Advanced Higher Maths

Unit 1.1 Partial Fractions Solutions

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Ex1 Type 1 - Basic

(a) $\dfrac{4x - 9}{(x - 2)(x - 3)} = \dfrac{A}{(x - 2)} + \dfrac{B}{(x - 3)}$ $4x - 9 = A(x - 3) + B(x - 2)$

$x = 2$ $-1 = -A$ $A = 1$

$x = 3$ $3 = B$ $B = 3$

$$\dfrac{4x - 9}{(x - 2)(x - 3)} = \dfrac{1}{(x - 2)} + \dfrac{3}{(x - 3)}$$

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(b) $\dfrac{3 - 8x}{x(1 - x)} = \dfrac{A}{x} + \dfrac{B}{(1 - x)}$ $3 - 8x = A(1 - x) + Bx$

$x = 0$ $3 = A$ $A = 3$

$x = 1$ $-5 = B$ $B = -5$

$$\dfrac{3 - 8x}{x(1 - x)} = \dfrac{3}{x} - \dfrac{5}{(1 - x)}$$

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(c) $\dfrac{x + 24}{x^2 - x - 12} = \dfrac{x + 24}{(x - 4)(x + 3)} = \dfrac{A}{(x - 4)} + \dfrac{B}{(x + 3)} = x + 24 = A(x + 3) + B(x - 4)$

$x = 4$ $28 = 7A$ $A = 4$

$x = -3$ $21 = -7B$ $B = -3$

$$\dfrac{x + 24}{x^2 - x - 12} = \dfrac{4}{(x - 4)} - \dfrac{3}{(x + 3)}$$

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(d) $\dfrac{2(3x+4)}{x^2+4x} = \dfrac{6x+8}{x(x+4)} = \dfrac{A}{x} + \dfrac{B}{(x+4)} = 6x+8 = A(x+4) + Bx$

$x = 0$ $8 = 4A$ $A = 2$

$x = -4$ $-16 = -4B$ $B = 4$

$$\frac{2(3x+4)}{x^2+4x} = \frac{2}{x} + \frac{4}{(x+4)}$$

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Ex2 Type 2 – Repeated factors

(a) $\dfrac{3x^2+1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{(x+1)} + \dfrac{C}{(x+1)^2} = 3x^2+1 = A(x+1)^2 + Bx(x+1) + Cx$

$x = 0$ $1 = A$ $A = 1$

$x = -1$ $4 = -C$ $C = -4$

$x = 1$ $4 = 4A + 2B + C$ $4 = 4 + 2B - 4$ $B = 2$

$$\frac{3x^2+1}{x(x+1)^2} = \frac{1}{x} + \frac{2}{(x+1)} - \frac{4}{(x+1)^2}$$

(b) $\dfrac{3x^2 + 2}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{(x-1)} + \dfrac{C}{(x-1)^2} = 3x^2 + 2 = A(x-1)^2 + Bx(x-1) + Cx$

$x = 0$ $\qquad$ $2 = A$ $\qquad$ $A = 2$

$x = 1$ $\qquad$ $5 = C$ $\qquad$ $C = 5$

$x = -1$ $\qquad$ $5 = 4A + 2B - C$ $\qquad$ $5 = 8 + 2B - 5$ $\qquad$ $B = 1$

$$\frac{3x^2 + 2}{x(x-1)^2} = \frac{2}{x} + \frac{1}{(x-1)} + \frac{5}{(x-1)^2}$$

(c) $\dfrac{x^2 - 2x + 10}{(x+2)(x-1)^2} = \dfrac{A}{(x+2)} + \dfrac{B}{(x-1)} + \dfrac{C}{(x-1)^2}$

$x^2 - 2x + 10 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$

$x = 1$ $\qquad$ $9 = 3C$ $\qquad$ $C = 3$

$x = -2$ $\qquad$ $18 = 9A$ $\qquad$ $A = 2$

$x = 0$ $\qquad$ $10 = A - 2B + 2C$ $\qquad$ $10 = 2 - 2B + 6$ $\qquad$ $B = -1$

$$\frac{x^2 - 2x + 10}{(x+2)(x-1)^2} = \frac{2}{(x+2)} - \frac{1}{(x-1)} + \frac{3}{(x-1)^2}$$

(d) $\dfrac{5x^2 - 6x - 21}{(2x-3)(x-4)^2} = \dfrac{A}{(2x-3)} + \dfrac{B}{(x-4)} + \dfrac{C}{(x-4)^2}$

$5x^2 - 6x - 21 = A(x-4)^2 + B(2x-3)(x-4) + C(2x-3)$

$x = 4$ $\qquad$ $35 = 5C$ $\qquad$ $C = 7$

$x = \dfrac{3}{2}$ $\qquad$ $\dfrac{-75}{4} = \dfrac{25}{4}A$ $\qquad$ $A = -3$

$x = 0$ $\qquad$ $-21 = 16A + 12B - 3C$ $\qquad$ $-21 = -48 + 12B - 21$ $\qquad$ $B = 4$

$$\dfrac{5x^2 - 6x - 21}{(2x-3)(x-4)^2} = \dfrac{-3}{(2x-3)} + \dfrac{4}{(x-4)} + \dfrac{7}{(x-4)^2}$$

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Ex3 Type 3 – Irreducible quadratic factor

(a) $\dfrac{8x-1}{(x-2)(x^2+1)} = \dfrac{A}{(x-2)} + \dfrac{Bx+C}{(x^2+1)} = 8x - 1 = A(x^2+1) + (Bx+C)(x-2)$

$x = 2$ $\qquad$ $15 = 5A$ $\qquad$ $A = 3$

$x = 0$ $\qquad$ $-1 = A - 2C$ $\qquad$ $-4 = -2C$ $\qquad$ $C = 2$

$x = 1$ $\qquad$ $7 = 2A - B - C$ $\qquad$ $7 = 6 - B - 2$ $\qquad$ $B = -3$

$$\dfrac{8x-1}{(x-2)(x^2+1)} = \dfrac{3}{(x-2)} - \dfrac{3x-2}{(x^2+1)}$$

(b) $\dfrac{7x^2 - x + 14}{(x-2)(x^2+4)} = \dfrac{A}{(x-2)} + \dfrac{Bx+C}{(x^2+4)} = 7x^2 - x + 14 = A(x^2+4) + (Bx+C)(x-2)$

$x = 2 \qquad 40 = 8A \qquad A = 5$

$x = 0 \qquad 14 = 4A - 2C \qquad 14 = 20 - 2C \qquad C = 3$

$x = 1 \qquad 20 = 5A - B - C \qquad 20 = 25 - B - 3 \qquad B = 2$

$$\frac{7x^2 - x + 14}{(x-2)(x^2+4)} = \frac{5}{(x-2)} + \frac{2x+3}{(x^2+4)}$$

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(c) $\dfrac{x(3x+2)}{(x-2)(x^2+4)} = \dfrac{A}{(x-2)} + \dfrac{Bx+C}{(x^2+4)} = x(3x+2) = A(x^2+4) + (Bx+C)(x-2)$

$x = 2 \qquad 16 = 8A \qquad A = 2$

$x = 0 \qquad 0 = 4A - 2C \qquad 0 = 8 - 2C \qquad C = 4$

$x = 1 \qquad 5 = 5A - B - C \qquad 5 = 10 - B - 4 \qquad B = 1$

$$\frac{x(3x+2)}{(x-2)(x^2+4)} = \frac{2}{(x-2)} + \frac{x+4}{(x^2+4)}$$

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$(d) \quad \dfrac{x^3 + 2x^2 + 61}{(x+3)^2(x^2+4)} = \dfrac{A}{(x+3)} + \dfrac{B}{(x+3)^2} + \dfrac{Cx+D}{(x^2+4)}$

$x^3 + 2x^2 + 61 = A(x+3)(x^2+4) + B(x^2+4) + (Cx+D)(x+3)^2$

$x = -3 \qquad 52 = 13B \qquad \boldsymbol{B = 4}$

$x = 0 \qquad 61 = 12A + 4B + 9D \qquad 61 = 12A + 16 + 9D \qquad 45 = 12A + 9D \qquad 15 = 4A + 3D \qquad (1)$

$x = 1 \qquad 64 = 20A + 5B + 16C + 16D \qquad 64 = 20A + 20 + 16C + 16D \qquad 11 = 5A + 4C + 4D \quad (2)$

$x = -1 \qquad 62 = 10A + 5B - 4C + 4D \qquad 62 = 10A + 20 - 4C + 4D \qquad 42 = 10A - 4C + 4D \ (3)$

(2) + (3) $\qquad 53 = 15A + 8D$

(1) $\qquad\qquad 15 = 4A + 3D$

Simultaneous equations: $\qquad 159 = 45A + 24D$

$\qquad\qquad\qquad\qquad\qquad 120 = 32A + 24D \qquad\qquad 39 = 13A \qquad \boldsymbol{A = 3}$

From (1) $\quad 3D = 15 - 12 \qquad \boldsymbol{D = 1}$

From (2) $\quad 4C = 11 - 15 - 4 \qquad \boldsymbol{C = -2}$

Summary: $\qquad \boldsymbol{A = 3} \qquad \boldsymbol{B = 4} \qquad \boldsymbol{C = -2} \qquad \boldsymbol{D = 1}$

$$\dfrac{x^3 + 2x^2 + 61}{(x+3)^2(x^2+4)} = \dfrac{3}{(x+3)} + \dfrac{4}{(x+3)^2} - \dfrac{2x-1}{(x^2+4)}$$

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Type : Improper Rational Fraction

Ex6

(a) $\dfrac{x^2 - x + 6}{x^2 + x - 2} = 1 - \dfrac{2x - 8}{(x + 2)(x - 1)}$

Proper Rational Fraction part

$\dfrac{2x - 8}{(x + 2)(x - 1)} = \dfrac{A}{(x + 2)} + \dfrac{B}{(x - 1)}$ $2x - 8 = A(x - 1) + B(x + 2)$

$x = -2$ $-12 = -3A$ $A = 4$

$x = 1$ $-6 = 3B$ $B = -2$

$$\dfrac{2x - 8}{(x + 2)(x - 1)} = \dfrac{4}{(x + 2)} - \dfrac{2}{(x - 1)}$$

Final answer:

$$\dfrac{x^2 - x + 6}{x^2 + x - 2} = 1 - \dfrac{4}{(x + 2)} + \dfrac{2}{(x - 1)}$$

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(b) $\dfrac{x^3 - x^2 - 5x - 7}{x^2 - 2x - 3} = x + 1 - \dfrac{4}{(x+1)(x-3)}$

Proper Rational Fraction part

$\dfrac{4}{(x+1)(x-3)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-3)}$ $\qquad 4 = A(x-3) + B(x+1)$

$x = -1 \quad 4 = -4A \qquad A = -1$

$x = 3 \qquad 4 = 4B \qquad B = 1$

$$\dfrac{4}{(x+1)(x-3)} = \dfrac{-1}{(x+1)} + \dfrac{1}{(x-3)}$$

Final answer:

$$\dfrac{x^3 - x^2 - 5x - 7}{x^2 - 2x - 3} = x + 1 + \dfrac{1}{(x+1)} - \dfrac{1}{(x-3)}$$

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(c) $\quad \dfrac{x^3 - 5x^2 + 11x - 12}{x^2 - 5x + 6} = x + \dfrac{5x - 12}{(x - 2)(x - 3)}$

Proper Rational Fraction part

$$\dfrac{5x - 12}{(x - 2)(x - 3)} = \dfrac{A}{(x - 2)} + \dfrac{B}{(x - 3)} \qquad 5x - 12 = A(x - 3) + B(x - 2)$$

$x = 2 \quad -2 = -A \quad A = 2$

$x = 3 \quad 3 = B \quad B = 3$

$$\dfrac{5x - 12}{(x - 2)(x - 3)} = \dfrac{2}{(x - 2)} + \dfrac{3}{(x - 3)}$$

Final answer:

$$\dfrac{x^3 - 5x^2 + 11x - 12}{x^2 - 5x + 6} = x + \dfrac{2}{(x - 2)} + \dfrac{3}{(x - 3)}$$

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(d) $\dfrac{2x^2 - 7}{x^2 - 4} = 2 + \dfrac{1}{(x+2)(x-2)}$

Proper Rational Fraction part

$$\dfrac{1}{(x+2)(x-2)} = \dfrac{A}{(x+2)} + \dfrac{B}{(x-2)} \qquad 1 = A(x-2) + B(x+2)$$

$x = -2 \quad 1 = -4A \qquad A = -\dfrac{1}{4}$

$x = 2 \qquad 1 = 4B \qquad B = \dfrac{1}{4}$

$$\dfrac{1}{(x+2)(x-2)} = -\dfrac{1}{4(x+2)} + \dfrac{1}{4(x-2)}$$

Final answer:

$$\dfrac{2x^2 - 7}{x^2 - 4} = 2 - \dfrac{1}{4(x+2)} + \dfrac{1}{4(x-2)}$$

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(e) $\dfrac{x^3 - 3x}{x^2 - x - 2} = x + 1 + \dfrac{2}{(x+1)(x-2)}$

Proper Rational Fraction part

$\dfrac{2}{(x+1)(x-2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-2)}$ $2 = A(x-2) + B(x+1)$

$x = -1 \quad 2 = -3A \quad A = -\dfrac{2}{3}$

$x = 2 \quad 1 = 3B \quad B = \dfrac{2}{3}$

$\dfrac{2}{(x+1)(x-2)} = -\dfrac{2}{3(x+1)} + \dfrac{2}{3(x-2)}$

Final answer:

$$\dfrac{x^3 - 3x}{x^2 - x - 2} = x + 1 - \dfrac{2}{3(x+1)} + \dfrac{2}{3(x-2)}$$

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(f) $\dfrac{x^2}{(x-1)^2} = 1 + \dfrac{2x-1}{(x-1)^2}$

Proper Rational Fraction part

$\dfrac{2x-1}{(x-1)^2} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-1)^2}$ $2x - 1 = A(x-1) + B$

$x = 1$ $1 = B$ $\boldsymbol{B = 1}$

$x = 0$ $-1 = -A + 1$ $\boldsymbol{A = 2}$

Simultaneous Equations:

$$\boldsymbol{A = 2} \quad \boldsymbol{B = 1}$$

$$\dfrac{2x-1}{(x+1)(x-2)} = \dfrac{2}{(x-1)} + \dfrac{1}{(x-1)^2}$$

Final answer:

$$\dfrac{x^2}{(x-1)^2} = 1 + \dfrac{2}{(x-1)} + \dfrac{1}{(x-1)^2}$$

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