HIGHER MATHEMATICS
COURSE NOTES

UNIT 3

FORMULAE LIST

Circle:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

The equation $(x - a)^2 + (y - b)^2 = r^2$ represents a circle centre (a, b) and radius r.

Scalar Product: $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$, where θ is the angle between **a** and **b**

or $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Trigonometric formulae:

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \sin 2A &= 2\sin A \cos A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}$$

Table of standard derivatives:

$f(x)$	$f'(x)$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$

Table of standard integrals:

$f(x)$	$\int f(x)\,dx$
$\sin ax$	$-\frac{1}{a}\cos ax + C$
$\cos ax$	$\frac{1}{a}\sin ax + C$

www.duggie.weebly.com

VECTORS

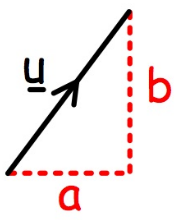
SCALAR quantities have SIZE (magnitude).

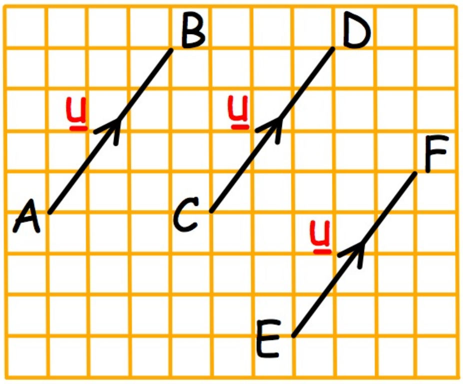
VECTOR quantities have SIZE and DIRECTION.

DIRECTED LINE SEGMENT

A line of a particular size and direction is used to represent a vector.

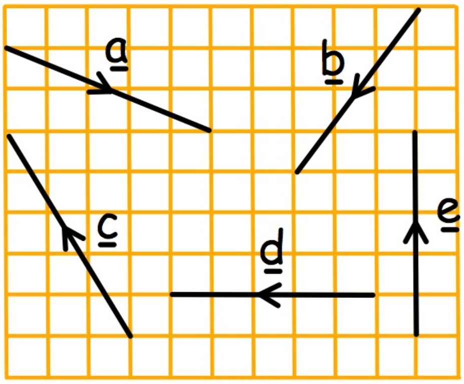
COMPONENT FORM $\underline{u} = \begin{pmatrix} a \\ b \end{pmatrix}$





$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

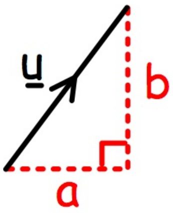
Three directed line segments,
same size and direction,
same component form,
same vector **u**. $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$



$\underline{a} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$

$\underline{c} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ $\underline{d} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$ $\underline{e} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$

MAGNITUDE Follows from Pyth. Thm.



$\underline{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ $\qquad |\underline{u}| = \sqrt{a^2 + b^2}$

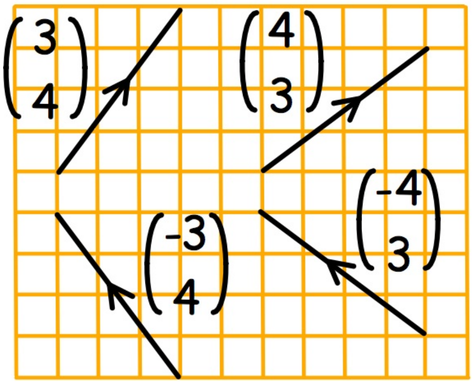
$\overrightarrow{AB} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

$$|\overrightarrow{AB}| = \sqrt{(-3)^2 + 6^2}$$

$$= \sqrt{45}$$

$$= \underline{\underline{3\sqrt{5} \text{ units}}}$$

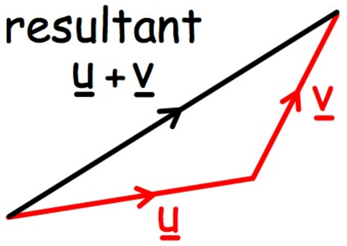
NOTE: different vectors can have the same magnitude.



all different vectors
same magnitude 5 units.

ADD and SUBTRACT

By "head-to-tail" triangle.

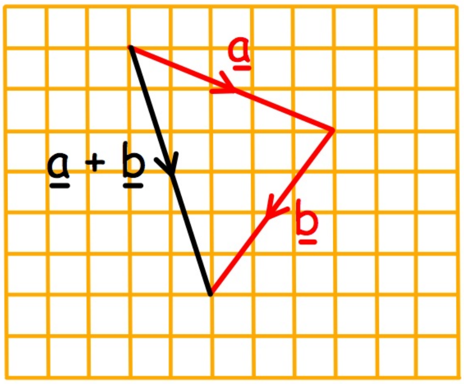


NOTE: $|\underline{u}| + |\underline{v}| > |\underline{u} + \underline{v}|$

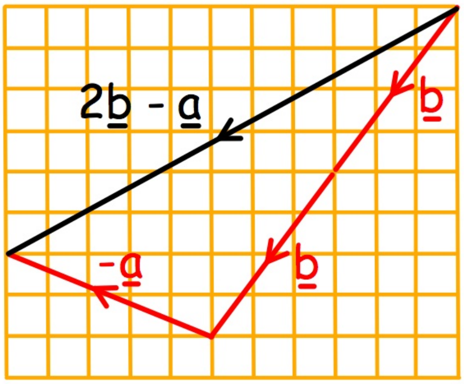
By components: add or subtract components.

MULTIPLY BY A SCALAR: multiply components.

$$k \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}$$



$$\underline{a} + \underline{b} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

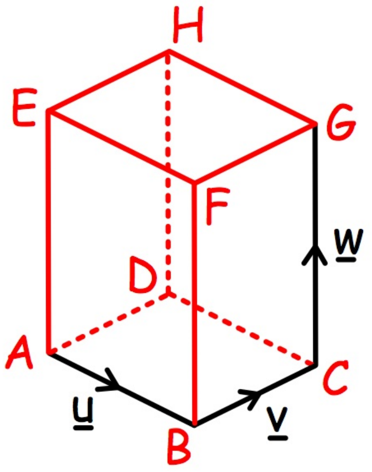


$$2\underline{b} - \underline{a} = 2\begin{pmatrix} -3 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

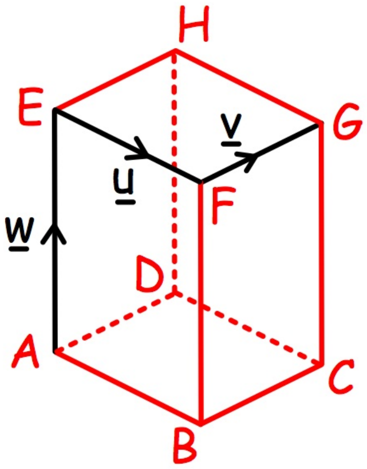
$$= \begin{pmatrix} -6 \\ -8 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} -11 \\ -6 \end{pmatrix}$$

3D VECTORS

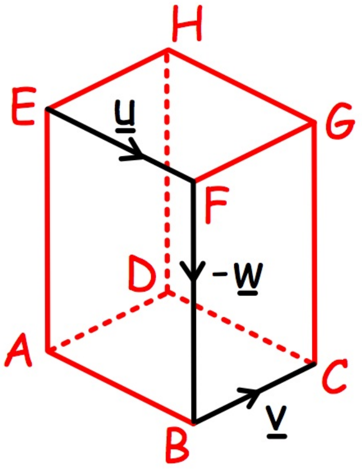


$$\overrightarrow{AG} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CG}$$
$$= \underline{u} + \underline{v} + \underline{w}$$



same result regardless of 'route'

$$\overrightarrow{AG} = \overrightarrow{AE} + \overrightarrow{EF} + \overrightarrow{FG}$$
$$= \underline{w} + \underline{u} + \underline{v}$$



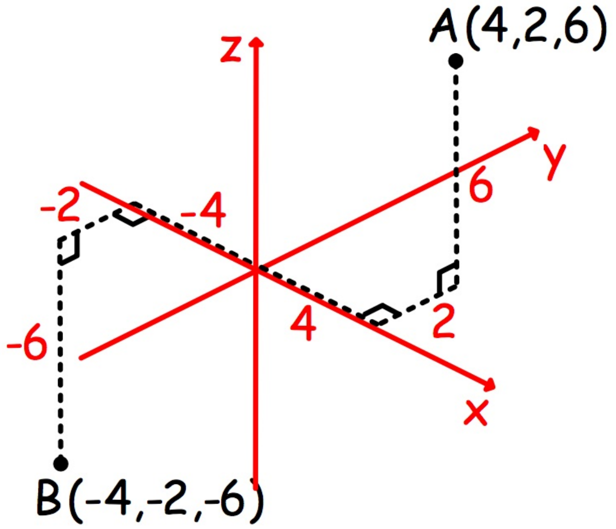
subtraction is adding the negative

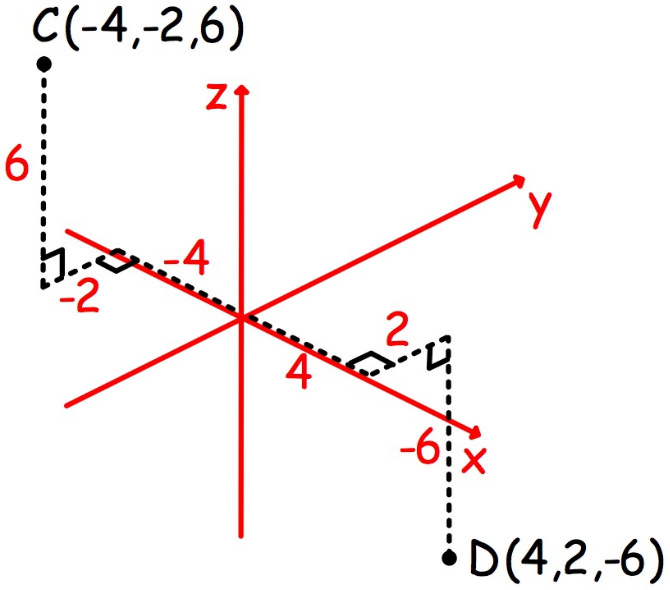
$$\overrightarrow{EC} = \overrightarrow{EF} + \overrightarrow{FB} + \overrightarrow{BC}$$
$$= \overrightarrow{EF} - \overrightarrow{BF} + \overrightarrow{BC}$$
$$= \underline{u} - \underline{w} + \underline{v}$$

NEGATIVE: direction reversed $\overrightarrow{FB} = -\overrightarrow{BF}$

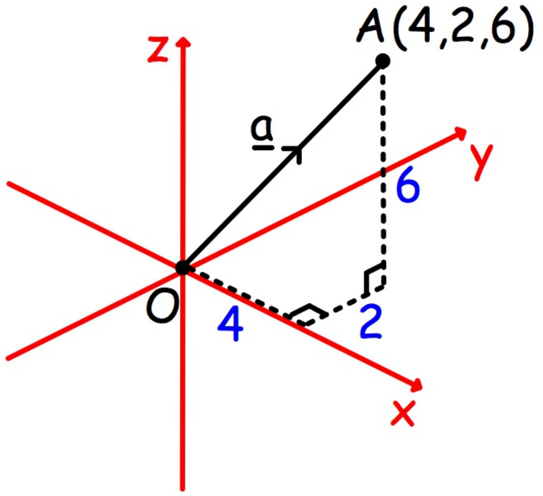
3D COORDINATES

Points (x,y,z) plotted on 3 mutually perpendicular axes.



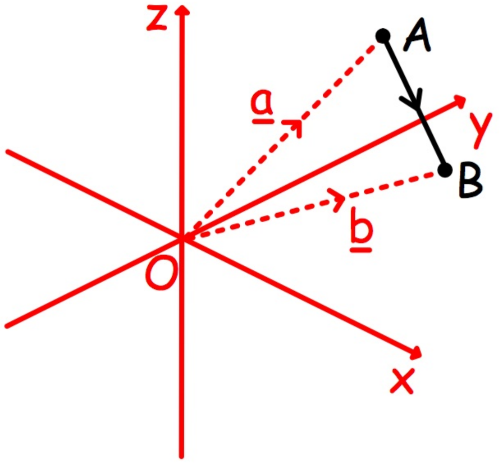


The POSITION VECTOR of point A is given by $\overrightarrow{OA}$.



$$\underline{a} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$

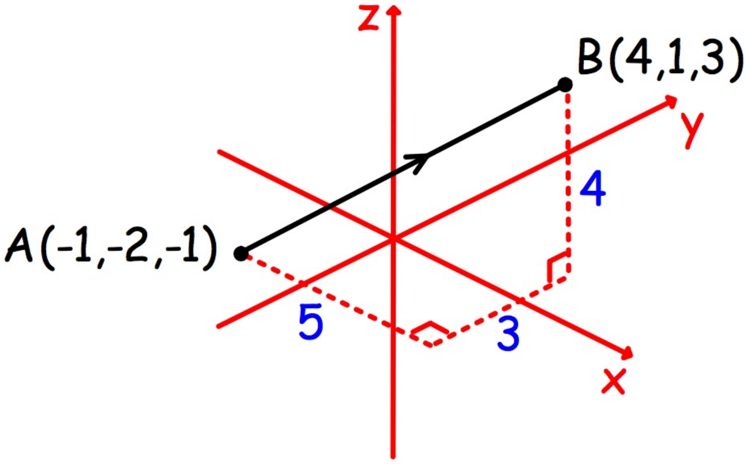
POSITION VECTORS



$$\overrightarrow{AB} = \underline{b} - \underline{a}$$

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$



$$\overrightarrow{AB} = \underline{b} - \underline{a}$$

$$= \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$$

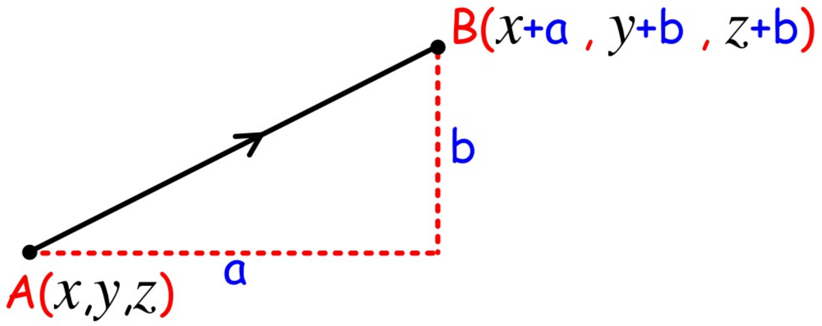
$$\overrightarrow{AB} = \underline{\underline{\begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}}}$$

OR

$$A(\overset{+5}{-1}, \overset{+3}{-2}, \overset{+4}{-1}) \longrightarrow B(4, 1, 3)$$

TRANSLATION $\vec{AB}$ represents a movement from A to B.

$$\vec{AB} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

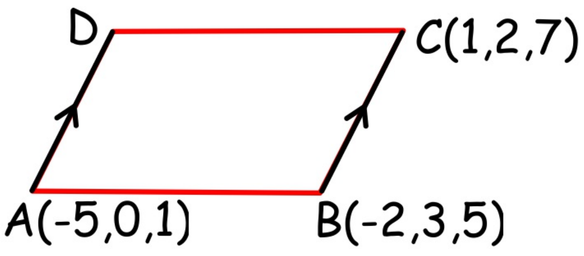


(1) If $\vec{AB} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$ and A(-1,-2,-1), find the coordinates of B.

$$A(-1,-2,-1) \longrightarrow B(\,-1+5\,,\,-2+3\,,\,-1+4\,)$$

$$\underline{\underline{B(4,1,3)}}$$

(2) For parallelogram ABCD, find the coordinates of D.



$$B(\overset{+3}{-2}\,,\,\overset{-1}{3}\,,\,\overset{+2}{5}) \longrightarrow C(\,1\,,\,2\,,\,7\,) \qquad \vec{BC} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

parallelogram: $\vec{AD} = \vec{BC} \Rightarrow \vec{AD} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$$A(\overset{+3}{-5}\,,\,\overset{-1}{0}\,,\,\overset{+2}{1}) \longrightarrow D$$

$$\underline{\underline{D(-2,-1,3)}}$$

MAGNITUDE   Follows from Pyth. Thm.

$$\underline{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad |\underline{u}| = \sqrt{a^2 + b^2 + c^2}$$

Find the distance from A(-2,3,5) to B(1,2,7).

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \qquad |\overrightarrow{AB}| = \sqrt{3^2 + (-1)^2 + 2^2}$$

$$= \underline{\underline{\sqrt{14} \text{ units}}}$$

NOTE: can use this instead of the Distance Formula.

UNIT VECTOR: has a magnitude of 1.

If $\underline{u} = \begin{pmatrix} {}^1/_2 \\ a \\ -{}^1/_2 \end{pmatrix}$ is a unit vector, find the value of a.

$$|\underline{u}| = 1 \qquad |\underline{u}|^2 = ({}^1/_2)^2 + a^2 + (-{}^1/_2)^2$$

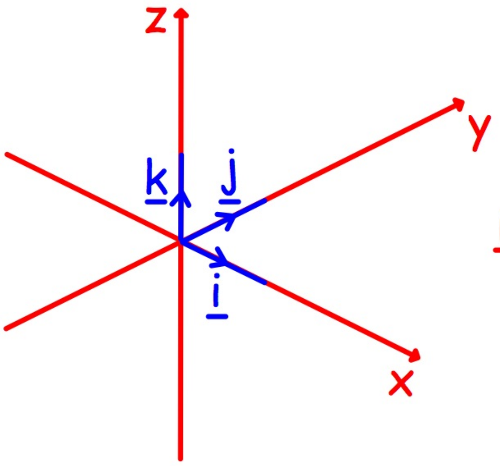
$$1 = {}^1/_4 + a^2 + {}^1/_4$$

$$a^2 = {}^1/_2$$

$$\underline{\underline{a = \pm {}^1/_{\sqrt{2}}}}$$

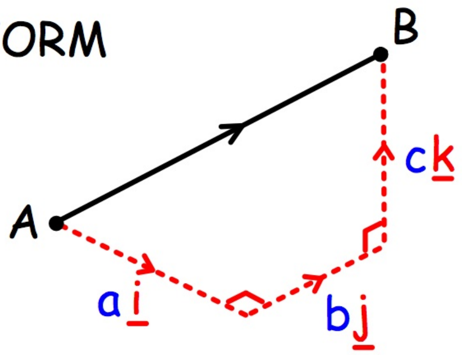
BASIS VECTORS

Three unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$ in the OX, OY and OZ directions are used as the basis of 3 dimensional space.



$$\underline{i} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \underline{j} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \underline{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\underline{i}$, $\underline{j}$, $\underline{k}$ FORM



$$\overrightarrow{AB} = \begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$\overrightarrow{AB} = a\underline{i} + b\underline{j} + c\underline{k}$$

$$\begin{pmatrix}2\\-3\\5\end{pmatrix} = 2\underline{i} - 3\underline{j} + 5\underline{k} \qquad \begin{pmatrix}1\\-3\\-2\end{pmatrix} = \underline{i} - 3\underline{j} - 2\underline{k}$$

$$\begin{pmatrix}4\\0\\-1\end{pmatrix} = 4\underline{i} - \underline{k} \qquad \begin{pmatrix}5\\3\\0\end{pmatrix} = 5\underline{i} + 3\underline{j}$$

ADD and SUBTRACT: add or subtract components.

MULTIPLY BY A SCALAR: multiply components.

$$k\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$$

If $\underline{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, find $|\underline{b} - 2\underline{a}|$.

$$\underline{b} - 2\underline{a} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ -4 \end{pmatrix}$$

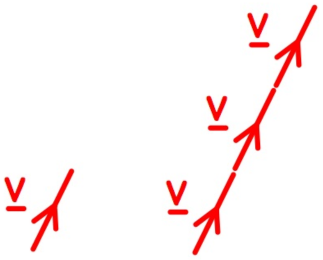
$$|\underline{b} - 2\underline{a}| = \sqrt{(-3)^2 + 4^2 + (-4)^2} = \underline{\underline{\sqrt{41} \text{ units}}}$$

PARALLEL:

$$k\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$$

$\underline{u} = k\underline{v}$

$\Rightarrow$ $\underline{u}$ and $\underline{v}$ are parallel



$\overrightarrow{AB} = \begin{pmatrix} 9 \\ -3 \\ 6 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}$. Show AB is parallel to CD.

OR

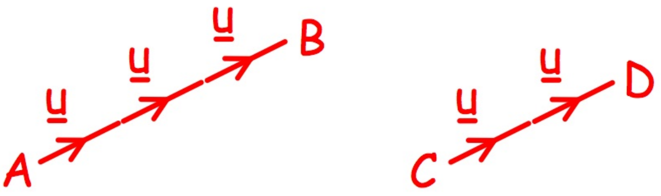
$\overrightarrow{AB} = {}^{3}/_{2}\,\overrightarrow{CD}$

$\overrightarrow{CD} = {}^{2}/_{3}\,\overrightarrow{AB}$

$\overrightarrow{AB} = 3\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ $\quad$ $\overrightarrow{CD} = 2\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

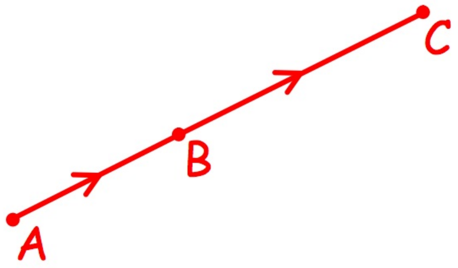
$2\overrightarrow{AB} = 3\overrightarrow{CD}$

$\Rightarrow$ $\underline{\underline{\text{AB and CD are parallel}}}$



COLLINEAR POINTS: points lie on the same line.

$\overrightarrow{AB} = k\,\overrightarrow{BC}$



$\Rightarrow$ lines AB and BC are parallel
and share common point B

$\Rightarrow$ points A, B and C are collinear

NOTE: lines parallel, points collinear

Show points A(-8,3,-7) , B(1,0,-1) and C(7,-2,3)
are collinear and find the ratio AB:BC.

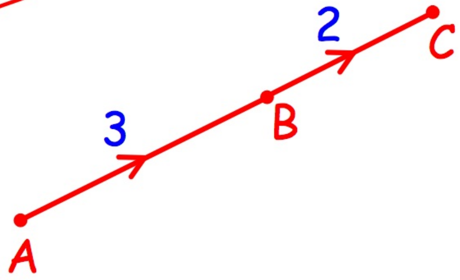
$$\overrightarrow{AB} = \begin{pmatrix} 9 \\ -3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \qquad \overrightarrow{BC} = \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

OR

$\overrightarrow{AB} = {}^{3}/_{2}\,\overrightarrow{BC}$

$\overrightarrow{BC} = {}^{2}/_{3}\,\overrightarrow{AB}$

$2\overrightarrow{AB} = 3\,\overrightarrow{BC}$



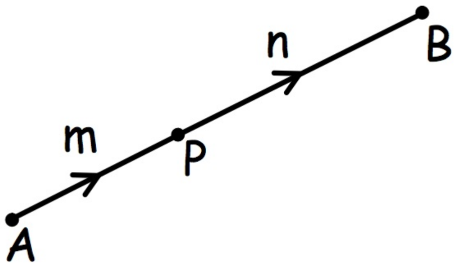
$\Rightarrow$ lines AB and BC are parallel
and share common point B

$\Rightarrow$ points A, B and C are collinear

AB:BC = 3:2

DIVIDING A LINE

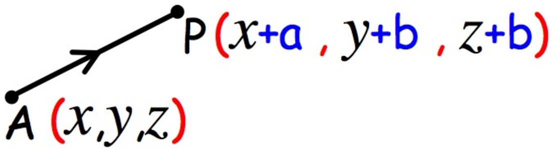
$\overrightarrow{AP}$ is a fraction of $\overrightarrow{AB}$:



(i) find $\overrightarrow{AB}$

(ii) find $\overrightarrow{AP}$ $\quad \overrightarrow{AP} = \frac{m}{m+n}\overrightarrow{AB}$

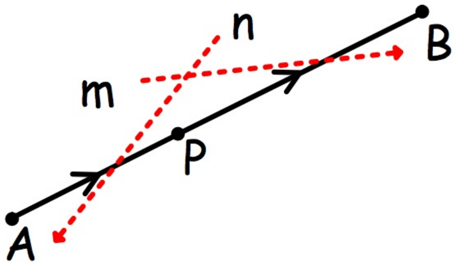
(iii) find P $\quad \overrightarrow{AP} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$



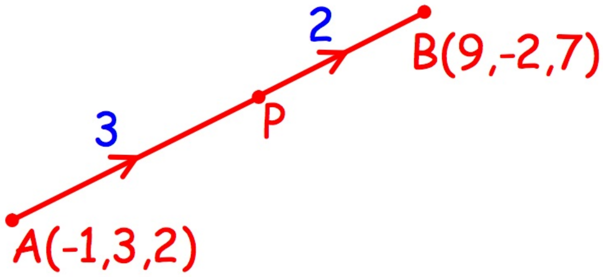
SECTION FORMULA

uses position vectors:

$$\underline{p} = \frac{m\underline{b} + n\underline{a}}{m+n}$$



(1) A(-1,3,2), B(9,-2,7). Find the coordinates of point P, which divides AB internally in the ratio 3:2 .



$$\overrightarrow{AB} = \begin{pmatrix} 10 \\ -5 \\ 5 \end{pmatrix}$$

$$\overrightarrow{AP} = {}^{3}/_{5}\ \overrightarrow{AB} = \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix}$$

$$A(\overset{+6}{-1}\ ,\ \overset{-3}{3}\ ,\ \overset{+3}{2}) \longrightarrow \underline{\underline{P(5, 0, 5)}}$$

OR

$$\underline{p} = \frac{3\underline{b} + 2\underline{a}}{3+2}$$

$$= \frac{1}{5}\left[3\begin{pmatrix} 9 \\ -2 \\ 7 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \right]$$

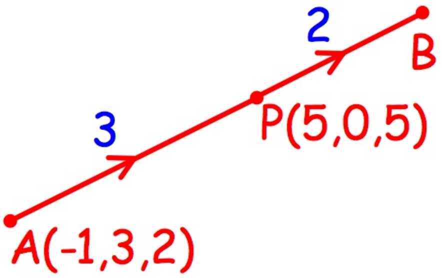
$$= \frac{1}{5}\begin{pmatrix} 25 \\ 0 \\ 25 \end{pmatrix}$$

$$\underline{p} = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix}$$

$$\underline{\underline{P(5, 0, 5)}}$$

(2) A(-1,3,2), P(5,0,5). Line AP is produced $^2/_3$ of its length to point B. Find the coordinates of B.

$$\overrightarrow{AP} = \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix}$$



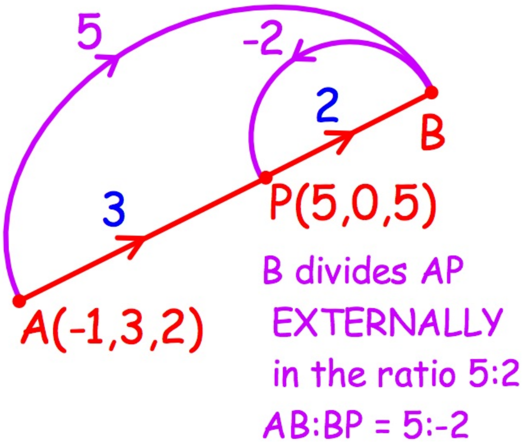
$$\overrightarrow{PB} = {}^2/_3 \ \overrightarrow{AP} \quad = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$$

$$P(\overset{+4}{5}\ ,\ \overset{-2}{0}\ ,\ \overset{+2}{5}) \longrightarrow \underline{\underline{B(9\ ,-2\ ,7\ )}}$$

OR

$$\underline{b} = \frac{5\underline{p} - 2\underline{a}}{5 + (-2)}$$

$$= \frac{1}{3}\left[5\begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \right]$$



B divides AP EXTERNALLY in the ratio 5:2
AB:BP = 5:-2

$$= \frac{1}{3}\begin{pmatrix} 27 \\ -6 \\ 21 \end{pmatrix}$$

$$\underline{b} = \begin{pmatrix} 9 \\ -2 \\ 7 \end{pmatrix}$$

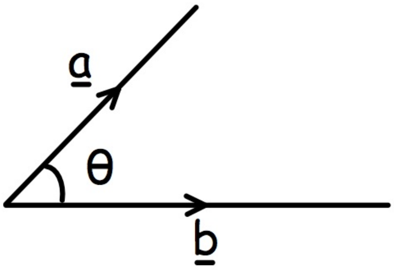
$$\underline{\underline{B(9\ ,-2\ ,7\ )}}$$

SCALAR PRODUCT (DOT PRODUCT)

Multiply two vectors for a scalar result.

$\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$ $\qquad \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\underline{a}.\underline{b} = |\underline{a}|\,|\underline{b}|\cos\theta\ ,\ \underline{a} \neq \underline{0}\ ,\ \underline{b} \neq \underline{0}$



NOTE: (i) vectors "pull away" from each other.

(ii) $\underline{a}.\underline{b} = \underline{b}.\underline{a}$

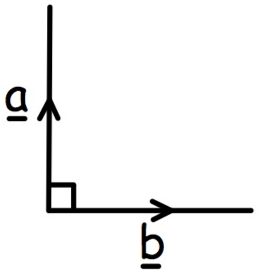
$\underline{a}.\underline{b}$ can be positive, zero or negative depending on θ.

θ acute ⇒ $\underline{a}.\underline{b}$ positive

θ obtuse ⇒ $\underline{a}.\underline{b}$ negative

θ = 90° $\qquad \underline{a}.\underline{b} = 0$

⇒ $\underline{a}$ is perpendicular to $\underline{b}$



(1) $\underline{p} = \begin{pmatrix}-2\\3\\1\end{pmatrix}$ $\underline{q} = \begin{pmatrix}4\\0\\3\end{pmatrix}$

$$\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$$

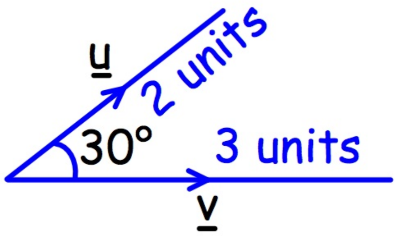
$$\begin{aligned}\underline{p}.\underline{q} &= -2\times4 + 3\times0 + 1\times3\\ &= -8 + 0 + 3\\ \underline{\underline{\underline{p}.\underline{q}}} &\underline{\underline{= -5}}\end{aligned}$$

(2) $\underline{r} = \begin{pmatrix}-5\\-1\\2\end{pmatrix}$ $\underline{s} = \begin{pmatrix}2\\-2\\4\end{pmatrix}$

$$\begin{aligned}\underline{r}.\underline{s} &= -5\times2 + (-1)\times(-2) + 2\times4\\ &= -10 + 2 + 8\\ \underline{\underline{\underline{r}.\underline{s}}} &\underline{\underline{= 0}}\end{aligned}$$

NOTE: ⇒ $\underline{r}$ is perpendicular to $\underline{s}$

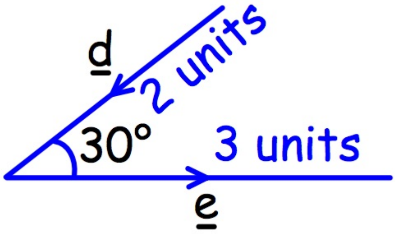
(3)



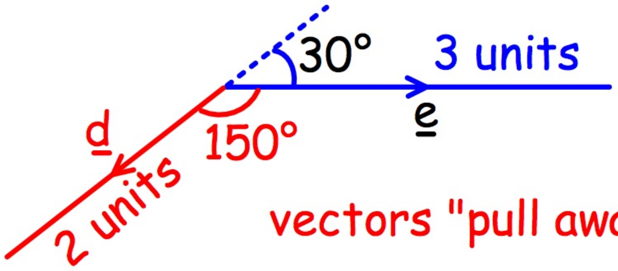
$$\underline{a}.\underline{b} = |\underline{a}|\,|\underline{b}|\cos\theta$$

$$\begin{aligned}\underline{u}.\underline{v} &= 2\times3\times\cos30°\\ &= 2\times3\times{}^{\sqrt{3}}/_{2}\\ \underline{\underline{\underline{u}.\underline{v}}} &\underline{\underline{= 3\sqrt{3}}}\end{aligned}$$

(4)



$$\begin{aligned}\underline{d}.\underline{e} &= 2\times3\times\cos150°\\ &= 2\times3\times(-{}^{\sqrt{3}}/_{2})\\ \underline{\underline{\underline{d}.\underline{e}}} &\underline{\underline{= -3\sqrt{3}}}\end{aligned}$$



vectors "pull away"

(5) $\underline{u} = -3\underline{i} + 3\underline{j} + 3\underline{k}$ $\quad$ $\underline{v} = \underline{i} + 5\underline{j} - \underline{k}$

Show that $\underline{u}+\underline{v}$ is perpendicular to $\underline{u}-\underline{v}$.

$$\underline{u} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} \qquad \underline{u}+\underline{v} = \begin{pmatrix} -2 \\ 8 \\ 2 \end{pmatrix} \qquad \underline{u}-\underline{v} = \begin{pmatrix} -4 \\ -2 \\ 4 \end{pmatrix}$$

$$\underline{v} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$$

$$\begin{aligned} (\underline{u}+\underline{v}).(\underline{u}-\underline{v}) &= -2\times(-4) + 8\times(-2) + 2\times4 \\ &= 8 + (-16) + 8 \\ &= 0 \end{aligned}$$

$$(\underline{u}+\underline{v}).(\underline{u}-\underline{v}) = 0$$

$$\Rightarrow \underline{\underline{\underline{u}+\underline{v} \text{ is perpendicular to } \underline{u}-\underline{v}}}$$

(6) $\underline{m} = \begin{pmatrix} -1 \\ k \\ -2 \end{pmatrix}$ $\quad$ $\underline{n} = \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix}$

Find k if $\underline{m}$ and $\underline{n}$ are perpendicular.

$$\begin{aligned} \underline{m}.\underline{n} &= -1\times(-4) + k\times2 + -2\times5 \\ &= 4 + 2k + (-10) \\ &= 2k - 6 \end{aligned}$$

$\underline{m}$ is perpendicular to $\underline{n}$

$$\Rightarrow \underline{m}.\underline{n} = 0$$

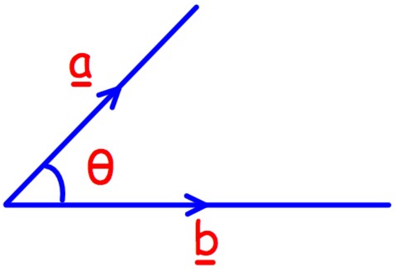
$$2k - 6 = 0$$

$$\underline{\underline{k = 3}}$$

ANGLE BETWEEN VECTORS

Combining formulae:

$$\left.\begin{aligned}\underline{a}.\underline{b} &= |\underline{a}||\underline{b}|\cos\theta \\ \underline{a}.\underline{b} &= a_1b_1 + a_2b_2 + a_3b_3\end{aligned}\right\} \quad \cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\underline{a}||\underline{b}|}$$



NOTE: vectors "pull away"

(1) $\underline{a} = 2\underline{i} + 2\underline{j} + \underline{k}$ $\qquad$ $\underline{b} = 2\underline{i} + 2\underline{k}$

Find the angle between the vectors.

$$\underline{a} = \begin{pmatrix}2\\2\\1\end{pmatrix} \qquad |\underline{a}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9}$$

$$|\underline{b}| = \sqrt{2^2 + 0^2 + 2^2} = \sqrt{8}$$

$$\underline{b} = \begin{pmatrix}2\\0\\2\end{pmatrix} \qquad \underline{a}.\underline{b} = 2\times2 + 2\times0 + 1\times2 = 6$$

$$\cos\theta = \frac{6}{(\sqrt{9}\times\sqrt{8})}$$

$$\frac{6}{\sqrt{72}} = \frac{6}{6\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{1}{\sqrt{2}}$$

$$\underline{\underline{\theta = 45°}}$$

(2) A(-4,2,5) , B(-3,0,4) , C(-2,0,1). Find $\angle ABC$.

vectors "pull away" from angle at B

$$\overrightarrow{BA} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \qquad |\overrightarrow{BA}| = \sqrt{(-1)^2 + 2^2 + 1^2} = \sqrt{6}$$

$$\overrightarrow{BC} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \qquad |\overrightarrow{BC}| = \sqrt{1^2 + 0^2 + (-3)^2} = \sqrt{10}$$

$\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\overrightarrow{BA} . \overrightarrow{BC} = -1 \times 1 + 2 \times 0 + 1 \times (-3)$$

$$= -1 + 0 + (-3)$$

$$\overrightarrow{BA} . \overrightarrow{BC} = -4$$

$\underline{a}.\underline{b} = |\underline{a}| |\underline{b}| \cos\theta$

$$\overrightarrow{BA} . \overrightarrow{BC} = |\overrightarrow{BA}| |\overrightarrow{BC}| \cos B$$

$$-4 = \sqrt{6} \times \sqrt{10} \times \cos B$$

$\cos\theta = \dfrac{\underline{a}.\underline{b}}{|\underline{a}| |\underline{b}|}$

$$\cos B = \frac{-4}{(\sqrt{6} \times \sqrt{10})}$$

$$\cos B = -0.51369...$$

$$B = 121.0909...$$

$$\underline{\underline{\angle ABC \approx 121.1^\circ}}$$

PROPERTIES OF THE SCALAR PRODUCT

$\underline{a}.\underline{a} = |\underline{a}|^2$ since $|\underline{a}||\underline{a}|\cos 0^\circ = |\underline{a}||\underline{a}| \times 1$

$\underline{a}.\underline{b} = 0 \Rightarrow \underline{a}$ is perpendicular to $\underline{b}$

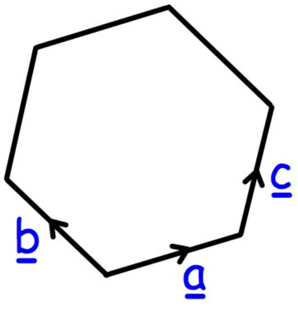
$\underline{a}.\underline{b} = \underline{b}.\underline{a}$

$\underline{a}.(\underline{b} + \underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$

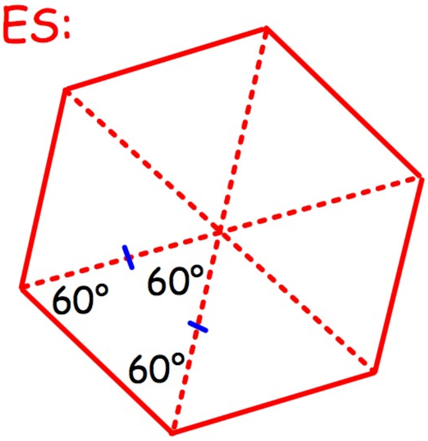
Regular hexagon side 2 units.

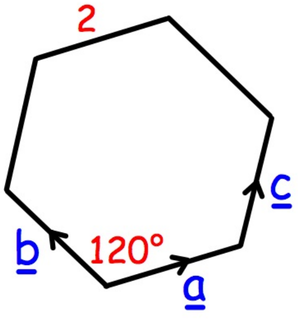
(a) Find $\underline{a}.(\underline{b} + \underline{c})$
and comment on the result.

(b) Find $\underline{b}.(\underline{a} + \underline{b} + \underline{c})$.

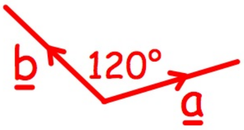


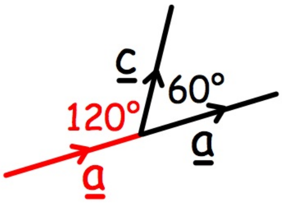
ANGLES:





(a)





$$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos 120°$$
$$= 2 \times 2 \times (-{}^{1}/_{2})$$
$$\underline{a}.\underline{b} = -2$$

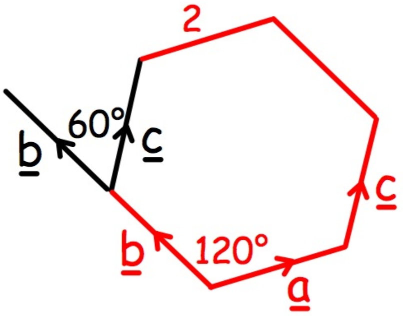
$$\underline{a}.\underline{c} = |\underline{a}||\underline{c}|\cos 60°$$
$$= 2 \times 2 \times {}^{1}/_{2}$$
$$\underline{a}.\underline{c} = 2$$

$$\underline{a}.(\underline{b} + \underline{c})$$
$$= \underline{a}.\underline{b} + \underline{a}.\underline{c}$$
$$= -2 + 2$$
$$= 0$$

$$\underline{a}.(\underline{b} + \underline{c}) = 0$$
$$\Rightarrow \underline{\underline{\underline{a} \text{ is perpendicular to } \underline{b}+\underline{c}}}$$

(b) $\underline{b}.\underline{c} = |\underline{b}||\underline{c}|\cos 60°$

$$= 2 \times 2 \times {}^{1}/_{2}$$
$$\underline{b}.\underline{c} = 2$$



$$\underline{b}.\underline{a} = \underline{a}.\underline{b} = -2$$

$\underline{b}.\underline{b} = |\underline{b}|^2 = 2^2 = 4$ since $|\underline{b}||\underline{b}|\cos 0° = |\underline{b}||\underline{b}| \times 1$

$$\underline{b}.(\underline{a} + \underline{b} + \underline{c})$$
$$= \underline{b}.\underline{a} + \underline{b}.\underline{b} + \underline{b}.\underline{c}$$
$$= -2 + 4 + 2$$
$$= \underline{\underline{4}}$$

FURTHER CALCULUS

DIFFERENTIATE TRIGONOMETRIC FUNCTIONS

In RADIANS only:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Displacement, d metres, of a particle at time t sec. is given by $d = t^2 + \cos t$.

Find the velocity of the particle after 2 seconds.

$$d(t) = t^2 + \cos t$$

$$d'(t) = 2t - \sin t$$

$$d'(2) = 2 \times 2 - \sin 2$$

calculator set to radians

$$= 3.0907...$$

$$\underline{\underline{\text{velocity} \approx 3.1 \text{ m/s}}}$$

CHAIN RULE

The rule to differentiate composite functions.

The order is important.

$$F(x) = \overset{\text{acts last}}{f}(g(x))$$

$$F'(x) = \underset{\text{differentiate first}}{f'}(g(x)) \times g'(x)$$

in Leibniz notation:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

(1) $h(x) = (2x - 3)^4$

$$h'(x) = 4(2x - 3)^3 \times 2$$

$$= \underline{\underline{8(2x - 3)^3}}$$

(2) $V(r) = \dfrac{4}{3r + 1}$

$$= 4(3r + 1)^{-1}$$

$$V'(r) = -4(3r + 1)^{-2} \times 3$$

$$= \underline{\underline{-\frac{12}{(3r + 1)^2}}}$$

(3) $w(x) = \sqrt{1 - x^2}$

$$= (1 - x^2)^{1/2}$$

$$w'(x) = {}^1/_2\,(1 - x^2)^{-1/2} \times (-2x)$$

$$\underline{\underline{-\frac{x}{\sqrt{1 - x^2}}}}$$

CHAIN RULE: TRIG. FUNCTIONS

The chain rule gives the results:

In RADIANS only:

$$\frac{d}{dx}\left(\sin(ax+b) \right) = a\cos(ax+b)$$

$$\frac{d}{dx}\left(\cos(ax+b) \right) = -\,a\sin(ax+b)$$

(1) $h(x) = \sin(2x + 3)$

$h'(x) = \underline{\underline{2\cos(2x + 3)}}$

(2) $V(t) = \cos 3t$

$V'(t) = \underline{\underline{-\,3\sin 3t}}$

$$\frac{d}{dx}\left((\sin x)^n \right) = n(\sin x)^{n-1} \times \cos x$$

$$\frac{d}{dx}\left((\cos x)^n \right) = n(\cos x)^{n-1} \times (-\sin x)$$

(1) $h(x) = \sin^3 x$

$= (\sin x)^3$

$h'(x) = 3(\sin x)^2 \times (\cos x)$

$= \underline{\underline{3\sin^2 x \cos x}}$

(2) $f(r) = \sqrt{\cos r}$

$= (\cos r)^{1/2}$

$f'(r) = {}^1/_2 (\cos r)^{-1/2} \times (-\sin r)$

$= \underline{\underline{-\frac{\sin r}{2\sqrt{\cos r}}}}$

A SPECIAL INTEGRAL:

To reverse the chain rule: $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$

NOTE: only for LINEAR FUNCTIONS ie. form $ax + b$

(1) $\int (2x + 3)^3\,dx$

$$= \frac{(2x+3)^4}{2 \times 4} + C$$

$$= {}^1/_8\,(2x+3)^4 + C$$

(2) $\int \sqrt{(1 - 2u)}\,du$

$$= \int (1-2u)^{1/2}\,du$$

$$= \frac{(1-2u)^{3/2}}{-2 \times {}^3/_2} + C$$

$$= -{}^1/_3\,(1-2u)^{3/2} + C$$

definite integrals:

(3) $\int_0^1 \dfrac{dx}{(x+1)^2}$

$$= \int_0^1 (x+1)^{-2}\,dx$$

$$= \left[\frac{(x+1)^{-1}}{1 \times (-1)}\right]_0^1$$

$$= \left[\frac{-1}{x+1}\right]_0^1$$

$$= \frac{-1}{1+1} - \frac{-1}{0+1}$$

$$= -{}^1/_2 - (-1)$$

$$= {}^1/_2$$

SPECIAL INTEGRALS: TRIG. FUNCTIONS

To reverse the chain rule:

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

NOTE: only for LINEAR FUNCTIONS ie. form $ax + b$

(1) $\int (x^2 - 3\cos x)\,dx$

$= \underline{\underline{\frac{x^3}{3} - 3\sin x + C}}$

(2) $\int (3 + \sin x)\,dx$

$= 3x + (-\cos x) + C$

$= \underline{\underline{3x - \cos x + C}}$

(3) $\int \cos(2x+3)\,dx$

$= \underline{\underline{{}^1/_2 \sin(2x+3) + C}}$

(4) $\int \sin 3u\,du$

$= \underline{\underline{-{}^1/_3 \cos 3u + C}}$

(5) $\int \cos(3w - {}^\pi/_4)\,dw$

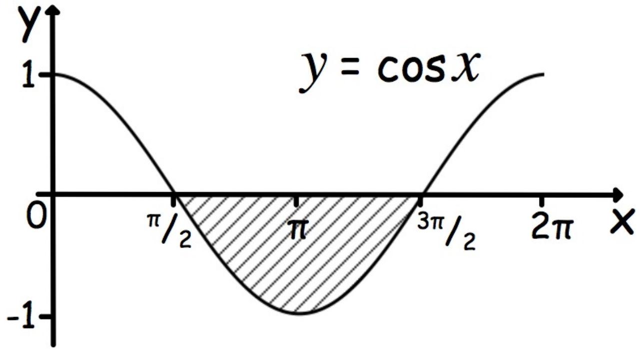
$= \underline{\underline{{}^1/_3 \sin(3w - {}^\pi/_4) + C}}$

(6) $\int \sin {}^1/_2 r\,dr$

$= \underline{\underline{-2\cos {}^1/_2 r + C}}$

AREAS: TRIG. FUNCTIONS

(1)



$$\int_{\pi/2}^{3\pi/2} \cos x \, dx$$

$$= \Big[\sin x\Big]_{\pi/2}^{3\pi/2}$$

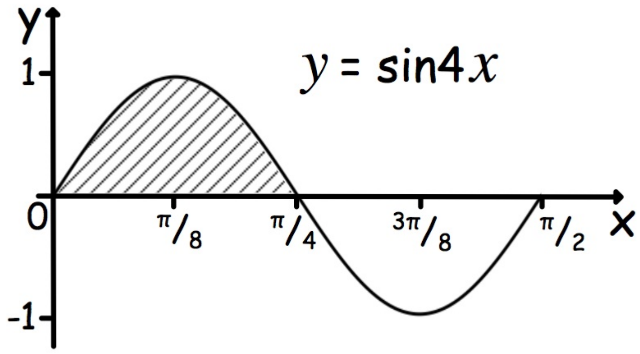
$$= \sin {}^{3\pi}/_{2} - \sin {}^{\pi}/_{2}$$

$$= -1 - 1$$

$$= -2$$

AREA = 2 units^2

(2)



$$\int_{0}^{\pi/4} \sin 4x \, dx$$

$$= \Big[-{}^{1}/_{4} \cos 4x\Big]_{0}^{\pi/4}$$

$$= -{}^{1}/_{4} \cos \overset{4\times{}^{\pi}/_{4}}{\pi} - \left(-{}^{1}/_{4} \cos \overset{4\times 0}{0}\right)$$

$$= -{}^{1}/_{4} \times (-1) - \left(-{}^{1}/_{4} \times 1\right)$$

$$= {}^{1}/_{2}$$

AREA = $^{1}/_{2}$ units^2

EXPONENTIALS and LOGARITHMS

Any POSITIVE number can be written as a power, a^x.

The logarithm of a number is the index (exponent) to which the base must be raised.

NOTE: can only log. a positive number ie. $N > 0$

$N = a^x$ INDEX or EXPONENT FORM

$\Leftrightarrow \log_a N = x$ LOGARITHMIC FORM

$49 = 7^2$ $\quad 1 = a^0$ $\quad {}^1/_8 = 2^{-3}$ $\quad 27 = 9^{3/2}$

$\log_7 49 = 2$ $\quad \log_a 1 = 0$ $\quad \log_2 {}^1/_8 = -3$ $\quad \log_9 27 = {}^3/_2$

NOTE: for $0 < N < 1$ $\quad \log_a N < 0$

for $N > 1$ $\quad \log_a N > 0$

eg. $\log {}^1/_2$ is negative: $\log {}^1/_2 = \log 2^{-1} = -\log 2$

Simplify

(1) $\log_3 27$

$= \log_3 3^3$

$= \underline{\underline{3}}$

(2) $\log_2 8$

$= \log_2 2^3$

$= \underline{\underline{3}}$

(3) $\log_4 8$

$= \log_4 4^{3/2}$

$= \underline{\underline{3/2}}$

(4) $\log_3 1/27$

$= \log_3 3^{-3}$

$= \underline{\underline{-3}}$

(5) $\log_5 1$

$= \log_5 5^0$

$= \underline{\underline{0}}$

(6) $\log_4 1/8$

$= \log_4 4^{-3/2}$

$= \underline{\underline{-3/2}}$

Solve

(1) $\log_2 x = 3$

$x = 2^3$

$\underline{\underline{x = 8}}$

(2) $\log_2 x = -3$

$x = 2^{-3}$

$\underline{\underline{x = 1/8}}$

(3) $\log_9 x = -1/2$

$x = 9^{-1/2}$

$x = \frac{1}{\sqrt{9}}$

$\underline{\underline{x = 1/3}}$

(4) $\log_x 8 = 3/2$

$8 = x^{3/2}$

$8^{2/3} = (x^{3/2})^{2/3}$

$(\sqrt[3]{8})^2 = x^1$

$\underline{\underline{x = 4}}$

LOG RULES:

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a {}^{x}/_{y} = \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

NOTE:

$\log_a 1 = 0$, since $a^0 = 1$

$\log_a a = 1$, since $a^1 = a$

Simplify

$$3 \log_4 2 - \log_4 6 + \log_4 3$$

$$= \log_4 2^3 - \log_4 6 + \log_4 3$$

$$= \log_4 8 - \log_4 6 + \log_4 3$$

$$= \log_4 \left(\frac{8 \times 3}{6} \right)$$

$$= \log_4 4$$

$$= \underline{\underline{1}}$$

Solve

$\log_2 3 + \log_2 x = 3$, $x > 0$

$$\log_2 3x = 3$$

$$3x = 2^3$$

$$\underline{\underline{x = {}^{8}/_{3}}}$$

Solve

$\log(x + 2) + \log(x - 3) = \log 14 \quad , \; x > 3$

$$\log(x + 2)(x - 3) = \log 14$$
$$(x + 2)(x - 3) = 14$$
$$x^2 - x - 6 = 14$$
$$x^2 - x - 20 = 0$$
$$(x + 4)(x - 5) = 0$$

$x = -4 \quad$ or $\quad x = 5$

$x > 3 , \quad \underline{\underline{x = 5}}$

Notice the base did not matter.

Using calculator function:

LOG common logarithms, base 10

10^x the corresponding ANTILOG function

Solve

(1) $10^x = 3$

$$x = \log_{10} 3$$
$$= 0.47712\ldots$$
$$\approx \underline{\underline{0.477}}$$

(2) $\log_{10} x = 0.4$

$$x = 10^{0.4}$$
$$= 2.51188\ldots$$
$$\approx \underline{\underline{2.51}}$$

EXPONENTIAL GROWTH and DECAY

DECAY

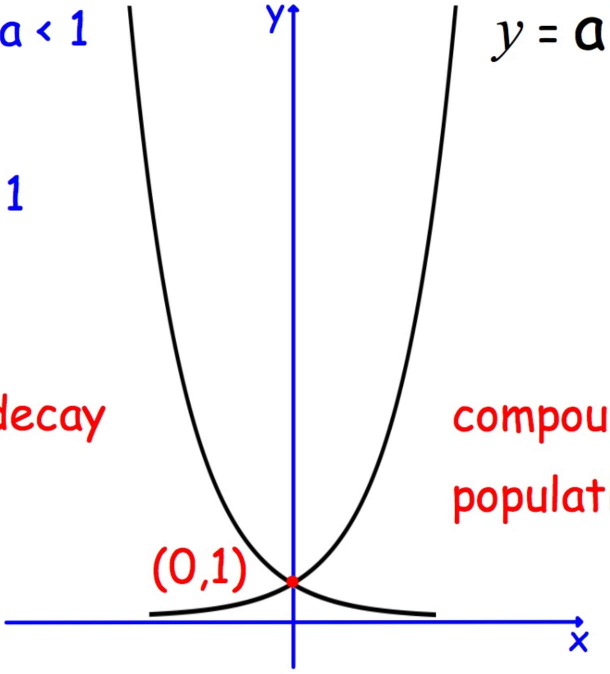
$y = a^x$, $0 < a < 1$

or

$y = a^{-x}$, $a > 1$

GROWTH

$y = a^x$, $a > 1$



radioactive decay

cooling

compound interest

population growths

FORMULAE $A_t = A_0 \times a^t$

$a > 1$ GROWTH

$0 < a < 1$ DECAY

$A_t = A_0 \times a^{kt}$

$k > 0$ GROWTH

$k < 0$ DECAY

initial amount A_0 ,

amount A_t after t iterations

EQUATIONS WITH UNKOWN EXPONONENT:

Log. both sides and use $\log_a x^n = n \log_a x$

$$4^x = 3$$

$$\log_{10} 4^x = \log_{10} 3$$

$$x \log_{10} 4 = \log_{10} 3$$

$$x = \frac{\log_{10} 3}{\log_{10} 4}$$

$$= 0.7924...$$

$$\approx \underline{\underline{0.792}}$$

Money is invested at 10% per year.

How many years for the investment to double ?

$$A_t = A_o\,(1.10)^t$$

$$200 = 100\,(1.10)^t \qquad \text{assume } A_o = 100$$

$$(1.10)^t = 2$$

$$\log_{10} (1.10)^t = \log_{10} 2$$

$$t \log_{10} (1.10) = \log_{10} 2$$

$$t = \frac{\log_{10} 2}{\log_{10} (1.10)} = 7.272...$$

$\underline{\underline{8 \text{ years}}}$ required

NATURAL GROWTH and DECAY

Base e ; an irrational number, e = 2.7 1828 1828 4590...

Calculator:

Ln natural logarithms, base e

e^x the corresponding ANTILOG function

The mass m grams of a radioactive isotope after t hours is

$$m_t = m_o e^{-0.02t}$$

Calculate

(a) the mass remaining in an 80 g sample after 10 years.

(b) the time for half the isotope to decay (half-life).

(a)

$$\begin{aligned} m_t &= m_o e^{-0.02t} \\ &= 80 \times e^{(-0.02 \times 10)} \\ &= 65.498... \\ &\approx \underline{\underline{65.5 \text{ grams}}} \end{aligned}$$

(b)

$$m_t = m_o e^{-0.02t}$$

$$50 = 100 e^{-0.02t} \qquad \text{assume } m_o = 100$$

$$e^{-0.02t} = 0.5$$

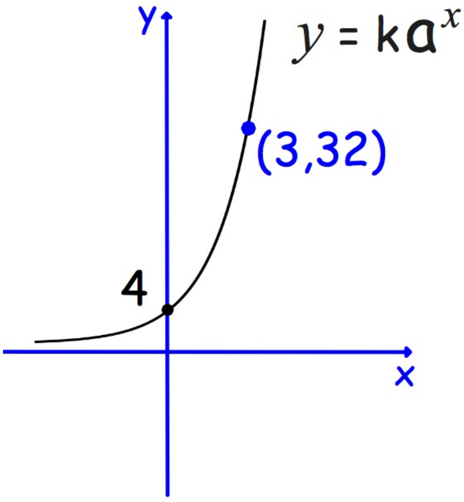
changing from index to log. form

$$-0.02t = \log_e 0.5$$

$$t = \frac{\log_e 0.5}{-0.02} = 34.657... \approx \underline{\underline{34.7 \text{ hours}}}$$

GRAPHS

(1) Find k and a



$y = ka^x$

$\overset{x\ y}{(0,4)}$ $\quad 4 = k \times a^0$

$4 = k \times 1$

$\underline{\underline{k = 4}}$

$y = 4a^x$

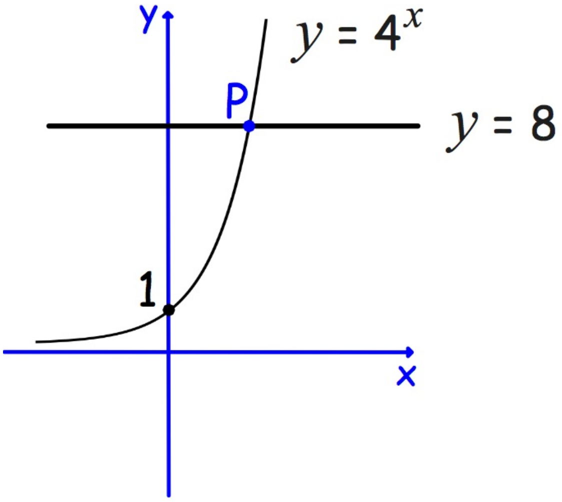
$\overset{x\ \ y}{(3,32)}$ $\quad 32 = 4a^3$

$8 = a^3$

$\underline{\underline{a = 2}}$

Equation $y = 4(2)^x$ or $y = 4 \times 2^x$

(2) Find the coordinates of the point of intersection P.



$4^x = 8$

$(2^2)^x = 2^3$

$2^{2x} = 2^3$

$2x = 3$

$x = {}^3/_2$

$\underline{\underline{P\left({}^3/_2 , 8\right)}}$

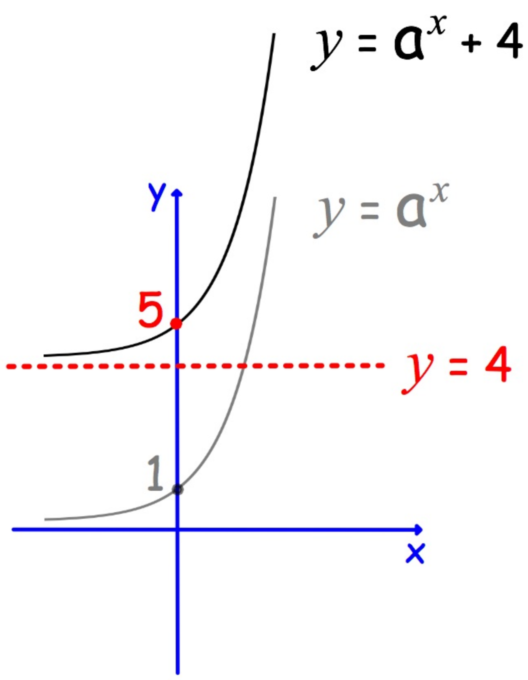
TRANSFORM GRAPHS

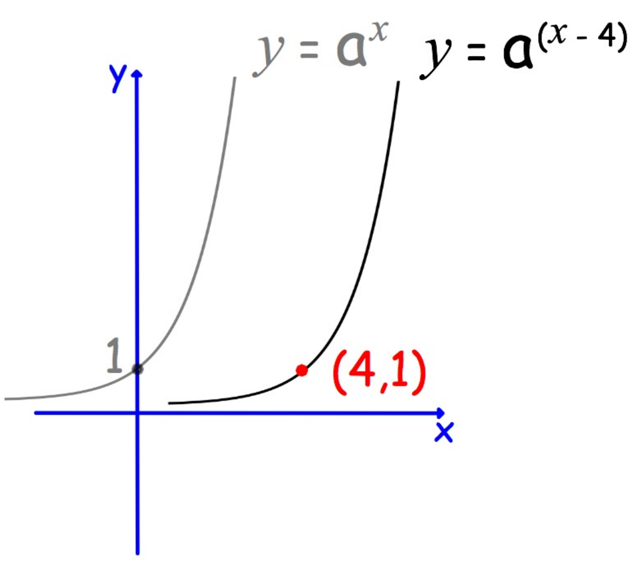
Draw the basic shape of the transformed graph.
Annotate with the images of key points.

	$y = f(x) + k$	$(x, y + k)$
	$y = f(x + k)$	$(x - k, y)$
	$y = kf(x)$	(x, ky)
	$y = f(kx)$	$(^1/_k x, y)$
REFLECT in X- axis	$y = -f(x)$	$(x, -y)$
REFLECT in Y- axis	$y = f(-x)$	$(-x, y)$
HALF-TURN about O	$y = -f(-x)$	$(-x, -y)$

(1) $f(x) = a^x + 4$

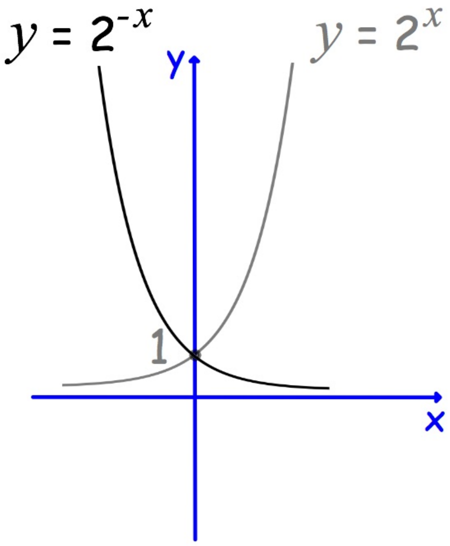
(2) $f(x) = a^{(x - 4)}$



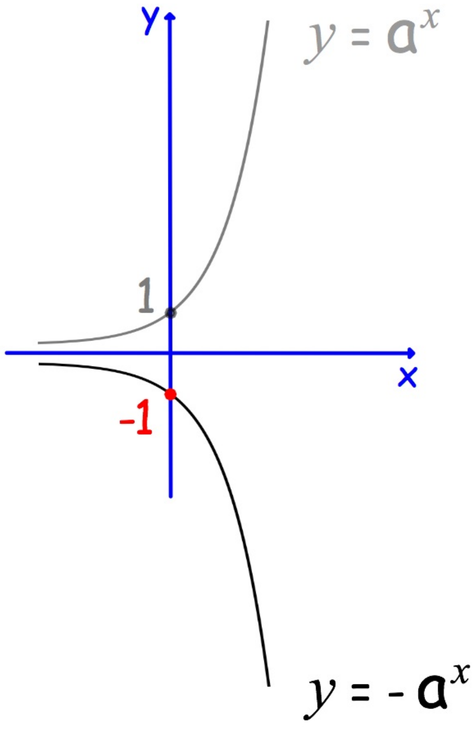


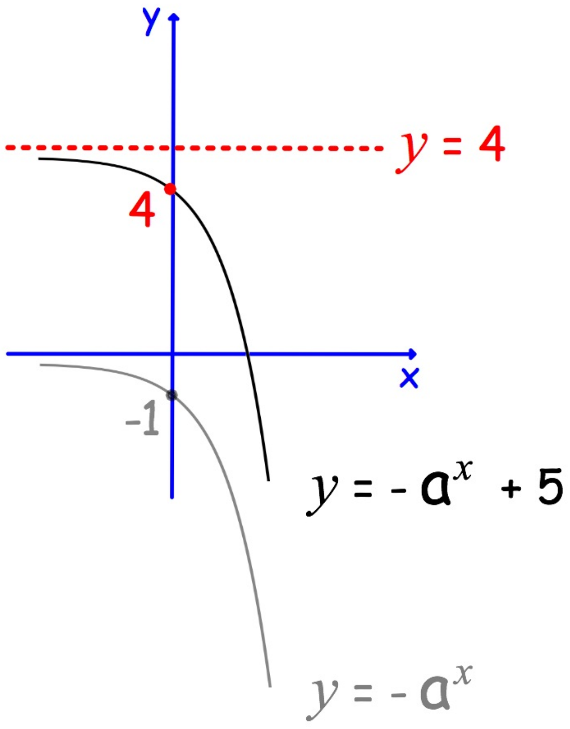
(3) $f(x) = (^1/_2)^x$

$$\begin{aligned} &(^1/_2)^x \\ &= (2^{-1})^x \\ &= 2^{-x} \end{aligned}$$



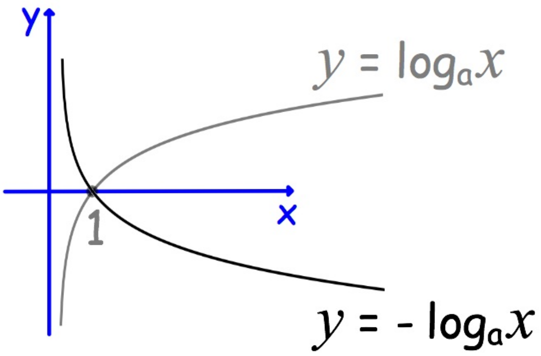
(4) $f(x) = 5 - a^x$



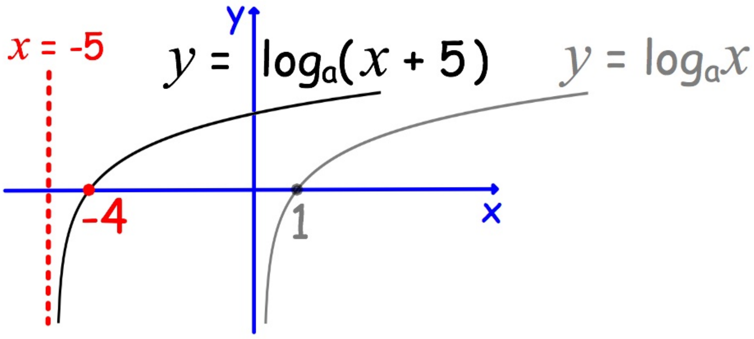


(5) $f(x) = \log_a(^1/_x)$

$$\log_a(^1/_x)$$
$$= \log_a x^{-1}$$
$$= -\log_a x$$



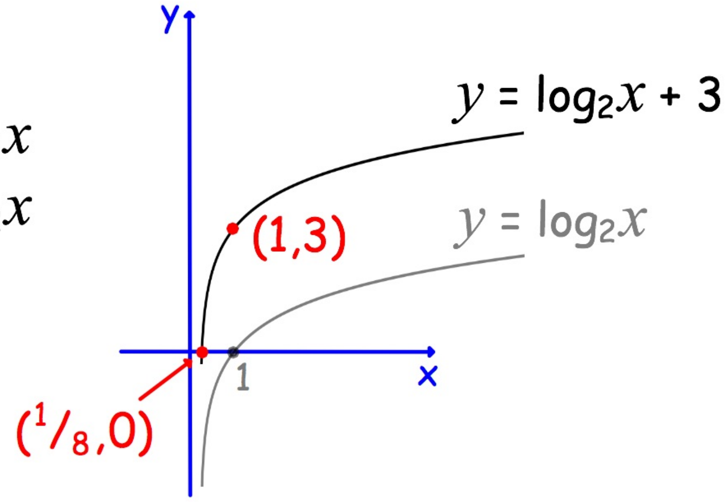
(6) $f(x) = \log_a(x + 5)$



(7) $f(x) = \log_2 8x$

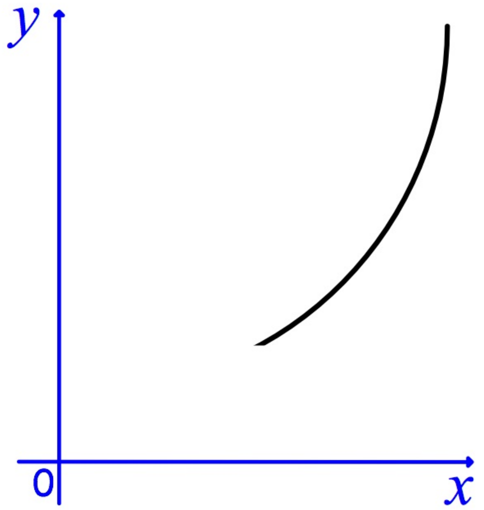
$$\log_2 8x$$
$$= \log_2 8 + \log_2 x$$
$$= 3 + \log_2 x$$

$2^3 = 8$



x-axis: $\log_2 8x = 0$
$$8x = 2^0$$
$$8x = 1$$
$$x = {}^1/_8$$

EXPERIMENTAL DATA: FORMULAE



When graphing experimental results exponential and power graphs look similar.

By plotting log. graphs they can be distinguished.

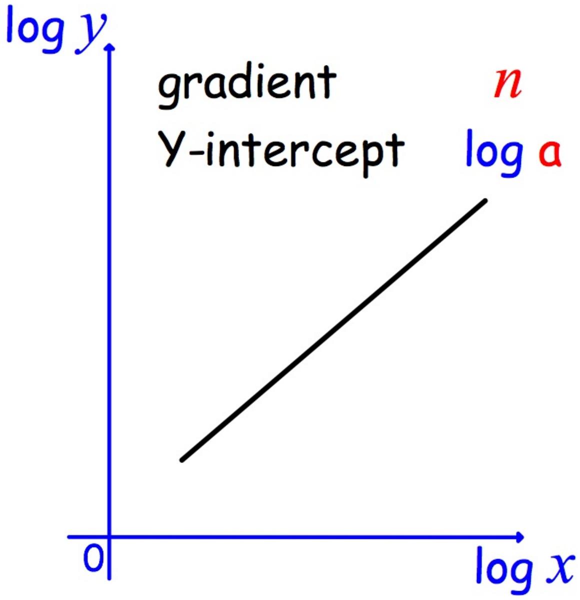
LOG. FORM shows linear relationship $y = mx + C$ and the graph will be a straight line.

$$y = ax^n$$

$$\log y = n \log x + \log a$$

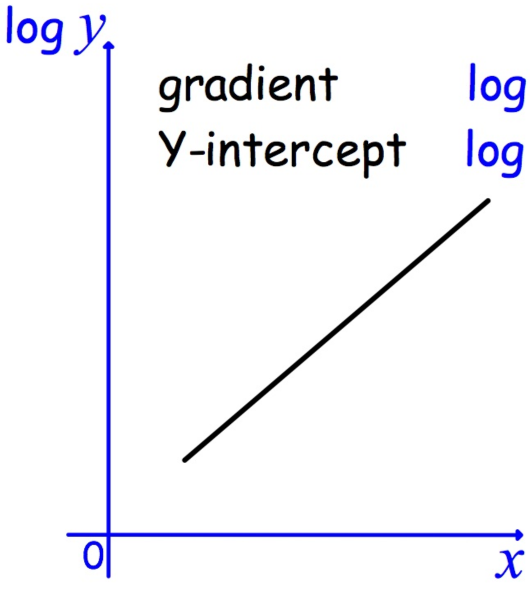
$$y = ab^x$$

$$\log y = (\log b)x + \log a$$



gradient n

Y-intercept $\log a$



gradient $\log b$

Y-intercept $\log a$

Log. rules are used to change from index to log. form.

$$y = ax^n$$

$$\log y = \log (ax^n)$$

$$\log y = \log x^n + \log a$$

$$\log y = \underbrace{n}_{\substack{\text{gradient} \\ n}} \log x + \underbrace{\log a}_{\substack{\text{Y-intercept} \\ \log a}}$$

$$y = ab^x$$

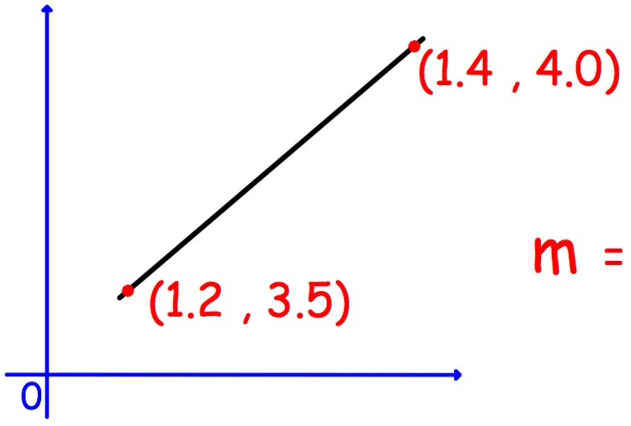
$$\log y = \log (ab^x)$$

$$\log y = \log b^x + \log a$$

$$\log y = x \log b + \log a$$

$$\log y = \underbrace{(\log b)}_{\substack{\text{gradient} \\ \log b}} x + \underbrace{\log a}_{\substack{\text{Y-intercept} \\ \log a}}$$

EQUATION OF THE LINE



$$m = \frac{4.0 - 3.5}{1.4 - 1.2} = \frac{0.5}{0.2} = 2.5$$

$$(\overset{a}{1.4} , \overset{b}{4.0})$$

or can use (1.2 , 3.5)

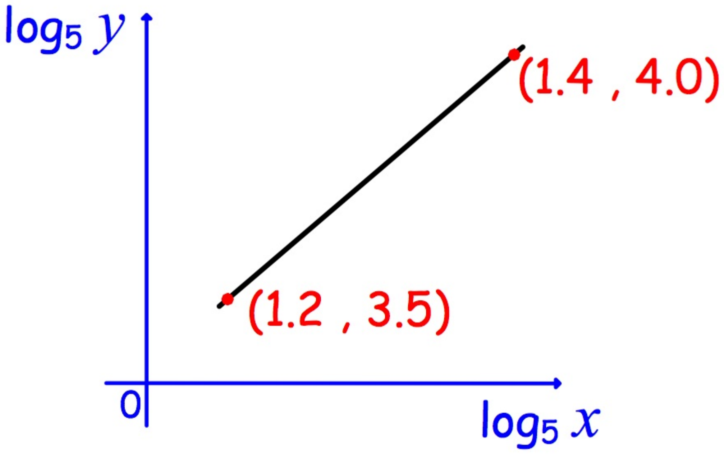
$$y - b = m(x - a)$$

$$y - 4.0 = 2.5 (x - 1.4)$$

$$y - 4.0 = 2.5x - 3.5$$

$$y = 2.5x + 0.5$$

(1) Find the formula connecting y and x.



equation of the line $\log y = 2.5 \log x + 0.5$

INDEX FORM $y = ax^n$

LOG. FORM $\log y = n \log x + \log a$

equation of the line $\log y = 2.5 \log x + 0.5$

$n = 2.5$

$\log_5 a = 0.5$

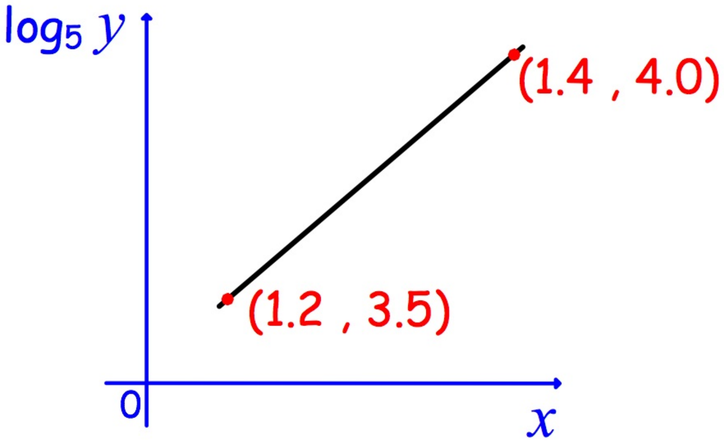
$a = 5^{0.5}$

$a = 2.236...$

$y = ax^n$

$$\underline{\underline{y = 2.2x^{2.5}}}$$

(2) Find the formula connecting y and x.



equation of the line $\log y = 2.5x + 0.5$

INDEX FORM $y = ab^x$

LOG. FORM $\log y = (\log b)x + \log a$

equation of the line $\log y = 2.5x + 0.5$

$\log_5 b = 2.5$ $\log_5 a = 0.5$

$b = 5^{2.5}$ $a = 5^{0.5}$

$b = 55.901...$ $a = 2.236...$

$y = ab^x$

$\underline{\underline{y = 2.2(55.9)^x}}$

WAVE FUNCTION

Functions of the form $y = a\cos x + b\sin x$

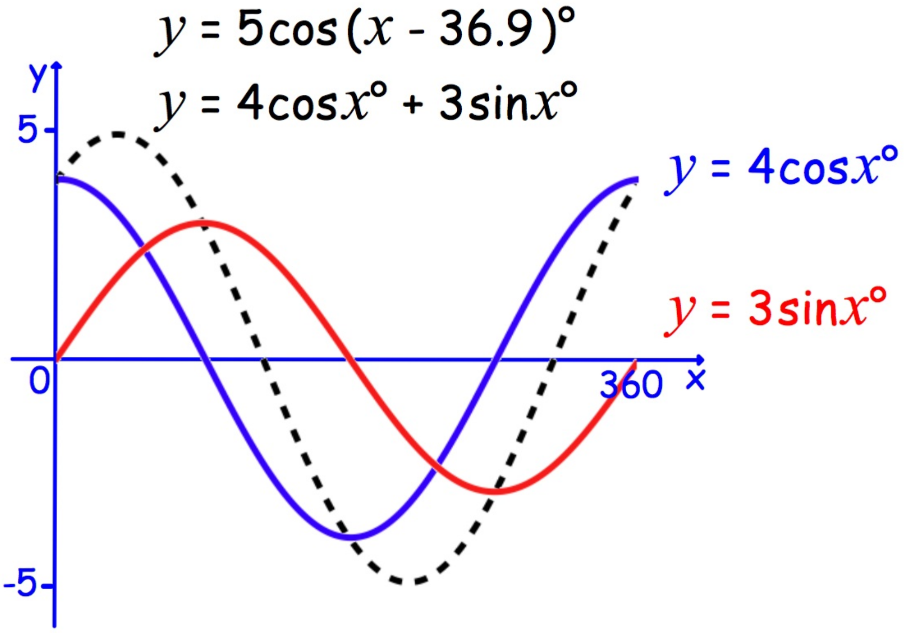
A sum of two functions,
the resultant wave has: increased amplitude, R
change of phase, a

so can be written in forms: $R\cos(x \pm a)$ or $R\sin(x \pm a)$

EXPANSION FORMULAE are used.

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$



(1) Write $4\cos x^\circ + 3\sin x^\circ$ in the form $R\cos(x - a)^\circ$

use expansion formulae:

$$\begin{aligned} 4\cos x^\circ + 3\sin x^\circ &= R\cos(x - a)^\circ \\ &= R\cos x^\circ \cos a^\circ + R\sin x^\circ \sin a^\circ \\ 4\cos x^\circ + 3\sin x^\circ &= (R\cos a^\circ)\cos x^\circ + (R\sin a^\circ)\sin x^\circ \end{aligned}$$

comparing sides: $R\sin a^\circ = 3$

$R\cos a^\circ = 4$

Solve for R and a using Trig. identities:

squaring

$$\begin{aligned} R^2\sin^2 a^\circ &= 9 \\ R^2\cos^2 a^\circ &= 16 \end{aligned}$$

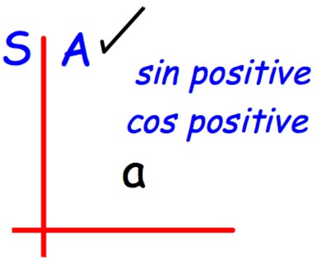
adding

$$\begin{aligned} R^2(\sin^2 a^\circ + \cos^2 a^\circ) &= 25 \\ R^2 \times 1 &= 25 \\ R &= 5 \end{aligned}$$

$$\frac{R\sin a^\circ}{R\cos a^\circ} = \frac{+3}{+4}$$

$$\tan a^\circ = {}^{3}/_{4}$$

$$a = 36.8698....$$



only one quadrant will satisfy the signs of both $R\sin a^\circ$ and $R\cos a^\circ$

$$4\cos x^\circ + 3\sin x^\circ = \underline{\underline{5\cos(x - 36.9)^\circ}}$$

(2) Write $\cos x^\circ - \sqrt{3}\sin x^\circ$ in the form $R\sin(x - a)^\circ$

$$\begin{aligned}
\cos x^\circ - \sqrt{3}\sin x^\circ &= R\sin(x - a)^\circ \\
&= R\sin x^\circ \cos a^\circ - R\cos x^\circ \sin a^\circ \\
-\sqrt{3}\sin x^\circ - (-1)\cos x^\circ &= (R\cos a^\circ)\sin x^\circ - (R\sin a^\circ)\cos x^\circ
\end{aligned}$$

$$\begin{aligned}
R\sin a^\circ &= -1 \\
R\cos a^\circ &= -\sqrt{3}
\end{aligned}$$

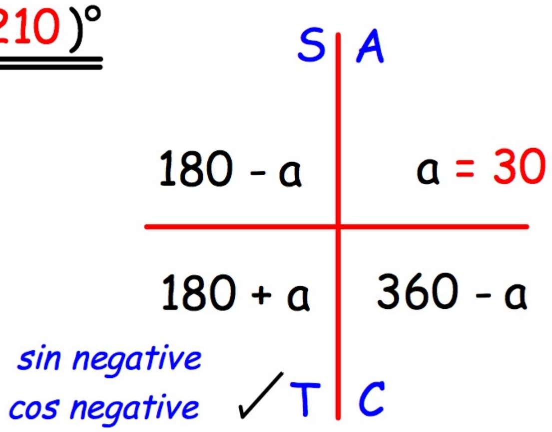
$$\begin{aligned}
R^2 &= (-1)^2 + (-\sqrt{3})^2 \\
&= 1 + 3 \\
&= 4 \\
R &= 2
\end{aligned}$$

$$\frac{R\sin a^\circ}{R\cos a^\circ} = \frac{-1}{-\sqrt{3}}$$

$$\tan a^\circ = {}^{1}/_{\sqrt{3}}$$

$$a = 210$$

$$\cos x^\circ - \sqrt{3}\sin x^\circ = \underline{\underline{2\sin(x - 210)^\circ}}$$



(3) Write $\cos 2x^\circ - \sin 2x^\circ$ in the form $R\sin(2x + a)^\circ$

NOTE: do not use expansions for sin2A or cos2A

$$\begin{aligned} \cos 2x^\circ - \sin 2x^\circ &= R\sin(2x + a)^\circ \\ &= R\sin 2x^\circ \cos a^\circ + R\cos 2x^\circ \sin a^\circ \\ (-1)\sin 2x^\circ + 1\cos 2x^\circ &= (R\cos a^\circ)\sin 2x^\circ + (R\sin a^\circ)\cos 2x^\circ \end{aligned}$$

$$\begin{aligned} R\sin a^\circ &= +1 \\ R\cos a^\circ &= -1 \end{aligned}$$

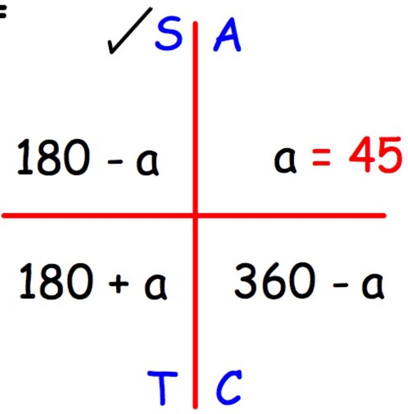
$$\begin{aligned} R^2 &= 1^2 + (-1)^2 \\ &= 1 + 1 \\ &= 2 \\ R &= \sqrt{2} \end{aligned}$$

$$\frac{R\sin a^\circ}{R\cos a^\circ} = \frac{+1}{-1}$$

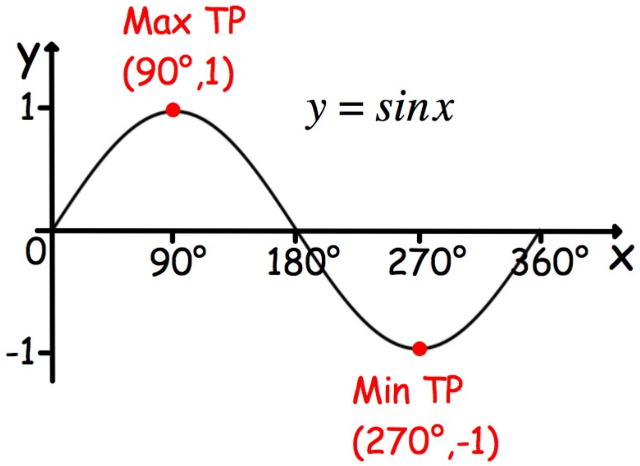
$$\begin{aligned} \tan a^\circ &= -1 \\ a &= 135 \end{aligned}$$

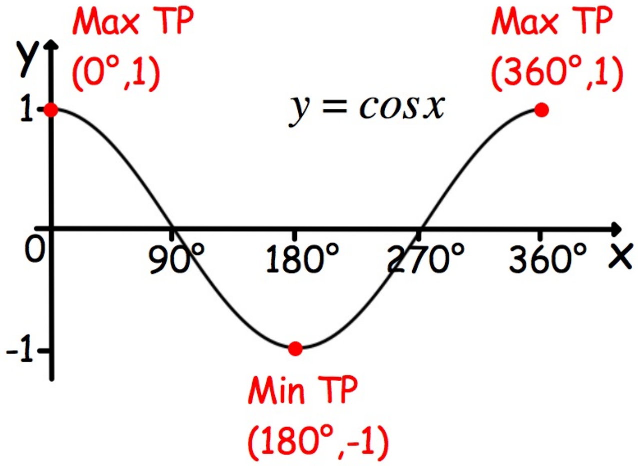
$$\cos 2x^\circ - \sin 2x^\circ = \underline{\underline{\sqrt{2}\sin(2x - 135)^\circ}}$$

sin positive
cos negative



MAXIMUM and MINIMUM VALUES



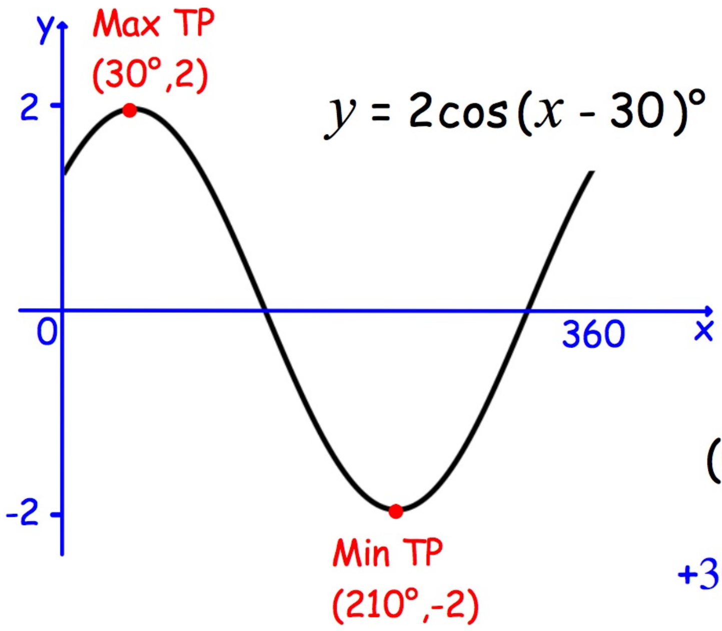


TRANSFORMATIONS: $R\cos(x \pm a)$

stretch R units vertically

- a shift a° RIGHT

+ a shift a° LEFT



$$(0°,\ 1) \xrightarrow{+30°,\ \times 2} (30°,\ 2)$$

MAX. TP

$$(180°,\ -1) \xrightarrow{+30°,\ \times 2} (210°,\ -2)$$

MIN. TP

NOTE: these are STATIONARY POINTS

(1) $5\sin(2x - 30)^\circ + 3 \quad , \quad 0 \le x \le 180$

MAXIMUM

$5\sin 90^\circ + 3$

$= 5 \times 1 + 3$

$= 8$

$2x - 30 = 90$

$2x = 120$

$x = 60$

MINIMUM

$5\sin 270^\circ + 3$

$= 5 \times (-1) + 3$

$= -2$

$2x - 30 = 270$

$2x = 300$

$x = 150$

MAX (60 , 8) and MIN (150 , -2)

(2) $5\cos(2x - 30)^\circ + 3 \quad , \quad 0 \le x \le 180$

MAXIMUM

$5\cos 0^\circ + 3$

$= 5 \times 1 + 3$

$= 8$

$2x - 30 = 0$

$2x = 30$

$x = 15$

MINIMUM

$5\cos 180^\circ + 3$

$= 5 \times (-1) + 3$

$= -2$

$2x - 30 = 180$

$2x = 210$

$x = 105$

MAX (15 , 8) and MIN (105 , -2)

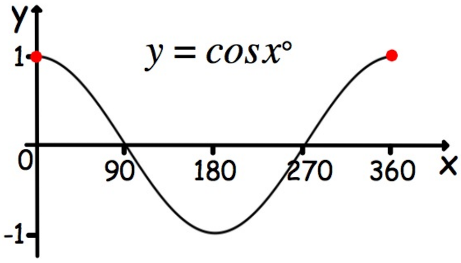
EQUATIONS

Equations of the form $a\cos x + b\sin x = c$

Express in the form $R\cos(x - a) = c$
or similar

(1)

$$4\cos x^\circ + 3\sin x^\circ = 5$$
$$5\cos(x - 36.9)^\circ = 5$$



$$\cos(x - 36.9)^\circ = 1$$
$$x - 36.9 = 0$$
$$\underline{\underline{x = 36.9}}$$

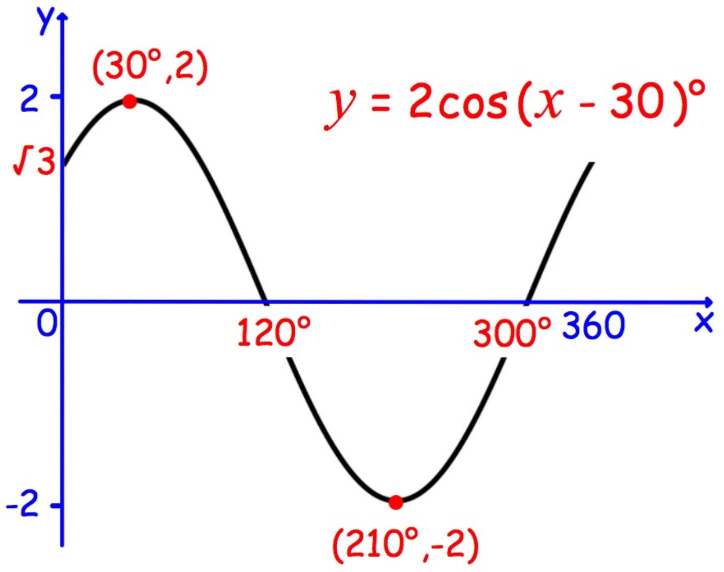
(2)

$$\cos x^\circ - \sqrt{3}\sin x^\circ = \sqrt{3}$$
$$2\sin(x - 210)^\circ = \sqrt{3}$$

$$\sin(x - 210)^\circ = \sqrt{3}/_2$$
$$x - 210 = 60\ ,\ 120$$
$$\underline{\underline{x = 270\ ,\ 330}}$$

sin positive ✓ S	✓ A sin positive
180 - a	$a = \sin^{-1}(\sqrt{3}/_2) = 60$
180 + a	360 - a
T	C

SKETCH $y = \sqrt{3}\cos x^\circ + \sin x^\circ$



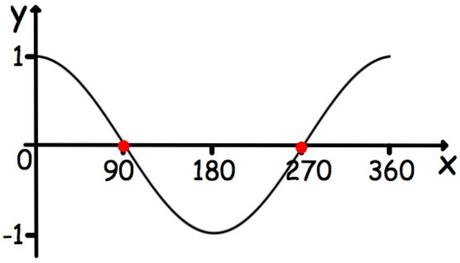
y-axis $x = 0$

$y = \sqrt{3}\cos x^\circ + \sin x^\circ$ or $y = 2\cos(x - 30)^\circ$

$$\begin{aligned} y &= \sqrt{3}\cos 0^\circ + \sin 0^\circ \\ &= \sqrt{3} \times 1 + 0 \\ &= \sqrt{3} \end{aligned}$$

$$\begin{aligned} y &= 2\cos(0 - 30)^\circ \\ &= 2\cos(-30)^\circ \\ &= 2 \times \sqrt{3}/2 \\ &= \sqrt{3} \end{aligned}$$

x-axis $y = 0$



$$\begin{aligned} 2\cos(x - 30)^\circ &= 0 \\ \cos(x - 30)^\circ &= 0 \\ x - 30 &= 90 \quad \text{or} \quad 270 \\ x &= 120 \quad \text{or} \quad 300 \end{aligned}$$