www.mathsrevision.com

Advanced Higher Maths

Unit 1.2 Differentiation Solutions Part 1

Mr. Lafferty BSC (Hons) Open MathSci & Mrs Bissett BSC (Hons) Glasgow

Ex1 First Principles (Higher Revision)

$(a)\quad f(x) = x^3 \qquad f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = \mathbf{3x^2}$$

$(b)\quad f(x) = x^2 + 2x \qquad f(x+h) = (x+h)^2 + 2(x+h) = x^2 + 2xh + h^2 + 2x + 2h$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x^2 + 2xh + h^2 + 2x + 2h) - x^2 + 2x}{h} = \lim_{h\to 0} 2x + h + 2 = \mathbf{2x+2}$$

$(c)\quad f(x) = 3x^2 + 4x - 5 \qquad f(x+h) = 3(x+h)^2 + 4(x+h) - 5 = 3x^2 + 6xh + 3h^2 + 4x + 4h - 5$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(3x^2 + 6xh + 3h^2 + 4x + 4h - 5) - (3x^2 + 4x - 5)}{h}$$

$$= \lim_{h\to 0} 6x + 3h + 4 = \mathbf{6x+4}$$

Ex2 Higher Revision

1. $f(x) = x^3 - x^2 + 5x - 6$ $\quad f'(x) = 3x^2 - 2x + 5$

2. $f(x) = 3x^2 + 7 - \frac{4}{x} = 3x^2 + 7 - 4x^{-1}$ $\quad f'(x) = 6x + 4x^{-2} = 6x + \frac{4}{x^2}$

3. $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ $\quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$

4. $f(x) = x^{\frac{3}{2}} - x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ $\quad f'(x) = \frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$

5. $f(x) = \frac{1}{x^2} - \frac{1}{x^3} = x^{-2} - x^{-3}$ $\quad f'(x) = -2x^{-3} + 3x^{-4} = \frac{-2}{x^3} + \frac{3}{x^4}$

6. $f(x) = \frac{\sqrt{x}}{x^2} + \frac{x^2}{\sqrt{x}} = x^{-\frac{3}{2}} + x^{\frac{3}{2}}$ $\quad f'(x) = -\frac{3}{2}x^{-\frac{5}{2}} + \frac{3}{2}x^{\frac{1}{2}} = -\frac{3}{2\sqrt{x^5}} + \frac{3}{2}\sqrt{x}$

7. $f(x) = (4x+5)^5$ $\quad$ *Chain Rule* $\quad f'(x) = 5(4x+5)^4 \times 4 = 20(4x+5)^4$

8. $f(x) = (2x^4 - 3)^{\frac{1}{2}}$ $\quad$ *Chain Rule* $\quad f'(x) = \frac{1}{2}(2x^4 - 3)^{-\frac{1}{2}} \times 8x^3 = 4x^3(2x^4 - 3)^{-\frac{1}{2}}$

$$= \frac{4x^3}{\sqrt{(2x^4 - 3)}}$$

9. $f(x) = \frac{3}{\sqrt{(4-x^2)}} = 3(4-x^2)^{-\frac{1}{2}}$ $\quad$ *Chain Rule* $\quad f'(x) = -\frac{3}{2}(4-x^2)^{-\frac{3}{2}} \times (-2x) = 3x(4-x^2)^{-\frac{3}{2}}$

$$= \frac{3x}{\sqrt{(4-x^2)^3}}$$

Common factor
3cancelled down

10. $f(x) = \frac{4}{(x^3+3x)^{\frac{1}{3}}} = 4(x^3+3x)^{-\frac{1}{3}}$ $\quad$ *Chain Rule* $\quad$ $f'(x) = -\frac{4}{3}(x^3+3x)^{-\frac{4}{3}} \times (3x^2+3) = -\frac{4(x^2+1)}{\sqrt[3]{(x^3+3x)^4}}$

11. $f(x) = cos^3x = (cosx)^3$ $\quad$ *Chain Rule* $\quad$ $f'(x) = 3(cosx)^2 \times (-sinx) = -3sinxcos^2x$

12. $f(x) = \sqrt{sinx} = (sinx)^{\frac{1}{2}}$ $\quad$ *Chain Rule* $\quad$ $f'(x) = \frac{1}{2}(sinx)^{-\frac{1}{2}} \times (cosx) = \frac{cosx}{2\sqrt{sinx}}$

Chain Rule Ex3 method $\left(\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right)$

1. $y = (x^2 + 4x - 5)^3$ $\quad u = x^2 + 4x - 5$ $\quad \frac{du}{dx} = 2x + 4$ $\quad y = u^3$ $\quad \frac{dy}{du} = 3u^2$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 \times (2x + 4) = (6x + 12)(x^2 + 4x - 5)^2$$

---

2. $y = \sqrt{x^3 + 5} = (x^3 + 5)^{\frac{1}{2}}$ $\quad u = x^3 + 5$ $\quad \frac{du}{dx} = 3x^2$ $\quad y = u^{\frac{1}{2}}$ $\quad \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times 3x^2 = \frac{3}{2}x^2(x^3 + 5)^{-\frac{1}{2}} = \frac{3x^2}{2\sqrt{(x^3 + 5)}}$$

---

3. $y = (1 + 2\sqrt{x})^4 = (1 + 2x^{\frac{1}{2}})^4$ $\quad u = 1 + 2x^{\frac{1}{2}}$ $\quad \frac{du}{dx} = x^{-\frac{1}{2}}$ $\quad y = u^4$ $\quad \frac{dy}{du} = 4u^3$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 4u^3 \times x^{-\frac{1}{2}} = 4x^{-\frac{1}{2}}(1 + 2x^{\frac{1}{2}})^3 = \frac{4(1 + 2\sqrt{x})^3}{\sqrt{x}}$$

---

4. $y = \frac{3}{\sqrt{(4-x^2)}} = 3(4 - x^2)^{-\frac{1}{2}}$ $\quad u = 4 - x^2$ $\quad \frac{du}{dx} = -2x$ $\quad y = 3u^{-\frac{1}{2}}$ $\quad \frac{dy}{du} = -\frac{3}{2}u^{-\frac{3}{2}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{3}{2}u^{-\frac{3}{2}} \times -2x = 3x(4 - x^2)^{-\frac{3}{2}} = \frac{3x}{\sqrt{(4 - x^2)^3}}$$

Ex4 Differentiating a Product method $\left(\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}\right)$

1. $y = x^2(x-3)^2$ $\quad u = x^2 \quad \frac{du}{dx} = 2x \quad v = (x-3)^2 \quad \frac{dv}{dx} = 2x - 6$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = (x-3)^2 \times 2x + x^2 \times (2x-6) = 2x(x-3)^2 + x^2(2x-6)$$

$$= 2x(x-3)^2 + 2x^2(x-3) = 2x(x-3)(2x-3)$$

---

2. $y = x(2x+3)^3$ $\quad u = x \quad \frac{du}{dx} = 1 \quad v = (2x+3)^3 \quad \frac{dv}{dx} = 6(2x+3)^2$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = (2x+3)^3 \times 1 + x \times 6(2x+3)^2 = (2x+3)^2(2x+3+6x)$$

$$= (8x+3)(2x+3)^2$$

---

3. $y = x\sqrt{(x-6)}$ $\quad u = x \quad \frac{du}{dx} = 1 \quad v = (x-6)^{\frac{1}{2}} \quad \frac{dv}{dx} = \frac{1}{2}(x-6)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = (x-6)^{\frac{1}{2}} \times 1 + x \times \frac{1}{2}(x-6)^{-\frac{1}{2}} = (x-6)^{\frac{1}{2}} + \frac{x}{2(x-6)^{\frac{1}{2}}}$$

$$= \frac{2(x-6)+x}{2(x-6)^{\frac{1}{2}}} = \frac{3(x-4)}{2\sqrt{x-6}}$$

---

4. $y = \sqrt{x}(x-3)^3$ $\quad u = x^{\frac{1}{2}}$ $\quad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad v = (x-3)^3$ $\quad \frac{dv}{dx} = 3(x-3)^2$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = (x-3)^3 \times \frac{1}{2}x^{-\frac{1}{2}} + x^{\frac{1}{2}} \times 3(x-3)^2 = \frac{(x-3)^3}{2x^{\frac{1}{2}}} + 3x^{\frac{1}{2}}(x-3)^2$$

$$= \frac{(x-3)^2(x-3+6x)}{2\sqrt{x}} = \frac{(x-3)^2(7x-3)}{2\sqrt{x}}$$

---

5. $y = (x+1)^2(x-1)^4$

$u = (x+1)^2$ $\quad \frac{du}{dx} = 2x+2$ $\quad v = (x-1)^4$ $\quad \frac{dv}{dx} = 4(x-1)^3$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = v = (x-1)^4 \times (2x+2) + (x+1)^2 \times 4(x-1)^3$$

$$= (x-1)^3[(x-1)(2x+2) + 4(x+1)^2]$$

$$= (x-1)^3(2x^2 - 2 + 4x^2 + 8x + 4)$$

$$= (x-1)^3(6x^2 + 8x + 2)$$

$$= 2(x-1)^3(3x^2 + 4x + 1)$$

$$= 2(x-1)^3(3x+1)(x+1)$$

---

6. $y = x^3\sqrt{(x-1)}$ $\quad u = x^3$ $\quad \frac{du}{dx} = 3x^2$ $\quad v = (x-1)^{\frac{1}{2}}$ $\quad \frac{dv}{dx} = \frac{1}{2}(x-1)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = (x-1)^{\frac{1}{2}} \times 3x^2 + x^3 \times \frac{1}{2}(x-1)^{-\frac{1}{2}} = 3x^2(x-1)^{\frac{1}{2}} + \frac{x^3}{2(x-1)^{\frac{1}{2}}}$$

$$= \frac{6x^2(x-1) + x^3}{2(x-1)^{\frac{1}{2}}} = \frac{x^2(7x-6)}{2\sqrt{(x-1)}}$$

---

7. $y = xsinx \qquad u = x \quad \frac{du}{dx} = 1 \qquad v = sinx \qquad \frac{dv}{dx} = cosx$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = sinx \times 1 + x \times cosx = sinx + xcosx$$

---

8. $y = x^2sinx \qquad u = x^2 \quad \frac{du}{dx} = 2x \qquad v = sinx \qquad \frac{dv}{dx} = cosx$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = sinx \times 2x + x^2 \times cosx = 2xsinx + x^2cosx$$

$$= x(2sinx + xcosx)$$

---

9. $y = sinxcosx \qquad u = sinx \quad \frac{du}{dx} = cosx \qquad v = cosx \qquad \frac{dv}{dx} = -sinx$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = cosx \times cosx + sinx \times -sinx = cos^2x - sin^2x$$

$$= cos2x$$

---

10. $y = sin2xcos5x \qquad u = sin2x \quad \frac{du}{dx} = 2cos2x \qquad v = cos5x \qquad \frac{dv}{dx} = -5sin5x$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = cos5x \times 2cos2x + sin2x \times -5sin5x = 2cos2xcos5x - 5sin2xsin5x$$

Ex5 Differentiating a Quotient method $\frac{d}{dx}\left(\frac{u}{v}\right) = \left(\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right)$

1. $y = \frac{x^2}{x+3}$ $\quad u = x^2 \quad \frac{du}{dx} = 2x \quad v = x + 3 \quad \frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x+3) \times 2x - x^2 \times 1}{(x+3)^2} = \frac{2x(x+3) - x^2}{(x+3)^2}$$

$$= \frac{x^2 + 6x}{(x+3)^2} = \frac{x(x+6)}{(x+3)^2}$$

---

2. $y = \frac{4-x}{x^2}$ $\quad u = 4 - x \quad \frac{du}{dx} = -1 \quad v = x^2 \quad \frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{x^2 \times (-1) - (4-x) \times 2x}{x^4} = \frac{-x^2 - 8x + 2x^2}{x^4}$$

$$= \frac{x^2 - 8x}{x^4} = \frac{x(x-8)}{x^4} = \frac{x-8}{x^3}$$

---

3. $y = \frac{4x}{(1-x)^3}$ $\quad u = 4x \quad \frac{du}{dx} = 4 \quad v = (1-x)^3 \quad \frac{dv}{dx} = -3(1-x)^2$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(1-x)^3 \times 4 - 4x \times -3(1-x)^2}{(1-x)^6} = \frac{4(1-x)^3 + 12x(1-x)^2}{(1-x)^6}$$

$$= \frac{(1-x)^2[4 - 4x + 12x]}{(1-x)^6} = \frac{4 + 8x}{(1-x)^4} = \frac{4(2x+1)}{(1-x)^4}$$

---

4. $y = \frac{2x^2}{x-2}$ $\quad u = 2x^2 \quad \frac{du}{dx} = 4x \quad v = x - 2 \quad \frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x-2) \times 4x - 2x^2 \times 1}{(x-2)^2} = \frac{4x^2 - 8x - 2x^2}{(x-2)^2}$$

$$= \frac{2x^2 - 8x}{(x-2)^2} = \frac{2x(x-4)}{(x-2)^2}$$

---

5. $y = \frac{(1-2x)^3}{x^3}$ $\quad u = (1-2x)^3 \quad \frac{du}{dx} = -6(1-2x)^2 \quad v = x^3 \quad \frac{dv}{dx} = 3x^2$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{x^3 \times -6(1-2x)^2 - (1-2x)^3 \times 3x^2}{x^6} = \frac{(1-2x)^2(-6x^3 - 3x^2 + 6x^3)}{x^6}$$

$$= \frac{-3x^2(1-2x)^2}{x^6} = \frac{-3(1-2x)^2}{x^4}$$

---

6. $y = \frac{\sqrt{(x+1)}}{x^2}$ $\quad u = (x+1)^{\frac{1}{2}}$ $\quad \frac{du}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}}$ $\quad v = x^2$ $\quad \frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{x^2 \times \frac{1}{2}(x+1)^{-\frac{1}{2}} - (x+1)^{\frac{1}{2}} \times 2x}{x^4} = \frac{\frac{x^2}{2(x+1)^{\frac{1}{2}}} - 2x(x+1)^{\frac{1}{2}}}{x^4}$$

$$= \frac{\frac{x^2 - 4x(x+1)}{2(x+1)^{\frac{1}{2}}}}{x^4} = \frac{-3x^2 - 4x}{2x^4(x+1)^{\frac{1}{2}}} = \frac{-x(3x+4)}{2x^4(x+1)^{\frac{1}{2}}} = -\frac{(3x+4)}{2x^3\sqrt{(x+1)}}$$

Ex6 Mixed New Functions with Chain, Product and Quotient Rules

1. $y = tan^3 2x = (tan2x)^3$ $u = tan2x$ $\frac{du}{dx} = 2sec^2 2x$ $y = u^3$ $\frac{dy}{du} = 3u^2$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 \times 2sec^2 2x = 6tan^2 2xsec^2 2x$$

---

2. $y = -2cosec^4 x$ $u = cosecx$ $\frac{du}{dx} = -cosecxcotx$ $y = u^4$ $\frac{dy}{du} = 4u^3$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -2[4u^3 \times -cosecxcotx] = -2[4cosecx^3 \times -cosecxcotx] = 8cosecx^4 cotx$$

---

3. $y = secxtanx$ $u = secx$ $\frac{du}{dx} = secxtanx$ $v = tanx$ $\frac{dv}{dx} = sec^2 x$

$$\left(\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}\right) = tanx \times secxtanx + secx \times sec^2 x = secx(tan^2 x + sec^2 x)$$

---

4. $y = x^2 cotx$ $u = x^2$ $\frac{du}{dx} = 2x$ $v = cotx$ $\frac{dv}{dx} = -cosec^2 x$

$$\left(\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}\right) = cotx \times 2x + x^2 \times -cosec^2 x = 2xcotx - x^2 cosec^2 x = x(2cotx - xcosec^2 x)$$

---

5. $y = \ln(3x + 2)$ Rule $\frac{d}{dx}\{ln[f(x)]\} = f'(x) \times \frac{1}{f(x)}$ $f(x) = (3x + 2)$ $f'(x) = 3$

$$= 3 \times \frac{1}{(3x + 2)} = \frac{3}{(3x + 2)}$$

6. $y = (x+2)e^{-x}$ $u = (x+2)$ $\frac{du}{dx} = 1$ $v = e^{-x}$ $\frac{dv}{dx} = -e^{-x}$

$$\left(\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}\right) = e^{-x} \times 1 + (x+2) \times -e^{-x} = -e^{-x}(1+x) = -(x+1)e^{-x}$$

---

7. $y = \frac{e^x}{x+2}$ $u = e^x$ $\frac{du}{dx} = e^x$ $v = x+2$ $\frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x+2) \times e^x - e^x \times 1}{(x+2)^2} = \frac{(x+1)e^x}{(x+2)^2}$$

---

8. $y = \frac{x^2}{lnx}$ $u = x^2$ $\frac{du}{dx} = 2x$ $v = lnx$ $\frac{dv}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{lnx \times 2x - x^2 \times \frac{1}{x}}{(lnx)^2} = \frac{2xlnx - x}{(lnx)^2} = \frac{x(2lnx - 1)}{(lnx)^2}$$

---

9. $y = \ln\left[\sqrt{(x^2+1)}\right]$ $Rule\ \frac{d}{dx}\{ln[f(x)]\} = f'(x) \times \frac{1}{f(x)}$

$$f(x) = (x^2+1)^{\frac{1}{2}} \qquad f'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x = \frac{x}{\sqrt{(x^2+1)}}$$

$$= \frac{x}{\sqrt{(x^2+1)}} \times \frac{1}{\sqrt{(x^2+1)}} = \frac{x}{(x^2+1)}$$

---

10. $y = xe^{-2x^2}$ $\quad u = x \quad \frac{du}{dx} = 1 \quad v = e^{-2x^2} \left[\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}\right] \quad \frac{dv}{dx} = -4xe^{-2x^2}$

$$\left(\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}\right) = e^{-2x^2} \times 1 + x \times -4xe^{-2x^2} = e^{-2x^2}(1-4x^2)$$

---

11. $y = \ln\left(\frac{1+x}{1-x}\right)$ $\quad Rule \ \frac{d}{dx}\{ln[f(x)]\} = f'(x) \times \frac{1}{f(x)}$

$$f(x) = \left(\frac{1+x}{1-x}\right) \qquad u = (1+x) \quad \frac{du}{dx} = 1 \qquad v = \ (1-x) \qquad \frac{dv}{dx} = -1$$

$$f'(x) = \left(\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right) = \frac{(1-x)\times 1 - (1+x)\times -1}{(1-x)^2} = \frac{(1-x)+(1+x)}{(1-x)^2} = \frac{2}{(1-x)^2}$$

$$= \frac{2}{(1-x)^2} \times \frac{1}{\frac{(1+x)}{(1-x)}} == \frac{2}{1-x^2}$$

---

12. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $\quad u = (e^x - e^{-x}) \quad \frac{du}{dx} = (e^x + e^{-x}) \quad v = \ (e^x + e^{-x}) \quad \frac{dv}{dx} = (e^x - e^{-x})$

$$\left(\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right) = \frac{(e^x + e^{-x}) \times (e^x + e^{-x}) - (e^x - e^{-x}) \times (e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}$$
