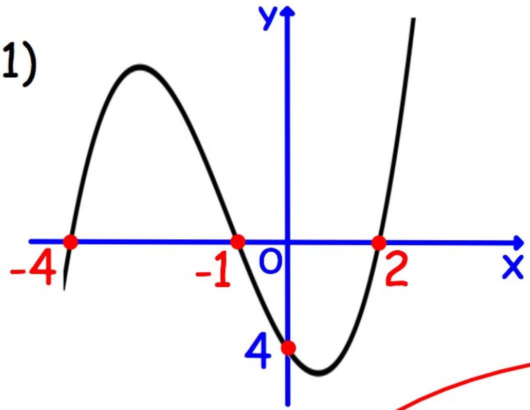


four roots



$y = k(x - a)(x - b)(x - c)$

(1)



$y = k(x - a)(x - b)(x - c)$

$y = k(x + 4)(x + 1)(x - 2)$

for (0,4) substitute $x = 0$ and $y = 4$

$4 = k(0 + 4)(0 + 1)(0 - 2)$

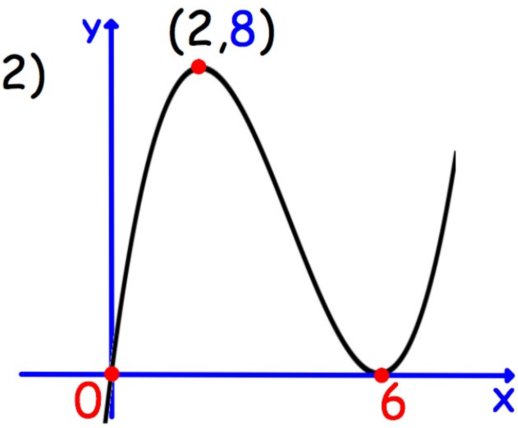
$4 = -8k$

$k = {}^{4}/_{-8}$

$k = -{}^{1}/_{2}$

$$\underline{\underline{y = -{}^{1}/_{2}(x + 4)(x + 1)(x - 2)}}$$

(2)



root $x = 0$

repeated root

$y = k\,x(x - b)^2$

$y = k\,x(x - 6)^2$

for (2,8) substitute $x = 2$ and $y = 8$

$8 = k \times 2 \times (2 - 6)^2$

$8 = 32k$

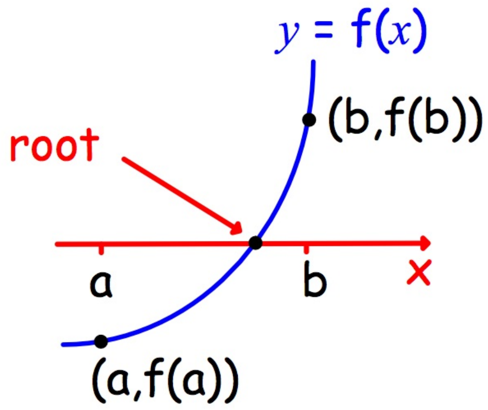
$k = {}^{8}/_{32}$

$k = {}^{1}/_{4}$

$$\underline{\underline{y = {}^{1}/_{4}\,x(x - 6)^2}}$$

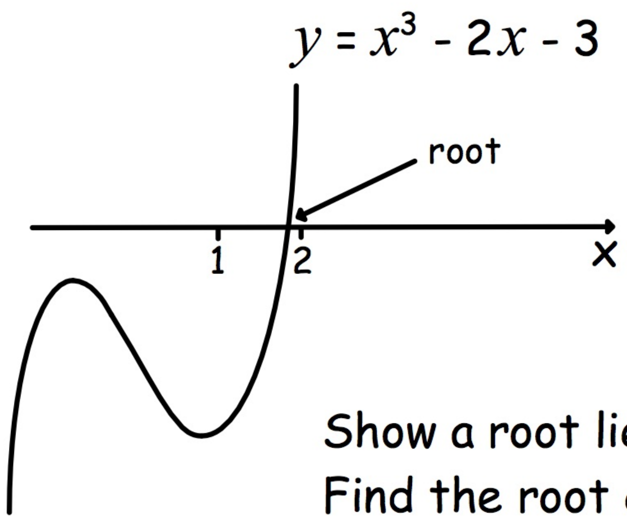
ITERATION

The root is where $f(x) = 0$



To show a root lies between $x = a$ and $x = b$,
show the y-coord. changes sign ie. find f(a) and f(b).

To improve the root repeat the process by using values of x between a and b.



Show a root lies between 1 and 2.
Find the root correct to one decimal place.

Show $x^3 - 2x - 3 = 0$ has a root between 1 and 2.

$f(x) = x^3 - 2x - 3$

$f(1) = 1^3 - 2\times1 - 3 \quad = -4$

$f(2) = 2^3 - 2\times2 - 3 \quad = +1$

function changed sign, so a root lies between 1 and 2.

consider half-way
between 1 and 2

$f(1.5) = 1.5^3 - 2\times1.5 - 3 \quad = -2.625$

sign changes
between 1.5 and 2

$f(1.6) = -2.104$

$f(1.7) = -1.487$

$f(1.8) = -0.768$

$f(1.9) = +0.059$

sign changes
between 1.8 and 1.9

$f(1.85) = -0.368375$

function changed sign, so a root lies between 1.85 and 1.9

ROOT ≈ 1.9

QUADRATIC THEORY

QUADRATIC EQUATIONS

$ax^2 + bx + c = 0$ can be solved by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $b^2 - 4ac$ is 0, 1, 4, 9.... then solve by factorising.

(1) $\dfrac{2}{x} = \dfrac{x+3}{5}$

$$\frac{2}{x} = \frac{x+3}{5}$$

$$\begin{aligned} 10 &= x(x+3) \\ 10 &= x^2 + 3x \\ 0 &= x^2 + 3x - 10 \end{aligned}$$

$$\begin{aligned} x^2 + 3x - 10 &= 0 \\ (x+5)(x-2) &= 0 \end{aligned}$$

$x = -5$ or $x = 2$

(2) Find the roots of $3x^2 - 4x - 9 = 0$

$ax^2 + bx + c = 0$

$a = 3$, $b = -4$, $c = -9$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+4 \pm \sqrt{124}}{6}$$

$$= \frac{4 - \sqrt{124}}{6} \quad \text{or} \quad \frac{4 + \sqrt{124}}{6}$$

$= -1.189...$ or $2.522...$

$x = -1.2$ or $x = 2.5$

$b^2 - 4ac$

$= (-4)^2 - 4 \times 3 \times (-9)$

$= 124$

QUADRATIC INEQUALITIES

$(x - a)(x - b) > 0$ or similar

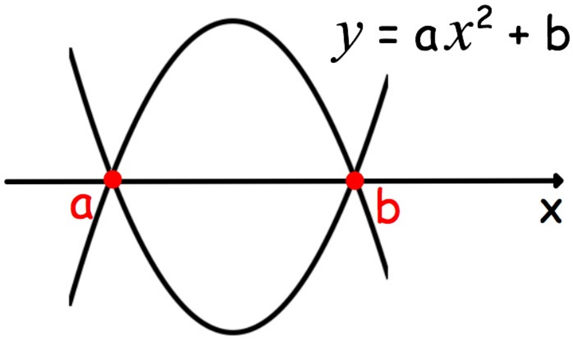
solution is one of two posibilities: (i) $a < x < b$

(ii) $x < a$ or $x > b$

SKETCH GRAPH: $a > 0$ ie. positive

$a < 0$ ie. negative

$y = ax^2 + bx + c$



$ax^2 + bx + c > 0$

curve above axis

$ax^2 + bx + c < 0$

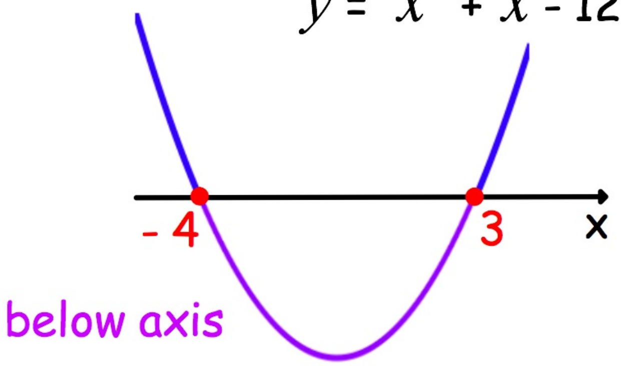
curve below axis

(1) $x^2 + x - 12 < 0$

$(x + 4)(x - 3) < 0$

$\underline{\underline{-4 < x < 3}}$

$y = x^2 + x - 12$

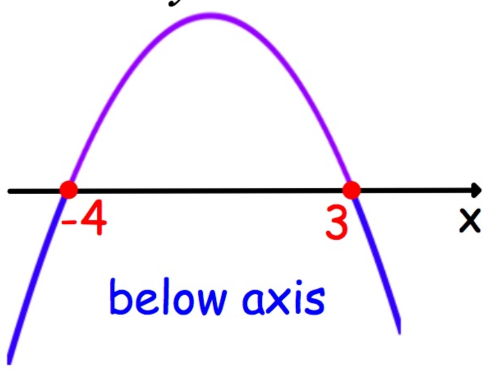


(2) $12 - x - x^2 \leq 0$

$(4 + x)(3 - x) \leq 0$

$\underline{\underline{x \leq -4 \text{ or } x \geq 3}}$

$y = 12 - x - x^2$



DISCRIMINANT

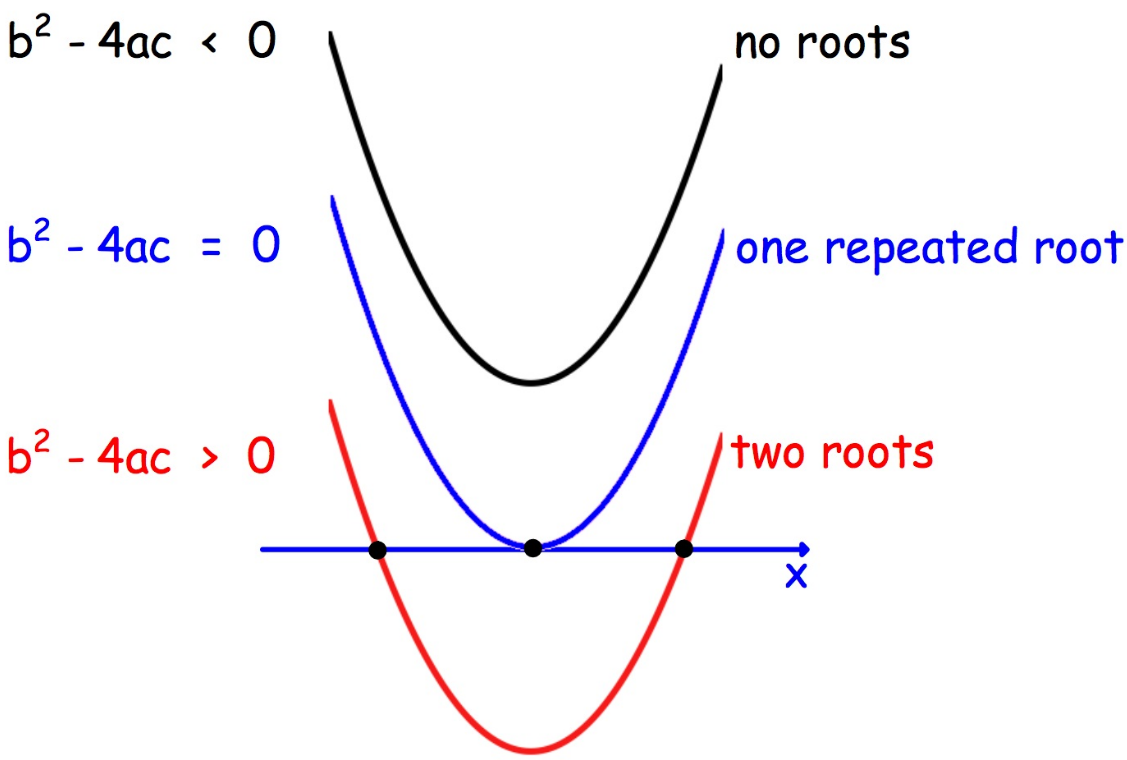
The quadratic equation $ax^2 + bx + c = 0$

can be rearranged to $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The quadratic formula finds the ROOTS of the equation.

$\pm\sqrt{\text{positive}}$	two solutions
$\pm\sqrt{0}$	one solution
$\pm\sqrt{\text{negative}}$	no solution

The DISCRIMINANT $b^2 - 4ac$
is used to determine the NATURE of the roots.



NATURE OF THE ROOTS

$b^2 - 4ac > 0$ TWO REAL AND DISTINCT ROOTS

$b^2 - 4ac = 0$ TWO REAL AND EQUAL ROOTS

$b^2 - 4ac < 0$ NO REAL ROOTS

NOTE:

(i) condition for REAL ROOTS $b^2 - 4ac \geq 0$

(ii) if $b^2 - 4ac$ is a square number 0, 1, 4, 9.... then
the roots are RATIONAL, otherwise IRRATIONAL.
(SURD)

(1) Find the nature of the roots of $3x^2 - 4x - 9 = 0$

$a = 3$, $b = -4$, $c = -9$

$b^2 - 4ac = (-4)^2 - 4 \times 3 \times (-9) = 124$

$b^2 - 4ac > 0 \Rightarrow$ two real and distinct roots
(and irrational)

(2) Find the nature of the roots of $2x^2 - x + 1 = 0$

$a = 2$, $b = -1$, $c = 1$

$b^2 - 4ac = (-1)^2 - 4 \times 2 \times 1 = -7$

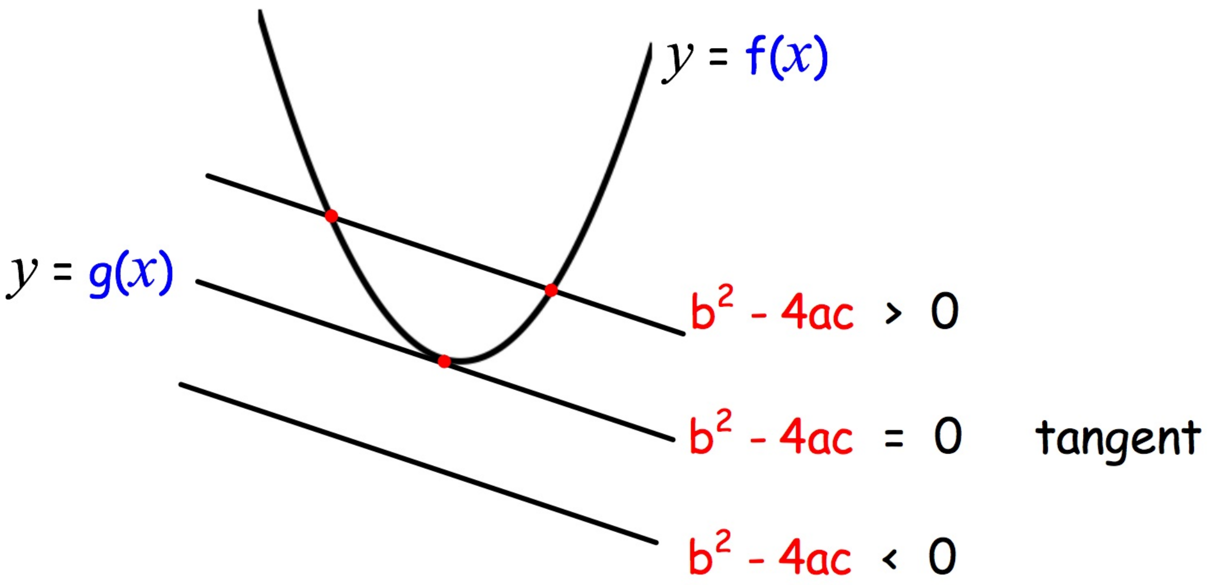
$b^2 - 4ac < 0 \Rightarrow$ no real roots

INTERSECTION OF A LINE AND CURVE

Substitution $\quad f(x) = g(x)$

results in $\quad ax^2 + bx + c = 0$

Discriminant $b^2 - 4ac$ distinguishes between:



TANGENCY

A tangent is a line that touches the curve at one point.

The substitution results in a quadratic equation
with one solution:

$$b^2 - 4ac = 0 \Rightarrow \text{EQUAL ROOTS}$$

so line is a tangent

Show that the line $y = 5x - 2$ is a tangent to the curve $y = 2x^2 + x$ and find the point of contact.

$$2x^2 + x = 5x - 2$$
$$2x^2 - 4x + 2 = 0$$
$$1x^2 - 2x + 1 = 0$$

$b^2 - 4ac = (-2)^2 - 4 \times 1 \times 1 = 0$

$b^2 - 4ac = 0 \Rightarrow$ line is a tangent

$$(x - 1)^2 = 0$$
$$x = 1$$

$$y = 5x - 2$$
$$= 5 \times 1 - 2$$
$$= 3$$

point of contact (1,3)

alternative:

$$2x^2 + x = 5x - 2$$
$$2x^2 - 4x + 2 = 0$$
$$1x^2 - 2x + 1 = 0$$
$$(x - 1)^2 = 0$$
$$x = 1$$

one point of contact

$\Rightarrow$ line is a tangent

$$y = 5x - 2$$
$$= 5 \times 1 - 2$$
$$= 3$$

point of contact (1,3)

UNKNOWNS

(1) Given $6x^2 + 12x + k = 0$ has REAL roots, find k.

$a = 6$, $b = 12$, $c = k$

$b^2 - 4ac = 12^2 - 4 \times 6 \times k = 144 - 24k$

for real roots $\Rightarrow$ $b^2 - 4ac \geq 0$

$$144 - 24k \geq 0$$
$$-24k \geq -144$$
$$\underline{\underline{k \leq 6}}$$

(2) Show that $(2k + 4)x^2 + (3k + 2)x + (k - 2) = 0$ always has REAL roots.

$a = (2k + 4)$, $b = (3k + 2)$, $c = (k - 2)$

$$\begin{aligned} b^2 - 4ac &= (3k + 2)^2 - 4(2k + 4)(k - 2) \\ &= (9k^2 + 12k + 4) - 4(2k^2 - 8) \\ &= 9k^2 + 12k + 4 - 8k^2 + 32 \\ &= k^2 + 12k + 36 \\ &= (k + 6)^2 \end{aligned}$$

for all values of k , $(k + 6)^2 \geq 0$

$b^2 - 4ac \geq 0 \Rightarrow$ real roots

<u>so roots are always real</u>

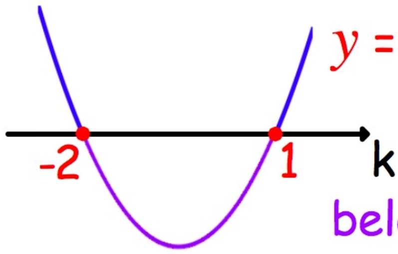
(3) Find the values of k for which the equation
$x^2 - 2kx + 2 - k = 0$ has no real roots.

$a = 1$, $b = (-2k)$, $c = (2 - k)$

no real roots
$\Rightarrow b^2 - 4ac < 0$

$$b^2 - 4ac = (-2k)^2 - 4 \times 1 \times (2 - k)$$
$$= 4k^2 - 8 + 4k$$
$$= 4k^2 + 4k - 8$$

$$4k^2 + 4k - 8 < 0$$
$$k^2 + k - 2 < 0$$
$$(k + 2)(k - 1) < 0$$
$$\underline{\underline{-2 < k < 1}}$$



(4) If the line gradient 3 , $y = 3x + C$, is a tangent to the curve $y = x^2 + 1$, find C.

$$x^2 + 1 = 3x + C$$
$$1x^2 - 3x + 1 - C = 0$$

$a = 1$, $b = -3$, $c = (1 - C)$

line a tangent
$\Rightarrow b^2 - 4ac = 0$

$$b^2 - 4ac = (-3)^2 - 4 \times 1 \times (1 - C)$$
$$= 9 - 4 + 4C$$
$$= 4C + 5$$

$$4C + 5 = 0$$
$$4C = -5$$
$$\underline{\underline{C = -5/4}}$$

INTEGRAL CALCULUS

ANTI-DIFFERENTIATION

Integration reverses the effect of differentiation.

RULE: $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad , \; n \neq -1$

ie. not for $\int \frac{1}{x} \, dx$

OTHER RULES:

$$\int [\, f(x) + g(x) \,] \, dx = \int f(x) \, dx + \int g(x) \, dx$$

$$\int k \, f(x) \, dx = k \int f(x) \, dx$$

TERMINOLOGY:

The INTEGRAL of x^n with respect to x

The INTEGRAND is x^n

The CONSTANT OF INTEGRATION is C

INDEFINITE INTEGRAL

DIFFERENTIAL EQUATIONS

Contain a derivative $\frac{dy}{dx}$. Solve by integrating.

(1) Solve the equation
if $y = 3$ when $x = 1$

$$\frac{dy}{dx} = 3x^2$$

$$y = \int 3x^2 \, dx$$

GENERAL SOLUTION $\quad y = x^3 + C$

$y = 3$ when $x = 1$ $\quad 3 = 1^3 + C$

$$C = 2$$

PARTICULAR SOLUTION $\quad \underline{\underline{y = x^3 + 2}}$

(2) The gradient of a curve is given by $f'(x) = x^2 - 5$.
It passes through the point (3,- 4).

Find the equation of the curve.

$$f(x) = \int (x^2 - 5) \, dx$$

$$= \frac{x^3}{3} - 5x + C$$

$f(3) = -4$ $\quad f(3) = \frac{3^3}{3} - 5 \times 3 + C$

$$-4 = 9 - 15 + C$$

$$C = 2$$

$$\underline{\underline{y = {}^1/_3\, x^3 - 5x + 2}}$$

INTEGRATION

(1) $\int (x^4 + x - 2)\ dx$

$= \dfrac{x^5}{5} + \dfrac{x^2}{2} - 2x + C$

$= \underline{\underline{\tfrac{1}{5}x^5 + \tfrac{1}{2}x^2 - 2x + C}}$

(2) $\int 6x^3\ dx$

$= \dfrac{6x^4}{4} + C$

$= \underline{\underline{\tfrac{3}{2}x^4 + C}}$

(3) $\int (u^{-2} + 2u - 1)\ du$

$= \dfrac{u^{-1}}{-1} + \dfrac{2u^2}{2} - u + C$

$= \underline{\underline{-u^{-1} + u^2 - u + C}}$

(4) $\int 6x^{-3}\ dx$

$= \dfrac{6x^{-2}}{-2} + C$

$= \underline{\underline{-3x^{-2} + C}}$

INDICES

$\dfrac{1}{a^p} = a^{-p}$ $\qquad \sqrt[n]{a^m} = a^{\frac{m}{n}}$

Use indices rules to express terms in the form x^n.

(1) $\displaystyle\int \frac{1}{3x^2}\ dx$ or $\displaystyle\int \frac{dx}{3x^2}$

$= \dfrac{1}{3}\displaystyle\int x^{-2}\ dx$

$= \dfrac{1}{3} \times \dfrac{x^{-1}}{-1} + C$

$= \underline{\underline{-\dfrac{1}{3x} + C}}$

(2) $\displaystyle\int \sqrt{x^3}\ dx$

$= \displaystyle\int x^{\frac{3}{2}}\ dx$

$= \dfrac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$

$= \underline{\underline{\dfrac{2}{5}x^{\frac{5}{2}} + C}}$

BRACKETS AND QUOTIENTS

Integrate sums and differences of terms x^n,
so 'break' brackets and 'split' quotients.

(1) $\int (2x - 3)^2 \, dx$

$$= \int (4x^2 - 12x + 9) \, dx$$

$$= \frac{4x^3}{3} - \frac{12x^2}{2} + 9x + C$$

$$= \underline{\underline{\tfrac{4}{3}x^3 - 6x^2 + 9x + C}}$$

(2) $\int \frac{x - 1}{\sqrt{x}} \, dx$

$$\frac{x^1}{x^{1/2}} - \frac{1}{x^{1/2}}$$

$$= x^{1/2} - x^{-1/2}$$

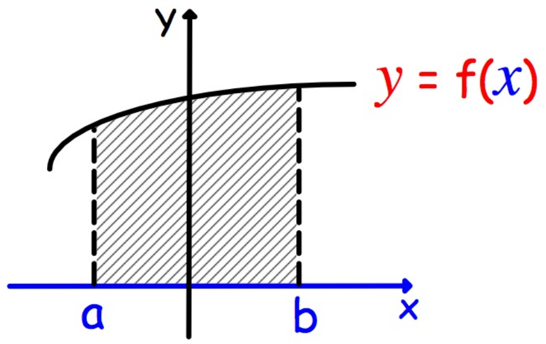
$$= \int (x^{1/2} - x^{-1/2}) \, dx$$

$$= \frac{2}{3}x^{3/2} - \frac{2}{1}x^{1/2} + C$$

$$= \underline{\underline{\frac{2}{3}x^{3/2} - 2x^{1/2} + C}}$$

AREA

The area between the curve and the x-axis is given by the DEFINITE INTEGRAL:



$$\text{AREA} = \int_a^b f(x)\, dx$$

lower limit is a
upper limit is b

EVALUATING DEFINITE INTEGRALS

if $f(x) = F'(x)$, $\int_a^b f(x)\, dx = F(b) - F(a)$

written $\left[F(x)\right]_a^b$

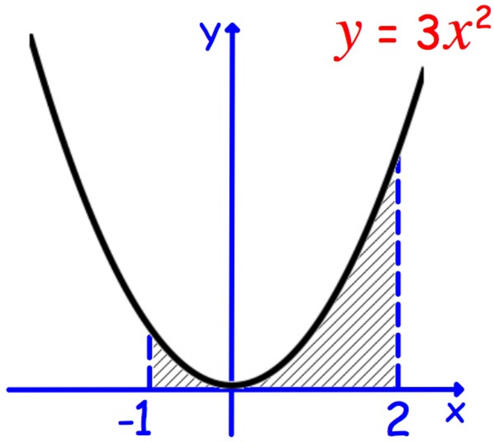
(1) $\int_{-1}^{2} 3x^2\, dx$

$= \left[x^3\right]_{-1}^{2}$

$= 2^3 - (-1)^3$

$= 8 - (-1)$

$= \underline{\underline{9}}$



Area 9 units^2

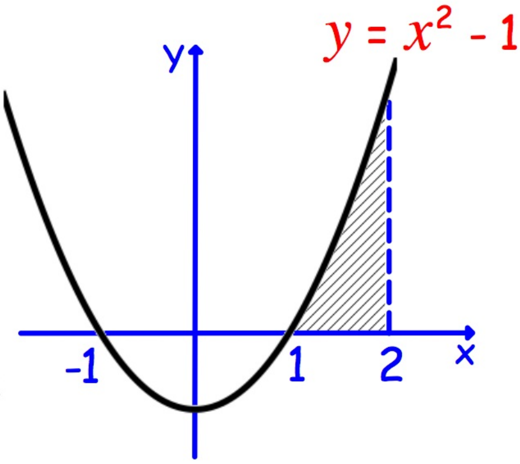
(2) $\int_{1}^{2} (x^2 - 1) \, dx$

$$= \left[\frac{x^3}{3} - x \right]_1^2$$

$$= \left(\frac{2^3}{3} - 2 \right) - \left(\frac{1^3}{3} - 1 \right)$$

$$= \frac{2}{3} - \left(-\frac{2}{3} \right)$$

$$= \frac{4}{3}$$



Area $\frac{4}{3}$ units^2

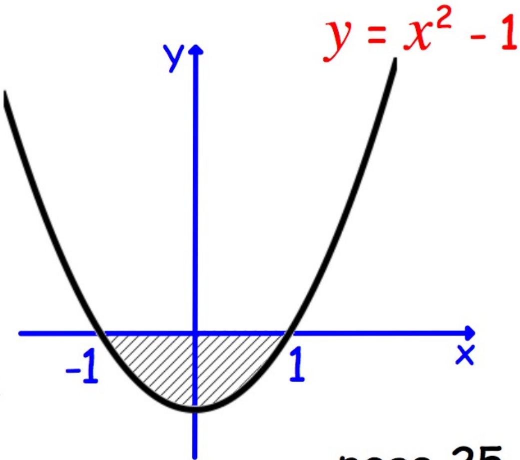
(3) $\int_{-1}^{1} (x^2 - 1) \, dx$

$$= \left[\frac{x^3}{3} - x \right]_{-1}^{1}$$

$$= \left(\frac{1^3}{3} - 1 \right) - \left(\frac{(-1)^3}{3} - (-1) \right)$$

$$= -\frac{2}{3} - \frac{2}{3}$$

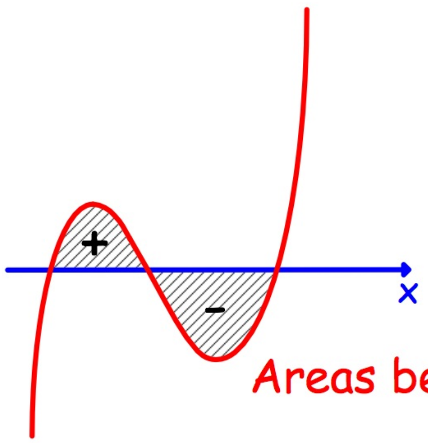
$$= -\frac{4}{3}$$



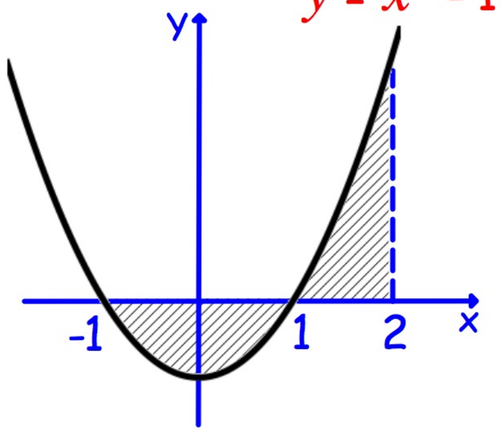
area cannot be negative,

Area $\frac{4}{3}$ units^2

Evaluate areas above and below the x-axis separately.



Areas below the x-axis appear to be negative.



$$\int_{-1}^{2} (x^2 - 1)\ dx = 0$$

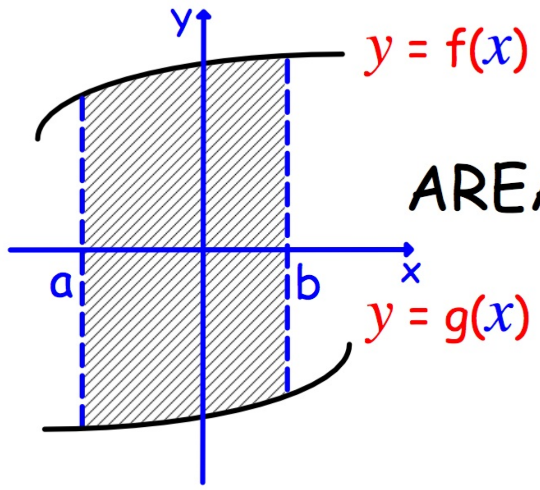
area below x-axis appears negative and cancels with area above x-axis

$$\int_{1}^{2} (x^2 - 1)\ dx = \frac{4}{3}$$

$$\int_{-1}^{1} (x^2 - 1)\ dx = -\frac{4}{3}$$

$$\text{Area} = \frac{4}{3} + \frac{4}{3} = \underline{\underline{\frac{8}{3}\ \text{units}^2}}$$

AREA BETWEEN CURVES

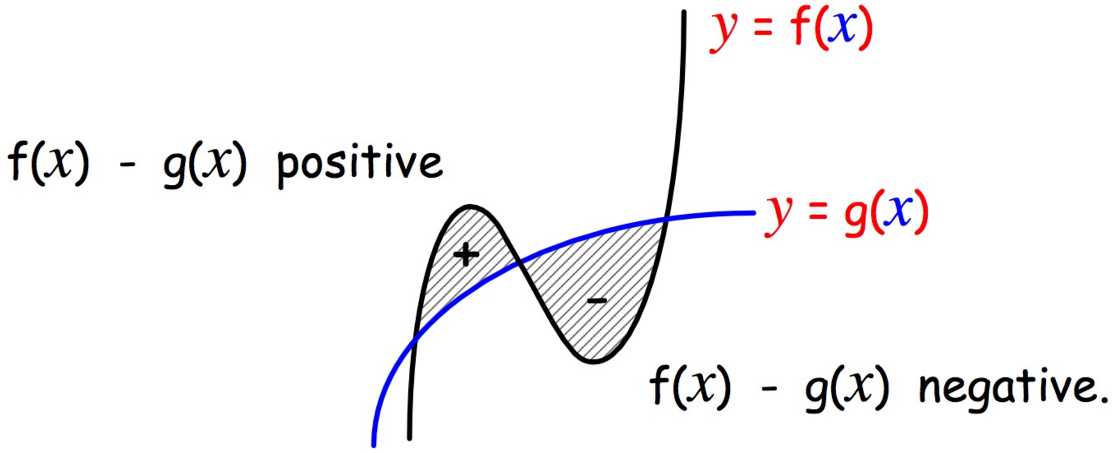


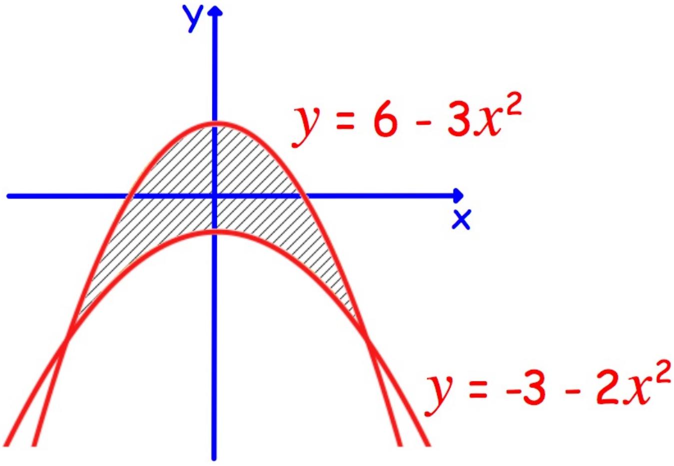
$$\text{AREA} = \int_a^b \{f(x) - g(x)\}\, dx$$

NOTE: TOP - BOTTOM always gives a positive area

ie. area is NOT negative below the x-axis.

If the curves cross over evaluate the areas separately.





LIMITS: intersection

TOP CURVE = BOTTOM CURVE

$$6 - 3x^2 = -3 - 2x^2$$

$$9 - x^2 = 0$$

$$(3 + x)(3 - x) = 0$$

$$x = -3 \text{ or } 3$$

INTEGRAND

TOP - BOTTOM

$$6 - 3x^2 - (-3 - 2x^2)$$

$$= 9 - x^2$$

from the symmetry:

$$\int_{-3}^{3} (9 - x^2)\,dx = 2\int_{0}^{3} (9 - x^2)\,dx$$

easier to evaluate with limit 0

$$\int_{0}^{3} (9 - x^2)\,dx$$

$$= \left[9x - \frac{x^3}{3}\right]_0^3$$

$$= 9 \times 3 - \frac{3^3}{3} - 0$$

$$= 18$$

Area = 2×18 units^2 = $\underline{\underline{36 \text{ units}^2}}$

TRIGONOMETRY: COMPOUND ANGLES

EXPANSION FORMULAE

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

NOTICE: $\cos(A + B) \neq \cos A + \cos B$

$$\cos A \cos B + \sin A \sin B = \cos(A - B)$$

$$\begin{aligned} \cos 110^\circ \cos 20^\circ + \sin 110^\circ \sin 20^\circ &= \cos(110^\circ - 20^\circ) \\ &= \cos 90^\circ \\ &= 0 \end{aligned}$$

$$\sin A \cos B + \cos A \sin B = \sin(A + B)$$

$$\begin{aligned} \sin 20^\circ \cos 10^\circ + \cos 20^\circ \sin 10^\circ &= \sin(20^\circ + 10^\circ) \\ &= \sin 30^\circ \\ &= {}^1/_2 \end{aligned}$$

EXACT VALUES

No calculator allowed.

Use combinations of 30°, 45°, 60°

Find the EXACT value of cos 105°.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(60° + 45°) = \cos 60° \cos 45° - \sin 60° \sin 45°$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \underline{\underline{\frac{1 - \sqrt{3}}{2\sqrt{2}}}}$$

rationalising denominator:

$$\frac{1 - \sqrt{3}}{2\sqrt{2}} \quad \begin{matrix} \times\sqrt{2} \\ \times\sqrt{2} \end{matrix}$$

$$= \underline{\underline{\frac{\sqrt{2} - \sqrt{6}}{4}}}$$

USING TRIG. RATIOS

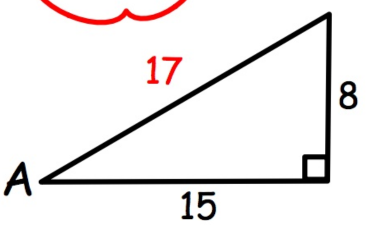
SOHACHTOA to sketch triangles

PYTH. THM. to find missing side

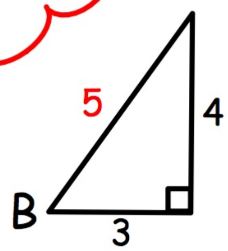
Find values of sin and cos of angle

If $\tan A = {}^{8}/_{15}$ and $\tan B = {}^{4}/_{3}$, find $\cos(A+B)$.

$$\begin{aligned} x^2 &= 15^2 + 8^2 \\ &= 289 \\ x &= 17 \end{aligned}$$



$$\begin{aligned} y^2 &= 4^2 + 3^2 \\ &= 25 \\ y &= 5 \end{aligned}$$



$\sin A = {}^{8}/_{17}$

$\cos A = {}^{15}/_{17}$

$\sin B = {}^{4}/_{5}$

$\cos B = {}^{3}/_{5}$

$$\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ &= {}^{15}/_{17} \times {}^{3}/_{5} - {}^{8}/_{17} \times {}^{4}/_{5} \\ &= {}^{45}/_{85} - {}^{32}/_{85} \\ &= \underline{\underline{{}^{13}/_{85}}} \end{aligned}$$

DOUBLE ANGLE FORMULAE

$\sin 2A = 2\sin A \cos A$

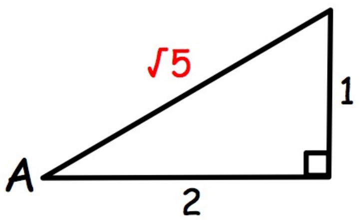
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

VARIATIONS:

$$\sin 2A = 2\sin A \cos A$$
$$\sin 4A = 2\sin 2A \cos 2A$$
$$\sin A = 2\sin {}^{A}/_{2} \cos {}^{A}/_{2}$$

(1) If $\tan A = {}^1/_2$, find $\sin 2A$.

$x^2 = 2^2 + 1^2$
$= 5$
$x = \sqrt{5}$



$\sin A = {}^1/_{\sqrt{5}}$

$\cos A = {}^2/_{\sqrt{5}}$

$$\sin 2A = 2\sin A \cos A$$
$$= {}^2/_1 \times {}^1/_{\sqrt{5}} \times {}^2/_{\sqrt{5}}$$
$$= \underline{\underline{{}^4/_5}}$$

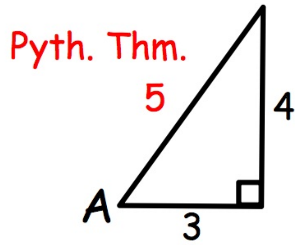
(2) If $\tan A = {}^4/_3$, find:

(i) $\sin 2A$ (ii) $\cos 2A$ (iii) $\sin 4A$ (iv) $\tan 2A$

$$\sin 2A = 2\sin A \cos A$$
$$= {}^2/_1 \times {}^4/_5 \times {}^3/_5$$
$$= \underline{\underline{{}^{24}/_{25}}}$$

$\sin A = {}^4/_5$

$\cos A = {}^3/_5$



$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= ({}^3/_5)^2 - ({}^4/_5)^2$$
$$= {}^9/_{25} - {}^{16}/_{25}$$
$$= \underline{\underline{-{}^7/_{25}}}$$

or can use:

$\cos 2A = 2\cos^2 A - 1$

$\cos 2A = 1 - 2\sin^2 A$

Using $\sin 2A = {}^{24}/_{25}$, $\cos 2A = -{}^7/_{25}$ and IDENTITIES:

$$\sin 2A = 2\sin A \cos A$$

$$\tan A = \frac{\sin A}{\cos A}$$

$$\sin 4A = 2\sin 2A \cos 2A$$
$$= {}^2/_1 \times {}^{24}/_{25} \times (-{}^7/_{25})$$
$$= \underline{\underline{-{}^{336}/_{625}}}$$

$$\tan 2A = \frac{\sin 2A}{\cos 2A}$$
$$= \frac{{}^{24}/_{25}}{-{}^7/_{25}}$$
$$= {}^{24}/_{-7}$$
$$= \underline{\underline{-{}^{24}/_7}}$$

TRIG. EQUATIONS: DOUBLE ANGLES

Equation contains: $\sin 2x$ or $\cos 2x$

AND $\sin x$ or $\cos x$

REPLACEMENTS: $\sin 2x$ by $2\sin x \cos x$

if equation has a $\cos x$ — $\cos 2x$ by $2\cos^2 x - 1$

if equation has a $\sin x$ — $\cos 2x$ by $1 - 2\sin^2 x$

FACTORISE: form $ax^2 + bx + c = 0$

$(\quad)(\quad) = 0$

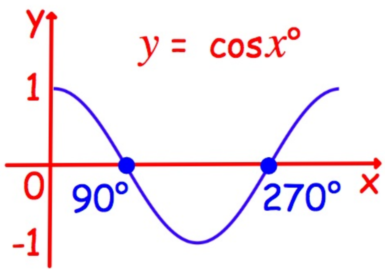
SOLVE: Trig. equations using CAST or graphs.

(1) Solve $\sin 2x° - 3\cos x° = 0$, $0 \leq x \leq 360$

$$2\sin x° \cos x° - 3\cos x° = 0$$

$$2\sin x° \cos x° - 3\cos x° = 0$$

$$\cos x° \,(2\sin x° - 3) = 0$$



$2\sin x° - 3 = 0$ or $\cos x° = 0$

$2\sin x° = 3$

$\sin x° = {}^3/_2$

no solution $\quad\quad x = 90$ or 270

$$\underline{\underline{x = 90, 270}}$$

(2) Solve $\cos 2x^\circ - 3\cos x^\circ + 2 = 0$, $0 \le x \le 360$

$$2\cos^2 x^\circ - 1 - 3\cos x^\circ + 2 = 0$$
$$2\cos^2 x^\circ - 3\cos x^\circ + 1 = 0$$
$$(2\cos x^\circ - 1)(\cos x^\circ - 1) = 0$$

$2\cos x^\circ - 1 = 0$ or $\cos x^\circ - 1 = 0$

$2\cos x^\circ = 1$

$\cos x^\circ = {}^1/_2$ or $\cos x^\circ = 1$

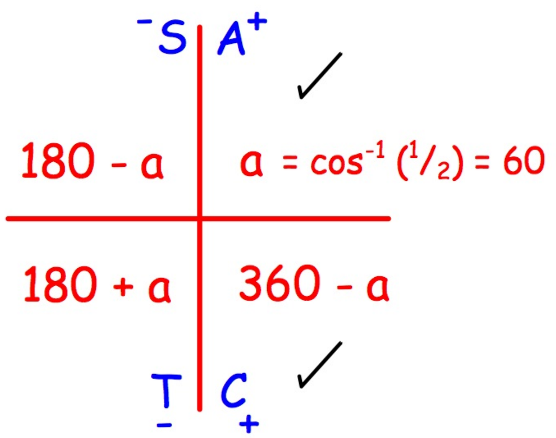
$x = 60$ or 300 $\qquad$ $x = 0$ or 360

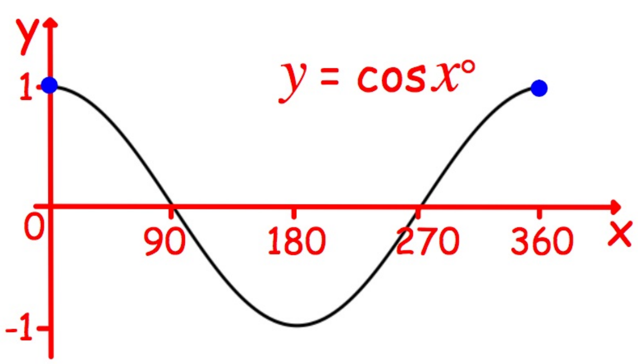
$$\underline{\underline{x = 0, 60, 300, 360}}$$

in radians: $\cos 2x - 3\cos x + 2 = 0$, $0 \le x \le 2\pi$

$$\underline{\underline{x = 0, {}^{\pi}/_3, {}^{5\pi}/_3, 2\pi}}$$

REMINDERS:





0° , 360°

$a = 60$

$360 - a = 300$

TRIG. FORMULAE: PROOFS

REPLACEMENTS may be made using the identities:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\sin 2A = 2\sin A \cos A$$

$$\begin{aligned} \cos 2A &= \sin^2 A - \cos^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}$$

OTHER FORMULAE:

$$\begin{aligned} \sin^2 A + \cos^2 A &= 1 \\ \cos^2 A &= 1 - \sin^2 A \\ \sin^2 A &= 1 - \cos^2 A \end{aligned}$$

$$\tan A = \frac{\sin A}{\cos A}$$

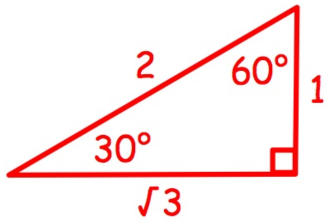
Sine Rule: $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

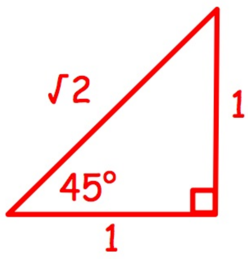
Cosine Rule

$$a^2 = b^2 + c^2 - 2bc \cos A \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Area Formula: $$\text{area } \Delta ABC = \tfrac{1}{2} ab \sin C$$

EXACT VALUES:

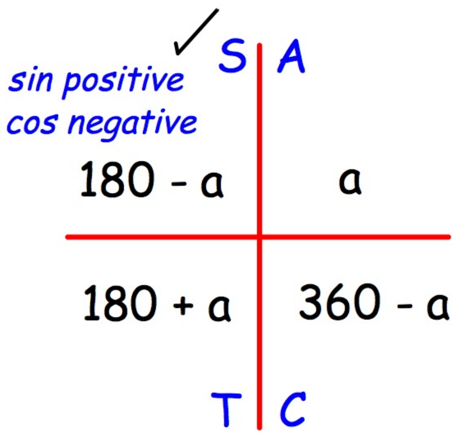




REMINDERS:

$\cos 120°$
$= \cos (180 - 60)°$
$= - \cos 60°$
$= -\frac{1}{2}$

$\sin 120°$
$= \sin (180 - 60)°$
$= + \sin 60°$
$= \frac{\sqrt{3}}{2}$



(1) Show that $\cos(x-30)° + \sin(x+120)° = \sqrt{3}\cos x°$

$\cos(x-30)° \quad + \quad \sin(x+120)°$

$= \cos x°\cos 30° + \sin x°\sin 30°$
$\quad + \sin x°\cos 120° + \cos x°\sin 120°$

$= \cos x° \times \frac{\sqrt{3}}{2} + \sin x° \times \frac{1}{2}$
$\quad + \sin x° \times (-\frac{1}{2}) + \cos x° \times \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}}{2}\cos x° + \frac{1}{2}\sin x° - \frac{1}{2}\sin x° + \frac{\sqrt{3}}{2}\cos x°$

$= \underline{\underline{\sqrt{3}\cos x°}}$

(2) Show $(1 - \sin^2 A)(1 - \tan^2 A) = \cos 2A$

$$(1 - \sin^2 A)(1 - \tan^2 A) = \cos^2 A \left(1 - \frac{\sin^2 A}{\cos^2 A}\right)$$

$$= \cos^2 A - \sin^2 A$$

$$= \underline{\underline{\cos 2A}}$$

(3) Use $\cos 2A$ identities to show that

$\sin^2 A = {}^1/_2(1 - \cos 2A)$ and $\cos^2 A = {}^1/_2(1 + \cos 2A)$

and hence show $\sin^2 A + 5\cos^2 A = 3 + 2\cos 2A$

$1 - 2\sin^2 A = \cos 2A$

$2\sin^2 A = 1 - \cos 2A$

$\sin^2 A = {}^1/_2(1 - \cos 2A)$

$2\cos^2 A - 1 = \cos 2A$

$2\cos^2 A = 1 + \cos 2A$

$\cos^2 A = {}^1/_2(1 + \cos 2A)$

$$\sin^2 A + 5\cos^2 A$$

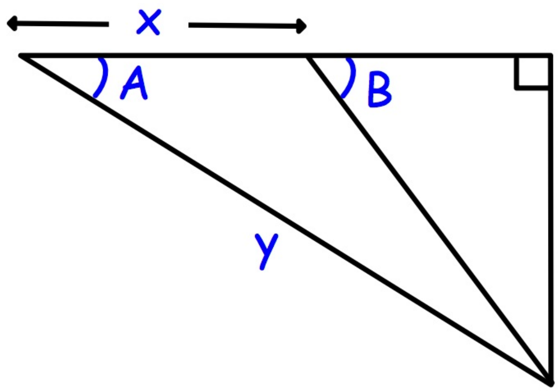
$$= {}^1/_2(1 - \cos 2A) + 5 \times {}^1/_2(1 + \cos 2A)$$

$$= {}^1/_2 - {}^1/_2 \cos 2A + {}^5/_2 + {}^5/_2 \cos 2A$$

$$= \underline{\underline{3 + 2\cos 2A}}$$

(4) Show that

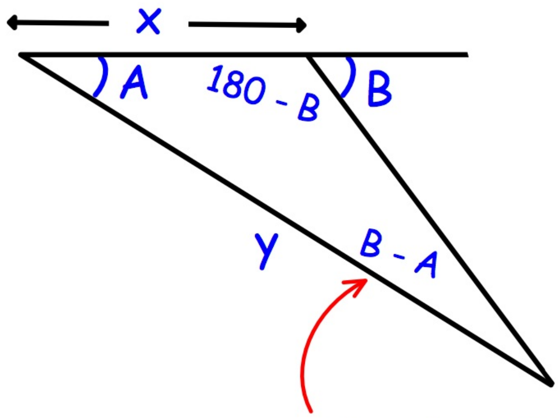
$$x = \frac{y\sin(B-A)}{\sin B}$$



SINE RULE:

$$\frac{x}{\sin(B-A)} = \frac{y}{\sin(180 - B)}$$

$$x = \frac{y\sin(B-A)}{\sin B}$$

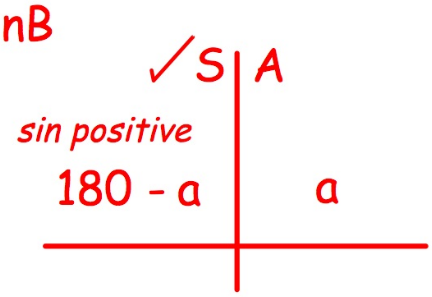


$180 - (A + 180 - B)$

$= 180 - A - 180 + B$

$= B - A$

$\sin(180 - B) = +\sin B$

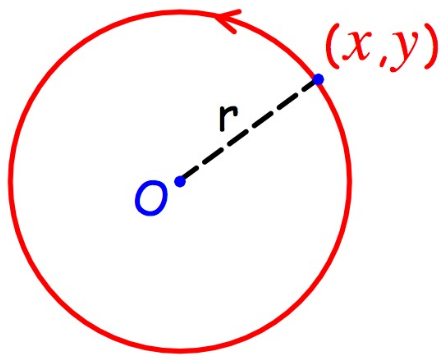


CIRCLES

LOCUS: all points r units from O.

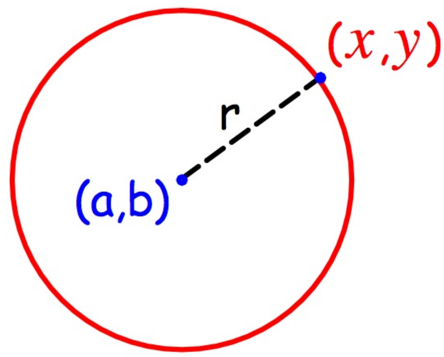
CIRCLE centre (0,0) , radius r.

$$x^2 + y^2 = r^2$$



CIRCLE centre (a,b) , radius r.

$$(x - a)^2 + (y - b)^2 = r^2$$



(1) Find the equation of the circle:

(a) centre (0,0) and passing through point (-2,1)

$$x^2 + y^2 = r^2$$

$$(-2)^2 + 1^2 = r^2$$

$$r^2 = 5$$

$$\underline{\underline{x^2 + y^2 = 5}}$$

(b) twice the radius and centre (2,-5)

$$r = \sqrt{5}$$

$$2r = 2\sqrt{5} = \sqrt{4} \times \sqrt{5} = \sqrt{20}$$

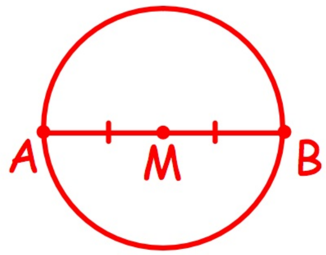
$$(x - a)^2 + (y - b)^2 = r^2$$

$\overset{a}{2}$, $\overset{b}{-5}$: (2 , -5)

$$(x - 2)^2 + (y + 5)^2 = (\sqrt{20})^2$$

$$\underline{\underline{(x - 2)^2 + (y + 5)^2 = 20}}$$

(2) Find the equation of the circle with diametrically opposite points A(-4,1) and B(2,3).



M_{AB} (-1,2)

$$(x - a)^2 + (y - b)^2 = r^2$$

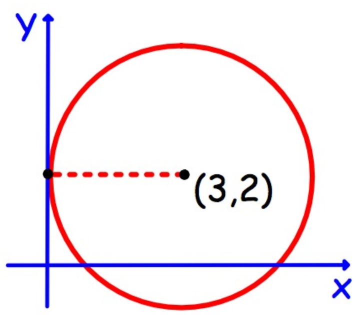
$\overset{a}{(-1}$, $\overset{b}{2)}$ $\qquad (x + 1)^2 + (y - 2)^2 = r^2$

B($\overset{x}{2}$, $\overset{y}{3}$) $\qquad (2 + 1)^2 + (3 - 2)^2 = r^2$

or use A(-4,1) $\qquad r^2 = 10$

$$\underline{\underline{(x + 1)^2 + (y - 2)^2 = 10}}$$

(3) Is the point (5,-1) inside, outside or on this circle?



($\overset{a}{3}$, $\overset{b}{2}$) $\qquad (x - a)^2 + (y - b)^2 = r^2$

r = 3 $\qquad (x - 3)^2 + (y - 2)^2 = 9$

($\overset{x}{5}$, $\overset{y}{-1}$) $\qquad (x - 3)^2 + (y - 2)^2$

$$= (5 - 3)^2 + (-1 - 2)^2$$

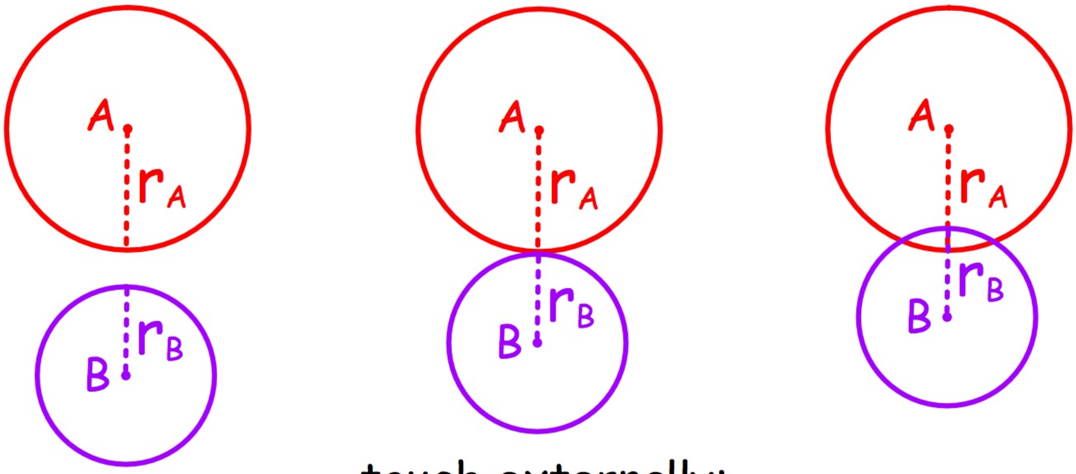
$$= 13$$

$$(x - 3)^2 + (y - 2)^2 > 9$$

$\Rightarrow$ <u>point outside circle</u>

INTERSECTING CIRCLES

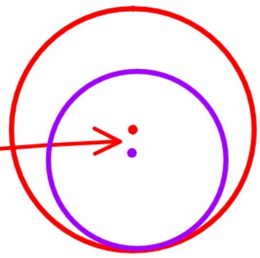
Difference between ($r_A + r_B$) and AB is the gap or overlap.



touch externally:

$r_A + r_B = AB$

touch internally: $r_A - r_B = AB$



Show that circles $x^2 + y^2 = 4$ and $(x - 3)^2 + (y - 4)^2 = 9$ touch externally.

$r_A = 2$ $\qquad$ $r_B = 3$

A (0,0) $\qquad$ B (3,4)

distance between centres:

$$AB^2 = (3 - 0)^2 + (4 - 0)^2$$
$$= 25$$
$$AB = 5$$

sum of radii:

$$r_A + r_B$$
$$= 2 + 3$$
$$= 5$$

$$AB = r_A + r_B$$

$\Rightarrow$ circles touch externally

GENERAL EQUATION OF A CIRCLE

$x^2 + y^2 + 2gx + 2fy + c = 0$

centre $(-g, -f)$ radius $\sqrt{g^2 + f^2 - c}$

so requires $g^2 + f^2 - c > 0$ ie. positive

(1) If $x^2 + y^2 - 2x + 8y + k = 0$ is a circle, find the possible values of k.

$x^2 + y^2 - 2x + 8y + k = 0$

$x^2 + y^2 + 2gx + 2fy + c = 0$

$g = -1$ $f = 4$ $c = k$

$g^2 + f^2 - c$

$= (-1)^2 + 4^2 - k$

$= 17 - k$

for a circle $g^2 + f^2 - c > 0$

$17 - k > 0$

$-k > -17$

$\underline{\underline{k < 17}}$

(2) Find the centre and radius of circle

$$x^2 + y^2 - 2x + 6y - 15 = 0$$

$$x^2 + y^2 - 2x + 6y - 15 = 0$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$g = -1 \qquad f = 3 \qquad c = -15$

$$g^2 + f^2 - c$$
$$= (-1)^2 + 3^2 - (-15)$$
$$= 25$$

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{25} = 5$$

centre $(\overset{-g}{1}, \overset{-f}{-3})$, radius 5 units

(3) Find the centre and radius of circle

$$x^2 + y^2 - 10y + 7 = 0$$

$$x^2 + y^2 + 0x - 10y + 7 = 0$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$g = 0 \qquad f = -5 \qquad c = 7$

$$g^2 + f^2 - c$$
$$= 0^2 + (-5)^2 - 7$$
$$= 18$$

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{18} = 3\sqrt{2}$$

centre $(\overset{-g}{0}, \overset{-f}{5})$, radius $3\sqrt{2}$ units

NOTE:

GIVEN equation $x^2 + y^2 + 2gx + 2fy + c = 0$
FIND centre and radius

GIVEN centre and radius
FIND equation using $(x - a)^2 + (y - b)^2 = r^2$

Find the equation of the circle CONCENTRIC with circle $x^2 + y^2 + 6x - 4y - 3 = 0$ but with twice the radius.

$$x^2 + y^2 + 6x - 4y - 3 = 0$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$g = 3 \qquad f = -2 \qquad c = -3$$

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{16} = 4$$

$$g^2 + f^2 - c$$

$$= 3^2 + (-2)^2 - (-3)$$

$$= 16$$

centre $(\overset{-g}{-3}, \overset{-f}{2})$, radius 4 units

centre $(\overset{a}{-3}, \overset{b}{2})$, radius 8 units

$$(x - a)^2 + (y - b)^2 = r^2$$

$$(x + 3)^2 + (y - 2)^2 = 64$$

LENGTH OF A TANGENT

(i) radius and centre from equation of circle

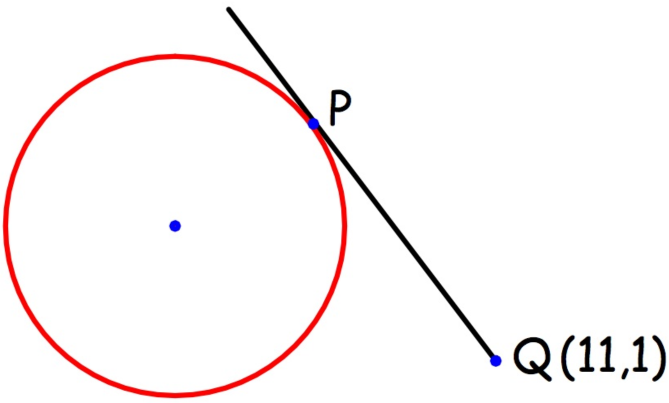
(ii) distance from centre by distance formula

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

(iii) length of tangent by Pyth. Thm.

Find the distance PQ.

$$x^2 + (y - 3)^2 = 25$$



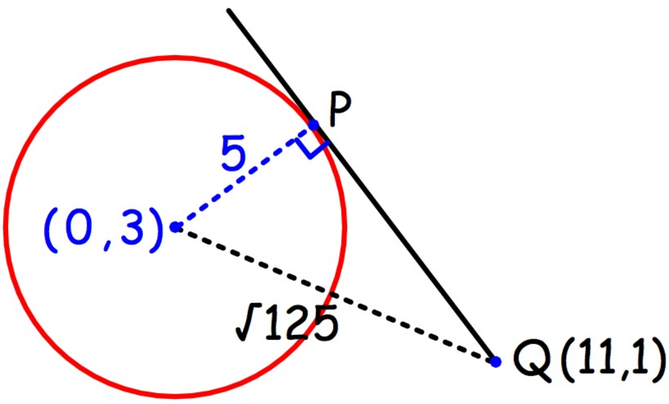
$$x^2 + (y - 3)^2 = 25$$

$$(x - a)^2 + (y - b)^2 = r^2$$

a b

centre (0,3)

radius 5



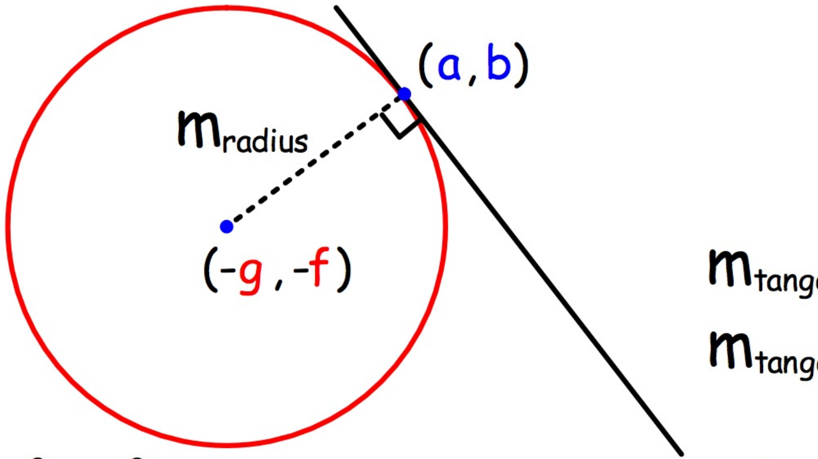
Distance formula

$$\begin{aligned} d^2 &= (11 - 0)^2 + (1 - 3)^2 \\ &= 11^2 + (-2)^2 \\ &= 125 \end{aligned}$$

Pyth. Thm.

$$\begin{aligned} PQ^2 &= d^2 - r^2 \\ &= 125 - 5^2 \\ &= 100 \\ \underline{PQ} &= \underline{10 \text{ units}} \end{aligned}$$

EQUATION OF A TANGENT



$$m_{tangent} \times m_{radius} = -1$$

$$m_{tangent}$$

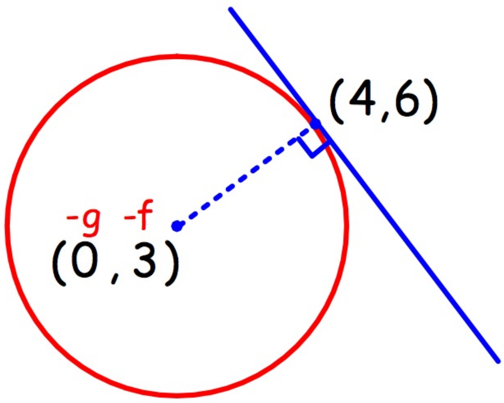
$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad y - b = m(x - a)$$

Find the equation of the tangent which meets the circle $x^2 + y^2 - 6y - 16 = 0$ at the point (4,6).

$$x^2 + y^2 + 0x - 6y - 16 = 0$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$g = 0 \qquad f = -3$$



$$m_{radius} = \frac{6 - 3}{4 - 0} = \frac{3}{4}$$

ppn gradient: $m_1 \times m_2 = -1$

$$m_{tangent} = -\frac{4}{3}$$

$$y - b = m(x - a)$$

P(4,6) $\quad y - 6 = \frac{-4}{3}(x - 4)$

$$3y - 18 = -4x + 16$$

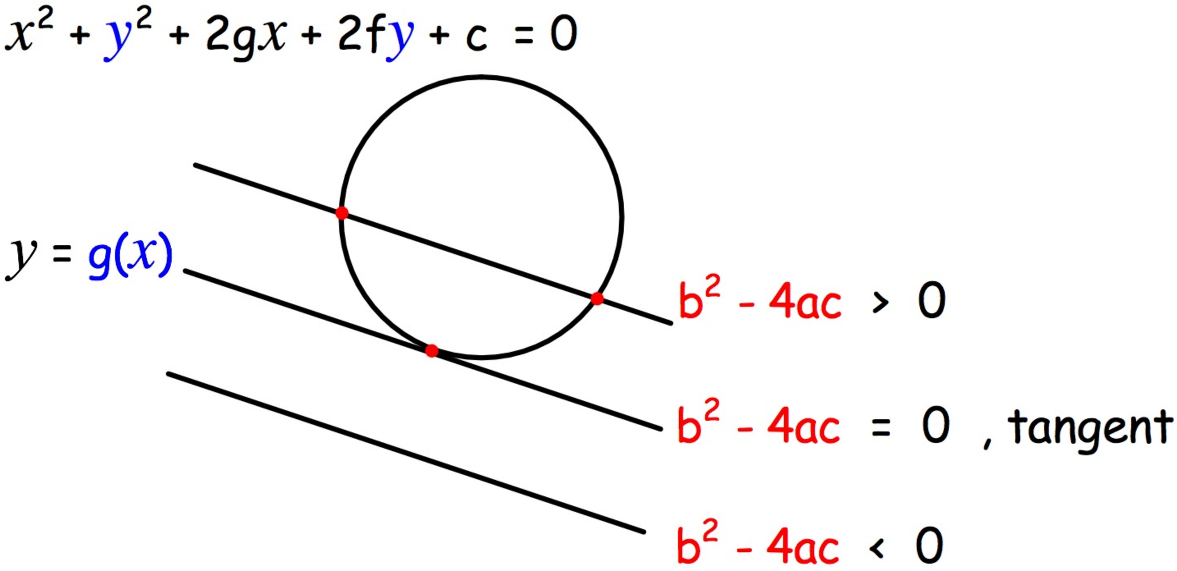
$$\underline{\underline{4x + 3y = 34}}$$

INTERSECTION OF A LINE AND CIRCLE

Substitute $y = g(x)$ into $x^2 + y^2 + 2gx + 2fy + c = 0$

results in $ax^2 + bx + c = 0$

Discriminant $b^2 - 4ac$ distinguishes between:



TANGENCY

A tangent is a line that touches the circle at one point.

The substitution results in a quadratic equation with one solution:

$b^2 - 4ac = 0 \Rightarrow$ EQUAL ROOTS
so line is a tangent

Show that the line $x + y = 4$ is a tangent to the circle $x^2 + y^2 + 6x + 2y - 22 = 0$ and find the point of contact.

$$x + y = 4$$

substitution: $$y = 4 - x$$

$$x^2 + y^2 + 6x + 2y - 22 = 0$$
$$x^2 + (4 - x)^2 + 6x + 2(4 - x) - 22 = 0$$
$$x^2 + 16 - 8x + x^2 + 6x + 8 - 2x - 22 = 0$$
$$2x^2 - 4x + 2 = 0$$

simplify quadratic equation: $$x^2 - 2x + 1 = 0$$

discriminant: $$1x^2 - 2x + 1 = 0$$
$$a = 1 \ , \ b = -2 \ , \ c = 1$$

$$b^2 - 4ac = (-2)^2 - 4 \times 1 \times 1 = 0$$

* $b^2 - 4ac = 0 \Rightarrow$ line is a tangent

$$x^2 - 2x + 1 = 0 \qquad y = 4 - x$$
$$(x - 1)^2 = 0 \qquad = 4 - 1$$
$$x = 1 \qquad = 3$$

point of contact (1,3)

* OR

one point of contact $\Rightarrow$ line is a tangent