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Advanced Higher Maths

Unit 1.2 Differentiation Solutions Part 2

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Ex1 Inverse Trig method $\left(\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right)$

1. $y = sin^{-1}\sqrt{x}$ $\quad u = x^{\frac{1}{2}}$ $\quad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad y = sin^{-1}u$ $\quad \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \times \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^{\frac{1}{2}^2}}} \times \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(1-x)}}$$

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2. $y = tan^{-1}\sqrt{x}$ $\quad u = x^{\frac{1}{2}}$ $\quad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad y = tan^{-1}u$ $\quad \frac{dy}{du} = \frac{1}{1+u^2}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{1+u^2} \times \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{(1+x)} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)}$$

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3. $y = xtan^{-1}x$ $\quad u = x$ $\quad \frac{du}{dx} = 1$ $\quad v = tan^{-1}x$ $\quad \frac{dv}{dx} = \frac{1}{1+x^2}$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = tan^{-1}x \times 1 + x \times \frac{1}{1+x^2} = tan^{-1}x + \frac{x}{1+x^2}$$

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4. $y = xtan^{-1}\left(\frac{x}{2}\right)$ $\quad u = x$ $\quad \frac{du}{dx} = 1$ $\quad v = tan^{-1}\left(\frac{x}{2}\right)$ $\quad \frac{dv}{dx} = \frac{1}{1+\left(\frac{x}{2}\right)^2} \times \frac{1}{2} = \frac{4}{2(4+x^2)} = \frac{2}{(4+x^2)}$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = tan^{-1}\left(\frac{x}{2}\right) \times 1 + x \times \frac{2}{(4+x^2)} = tan^{-1}\left(\frac{x}{2}\right) + \frac{2x}{4+x^2}$$

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5. $y = xsin^{-1}x + \sqrt{1-x^2}$

First part Product

$$u = x \qquad \frac{du}{dx} = 1 \qquad v = sin^{-1}x \qquad \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = sin^{-1}x \times 1 + x \times \frac{1}{\sqrt{1-x^2}} = sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$$

Second part

$$f(x) = \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}} \qquad f'(x) = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x = \frac{-x}{\sqrt{1-x^2}}$$

Final answer:

$$sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = sin^{-1}x$$

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6. $y = cos^{-1}(2x-1)$ method $\left(\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right)$

$$u = 2x-1 \qquad \frac{du}{dx} = 2 \qquad y = cos^{-1}u \qquad \frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{-1}{\sqrt{1-u^2}} \times 2 = \frac{-2}{\sqrt{1-u^2}} = \frac{-2}{\sqrt{1-(2x-1)^2}} = \frac{-2}{\sqrt{1-(4x^2-4x+1)}}$$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{(4x-4x^2)}} = \frac{-1}{\sqrt{(x-x^2)}} = \frac{-1}{\sqrt{x(1-x)}}$$

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7. $y = \sin^{-1}\left(\frac{x-1}{x+1}\right)$ method $\left(\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right)$

$$u = \frac{x-1}{x+1} \qquad \frac{du}{dx} = see\ below \qquad y = sin^{-1}u \qquad \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$f(x) = \left(\frac{x-1}{x+1}\right) \qquad u = (x-1) \quad \frac{du}{dx} = 1 \qquad v = \ (x+1) \qquad \frac{dv}{dx} = 1$$

$$f'(x) = \left(\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right) = \frac{(x+1)\times 1 - (x-1)\times 1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} \times \frac{2}{(x+1)^2} = \frac{(x+1)}{\sqrt{(x+1)^2-(x-1)^2}} \times \frac{2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{(x+1)}{\sqrt{(x^2+2x+1)-(x^2-2x+1)}} \times \frac{2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{(x+1)}{\sqrt{4x}} \times \frac{2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}(x+1)}$$

8. $y = \tan^{-1}\left(\frac{x-1}{x+1}\right)$ method $\left(\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right)$

$$u = \frac{x-1}{x+1} \qquad \frac{du}{dx} = see\ below \qquad y = tan^{-1}u \qquad \frac{dy}{du} = \frac{1}{1+u^2}$$

$$f(x) = \left(\frac{x-1}{x+1}\right) \qquad u = (x-1) \quad \frac{du}{dx} = 1 \qquad v = (x+1) \qquad \frac{dv}{dx} = 1$$

$$f'(x) = \left(\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right) = \frac{(x+1)\times 1 - (x-1)\times 1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{(1+u^2)} \times \frac{2}{(x+1)^2} = \frac{1}{1+\left(\frac{x-1}{x+1}\right)^2} = \frac{(x+1)^2}{(x+1)^2+(x-1)^2} \times \frac{2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{2}{(x^2+2x+1)+(x^2-2x+1)}$$

$$\frac{dy}{dx} = \frac{2}{2x^2+2} = \frac{1}{1+x^2}$$

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9. $y = \tan^{-1}\left(\frac{2x}{\sqrt{1-x^2}}\right)$ method $\left(\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right)$

$$u = \frac{2x}{\sqrt{1-x^2}} \qquad \frac{du}{dx} = see\ below \qquad y = tan^{-1}u \qquad \frac{dy}{du} = \frac{1}{1+u^2}$$

$$f(x) = \left(\frac{2x}{\sqrt{1-x^2}}\right) \qquad u = 2x \quad \frac{du}{dx} = 2 \qquad v = (1-x^2)^{\frac{1}{2}} \qquad \frac{dv}{dx} = -x(1-x^2)^{-\frac{1}{2}}$$

$$f'(x) = \left(\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right) = \frac{(1-x^2)^{\frac{1}{2}} \times 2 - 2x \times -x(1-x^2)^{-\frac{1}{2}}}{(1-x^2)} = \frac{2(1-x^2)^{\frac{1}{2}} + 2x^2(1-x^2)^{-\frac{1}{2}}}{(1-x^2)}$$

$$= \frac{\frac{2(1-x^2)+2x^2}{(1-x^2)^{\frac{1}{2}}}}{(1-x^2)} = \frac{2}{(1-x^2)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{1}{(1+u^2)} \times \frac{2}{(1-x^2)^{\frac{3}{2}}} = \frac{1}{1+\left(\frac{2x}{\sqrt{1-x^2}}\right)^2} \times \frac{2}{(1-x^2)^{\frac{3}{2}}} = \frac{(1-x^2)}{(1-x^2+4x^2)} \times \frac{2}{(1-x^2)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{2}{(1+3x^2)\sqrt{(1-x^2)}}$$

10. $y = \sin^{-1}\left(\frac{2x}{\sqrt{1-x^2}}\right)$ method $\left(\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right)$

$$u = \frac{2x}{\sqrt{1-x^2}} \qquad \frac{du}{dx} = see\ below \qquad y = sin^{-1}u \qquad \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$f(x) = \left(\frac{2x}{\sqrt{1-x^2}}\right) \qquad u = 2x \quad \frac{du}{dx} = 2 \qquad v = \ (1-x^2)^{\frac{1}{2}} \qquad \frac{dv}{dx} = -x(1-x^2)^{-\frac{1}{2}}$$

$$\frac{du}{dx} = f'(x) = \left(\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right) = \frac{(1-x^2)^{\frac{1}{2}} \times 2 - 2x \times -x(1-x^2)^{-\frac{1}{2}}}{(1-x^2)} = \frac{2(1-x^2)^{\frac{1}{2}} + 2x^2(1-x^2)^{-\frac{1}{2}}}{(1-x^2)}$$

$$= \frac{\frac{2(1-x^2) + 2x^2}{(1-x^2)^{\frac{1}{2}}}}{(1-x^2)} = \frac{2}{(1-x^2)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \times \frac{2}{(1-x^2)^{\frac{3}{2}}} = \frac{1}{\sqrt{1-\left(\frac{2x}{\sqrt{1-x^2}}\right)^2}} \times \frac{2}{(1-x^2)^{\frac{3}{2}}} = \frac{(1-x^2)^{\frac{1}{2}}}{\sqrt{(1-x^2-4x^2)}} \times \frac{2}{(1-x^2)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{(1-5x^2)(1-x^2)}}$$

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11. $y = \tan^{-1}(secx)$ method $\left(\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right)$

$$u = secx \qquad \frac{du}{dx} = secxtanx \qquad y = tan^{-1}u \qquad \frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{1+u^2} \times secxtanx = \frac{1}{1+secx^2} \times secxtanx = \frac{secxtanx}{1+secx^2}$$

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12. $y = \cos(sin^{-1}x)$ method $\left(\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right)$

$$u = sin^{-1}x \qquad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad y = cosu \qquad \frac{dy}{du} = -sinu$$

Inverse of each other ie like doing nothing!

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -sinu \times \frac{1}{\sqrt{1-x^2}} = -\sin(sin^{-1}x) \times \frac{1}{\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

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13. $y = tan^{-1}(e^x)$ method $\left(\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right)$

$$u = e^x \qquad \frac{du}{dx} = e^x \qquad y = tan^{-1}u \qquad \frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{1+u^2} \times e^x = \frac{1}{1+(e^x)^2} \times e^x = \frac{e^x}{1+e^{2x}}$$

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14. $y = sin^{-1}(x^2) - xe^x$ $\quad$ *Chain Rule and Product* $\left(\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right)$

First Part $y = sin^{-1}(x^2)$

$$u = x^2 \qquad \frac{du}{dx} = 2x \qquad y = sin^{-1}u \qquad \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \times 2x = \frac{1}{\sqrt{1-(x^2)^2}} \times 2x = \frac{2x}{\sqrt{1-x^4}}$$

Second Part $y = xe^x$

$$y = xe^x$$

$$u = x \qquad \frac{du}{dx} = 1 \qquad v = e^x \qquad \frac{dv}{dx} = e^x$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = e^x \times 1 + x \times e^x = e^x(1+x)$$

Final Answer

$$\frac{dy}{dx} = \frac{2x}{\sqrt{1-x^4}} - e^x(1+x)$$

Ex2 – Differentiating a function defined Implicitly

1. $x^2 - y^2 = 0$ $\quad$ *differentiating implicitly* $\quad 2x - 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x}{y}$

(b) $y^2 = 2x + 2y \quad 2y\frac{dy}{dx} = 2 + 2\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx}(2y-2) = 2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2}{(2y-2)} = \frac{1}{(y-1)}$

(c) $xy^2 = 9 \quad Product \quad y^2 \times 1 + x \times 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-y^2}{2xy} = \frac{-y}{2x}$

(d) $4x^2 - y^3 + 2x + 3y = 0 \quad 8x - 3y^2\frac{dy}{dx} + 2 + 3\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx}(3y^2 - 3) = \frac{8x+2}{(3y^2-3)} = \frac{2(4x+1)}{3(y^2-1)}$

(e) $x^3y + xy^3 = x - y \quad \Rightarrow \quad \left(3x^2y + x^3\frac{dy}{dx}\right) + (y^3 \times 1 + x \times 3y^2\frac{dy}{dx}) = 1 - \frac{dy}{dx}$

$$3x^2y + y^3 + x^3\frac{dy}{dx} - 3xy^2\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx}(x^3 + 3xy^2 + 1) = 1 - 3x^2y - y^3$$

$$\frac{dy}{dx} = \frac{1 - 3x^2y - y^3}{(x^3 + 3xy^2 + 1)}$$

(f) $sinxcosy = 1 \quad Product \quad \Rightarrow \quad cosy \times cosx + sinx \times -siny\frac{dy}{dx} = 0$

$$cosycosx - sinxsiny\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = \frac{cosycosx}{sinxsiny} = cotxcoty$$

(g) $e^{xy} = 2 \quad Chain\ Rule\ and\ Product \quad \Rightarrow \quad e^{xy} \times \left(y \times 1 + x\frac{dy}{dx}\right) = 0$

$$ye^{xy} + xe^{xy}\frac{dy}{dx} = 0$$

$$ye^{xy} + xe^{xy}\frac{dy}{dx} = \frac{-ye^{xy}}{xe^{xy}} = \frac{-y}{x}$$

(h) $e^x lny = x \quad Product \quad \Rightarrow \quad lny \times e^x + e^x\frac{1}{y}\frac{dy}{dx} = 1$

$$lny\ e^x + e^x\frac{1}{y}\frac{dy}{dx} = 1$$

$$lny\ e^x + e^x\frac{1}{y}\frac{dy}{dx} = \frac{y(1 - lny\ e^x)}{e^x} = ye^{-x} - ylny$$

Q2. $x^3 - 2y^3 = 3xy \qquad at\ the\ point\ \ (2,1)$

$$3x^2 - 6y^2\frac{dy}{dx} = y \times 3 + 3x\frac{dy}{dx}$$

$$\frac{dy}{dx}(6y^2 + 3x) = 3x^2 - 3y$$

$$\frac{dy}{dx} = \frac{3x^2 - 3y}{(6y^2 + 3x)}$$

$For\ x = 2\ and\ y = 1 \qquad then \qquad \frac{dy}{dx} = \frac{3(2)^2 - 3(1)}{(6(1)^2 + 3(2))} = \frac{9}{12} = \frac{3}{4}$

$Using \quad y - b = (x - a) \quad \Rightarrow \quad y - 1 = \frac{3}{4}\ (x - 2) = 3x - 4y = 2$

(b). $x^2y^2 = x^2 + 5y^2$ *at the point* $\left(3, \frac{3}{2}\right)$ *Product*

$$y^2 \times 2x + x^2 \times 2y\frac{dy}{dx} = 2x + 10y\frac{dy}{dx}$$

$$2xy^2 + 2x^2y\frac{dy}{dx} = 2x + 10y\frac{dy}{dx}$$

$$\frac{dy}{dx}(2x^2y - 10y) = 2x - 2xy^2$$

$$\frac{dy}{dx} = \frac{2x - 2xy^2}{(2x^2y - 10y)}$$

For $x = 3$ *and* $y = \frac{3}{2}$ *then* $\frac{dy}{dx} = \frac{2(3) - 2(3)\left(\frac{3}{2}\right)^2}{2(3)^2\left(\frac{3}{2}\right) - 10\left(\frac{3}{2}\right)} = \frac{-\frac{15}{2}}{12} = -\frac{5}{8}$

Using $y - b = (x - a) \quad \Rightarrow \quad y - \frac{3}{2} = -\frac{5}{8}(x - 3) = 5x + 8y = 27$

(c). $y\,(x+y)^2 = 3(x^3 - 5) \quad at\ the\ point\ \ (2,1) \qquad LHS\ Product$

$$(x+y)^2 \times \frac{dy}{dx} + y \times 2(x+y)(1+\frac{dy}{dx}) = 9x^2$$

$$(x+y)^2\frac{dy}{dx} + 2y(x+y+x\frac{dy}{dx}+y\frac{dy}{dx}) = 9x^2$$

$$(x+y)^2\frac{dy}{dx} + 2xy + 2y^2 + 2xy\frac{dy}{dx} + 2y^2\frac{dy}{dx} = 9x^2$$

$$\frac{dy}{dx}((x+y)^2 + 2xy + 2y^2) = \frac{9x^2-2xy-2y^2}{((x+y)^2+2xy+2y^2)}$$

$For\ x = 2\ and\ y = 1 \quad then \qquad \frac{dy}{dx} = \frac{9(2)^2-2(2)(1)-2(1)^2}{((2+1)^2+2(2)(1)+2(1)^2)} = \frac{30}{15} = 2$

$Using \quad y - b = (x-a) \quad \Rightarrow \quad y - 1 = 2\,(x-2) = 2x - y = 3$

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3. $xy(x+y) = 84 \quad \textit{at the point } (3,4) \quad \textit{LHS Product}$

$$x^2y + xy^2 = 84$$

$$\left(y \times 2x + x^2 \times \frac{dy}{dx}\right) + \left(y^2 \times 1 + x \times 2y\frac{dy}{dx}\right) = 0$$

$$\left(2xy + x^2\frac{dy}{dx}\right) + \left(y^2 + 2xy\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2$$

$$\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$$

$$For\ x = 3\ and\ y = 4 \quad then \quad \frac{dy}{dx} = \frac{-2(3)(4) - (4)^2}{(3)^2 + 2(3)(4)} = \frac{-24 - 16}{9 + 24} = \frac{-40}{33}$$

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4. $x^2 + 3xy + y^2 = x + y + 8 \quad \textit{at the point } (1,2)$

$$2x + \left(y \times 3 + 3x \times \frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$2x + 3y + 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$+3x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 2x + 3y$$

$$\frac{dy}{dx}(3x + 2y - 1) = 1 - 2x - 3y$$

$$\frac{dy}{dx} = \frac{1-2x-3y}{(3x+2y-1)}$$

$For\ x = 1\ and\ y = 2 \quad then \quad \frac{dy}{dx} = \frac{dy}{dx} = \frac{1-2(1)-3(2)}{(3(1)+2(2)-1)} = \frac{-7}{6}$

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5. $4x^2 + y^3 = 2x + 7y \quad at\ the\ point\ \ (-1,2)$

$$8x + 3y^2\frac{dy}{dx} = 2 + 7\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 7\frac{dy}{dx} = 2 - 8x$$

$$\frac{dy}{dx}(3y^2 - 7) = 2 - 8x$$

$$\frac{dy}{dx} = \frac{2-8x}{(3y^2-7)} \qquad For\ \ x = -1\ \ and\ \ y = 2 \qquad \boldsymbol{\frac{dy}{dx} = \frac{2-8(-1)}{(3(2)^2-7)} = \frac{10}{5} = 2}$$

Second Derivative from

$$\left(\frac{dy}{dx} \times 6y\frac{dy}{dx} + 3y^2\frac{d^2y}{dx^2}\right) - \left(7\frac{d^2y}{dx^2}\right) = -8$$

$$\left(\frac{dy}{dx} \times 6y\frac{dy}{dx} + 3y^2\frac{d^2y}{dx^2}\right) - \left(7\frac{d^2y}{dx^2}\right) = -8$$

$$6y\left(\frac{dy}{dx}\right)^2 + 3y^2\frac{d^2y}{dx^2} - 7\frac{d^2y}{dx^2} = -8 \qquad \Rightarrow \qquad \frac{d^2y}{dx^2}(3y^2 - 7) = -8 - 6\left(\frac{dy}{dx}\right)^2$$

$$\boldsymbol{\frac{d^2y}{dx^2} = \frac{-8-6y\left(\frac{dy}{dx}\right)^2}{(3y^2-7)} = \frac{-8-6(2)(2)^2}{(3(2)^2-7)} = \frac{-56}{5}}$$

6. $3x^2 + 2xy - 5y^2 + 16y = 0$ $\quad Show\ (-2,3) and\ (0,0) are\ stationary\ points.$

$$6x + \left(y \times 2 + 2x\frac{dy}{dx}\right) - 10y\frac{dy}{dx} + 16\frac{dy}{dx} = 0$$

$$6x + 2y + 2x\frac{dy}{dx} - 10y\frac{dy}{dx} + 16\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2x - 10y + 16) = -6x - 2y$$

$$\frac{dy}{dx} = \frac{-6x-2y}{(2x-10y+16)} \quad if\ stationary\ then\ \frac{dy}{dx} = 0$$

$For\ (0,0)$ $\frac{dy}{dx} = \frac{-6x-2y}{(2x-10y+16)} = \frac{0}{16} = 0$ $\quad For\ (-1,3)$ $\frac{dy}{dx} = \frac{-6(-1)-2(3)}{(2(-1)-10(3)+16)} = \frac{0}{-16} = 0$

Second Derivative

$$6 + 2\frac{dy}{dx} + \left(\frac{dy}{dx} \times 2 + 2x \times \frac{d^2y}{dx^2}\right) - \left(\frac{dy}{dx} \times 10\frac{dy}{dx} + 10y \times \frac{d^2y}{dx^2}\right) + 16\frac{d^2y}{dx^2} = 0$$

$$6 + 4\frac{dy}{dx} + 2x\frac{d^2y}{dx^2} - 10\left(\frac{dy}{dx}\right)^2 - 10y\frac{d^2y}{dx^2} + 16\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}(2x - 10y + 16) = -6 - 4\frac{dy}{dx} + 10\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-6 - 4\frac{dy}{dx} + 10\left(\frac{dy}{dx}\right)^2}{(2x - 10y + 16)}$$

$For\ (0,0)$ $\quad \frac{d^2y}{dx^2} = \frac{-6-4(0)+10(0)^2}{(2(0)-10(0)+16)} = \frac{-6}{16}$ $\quad Hence\ Max\ TP$

$For\ (-1,3)$ $\quad \frac{d^2y}{dx^2} = \frac{-6 - 4(0) + 10(0)^2}{(2(-1) - 10(-3) + 16)} = \frac{6}{44}$ $\quad Hence\ Mini\ TP$

Ex3 Context Questions

1. $P = (2m+3)^4$ $\quad Find\ \frac{dm}{dt}\ \ when\ m = 1,\ given\ that\ \ \frac{dP}{dt} = 2$

$Using\ implicit\ technique\ :\quad \frac{dP}{dt} = 4(2m+3)^3 \times\ 2 \times \frac{dm}{dt} = 8(2m+3)^3\frac{dm}{dt}$

$$m = 1 \qquad 2 = 8(2(1)+3)^3\frac{dm}{dt}$$

$$\frac{dm}{dt} = \frac{2}{1000}$$

$$\frac{dm}{dt} = \frac{1}{500}$$

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2. $r = \frac{1+p}{1+p^2}$ $\quad Find\ \frac{dp}{dt}\ \ when\ p = 2,\ given\ that\ \ \frac{dr}{dt} = 14$

$$\frac{dr}{dt} = \frac{(1+p^2)\times\frac{dP}{dt} - (1+p)\times 2p\frac{dP}{dt}}{(1+p^2)^2}$$

$$p = 2 \qquad 14 = \frac{(1+(2)^2)\times\frac{dP}{dt} - (1+2)\times 2(2)\frac{dP}{dt}}{(1+(2)^2)^2}$$

$$350 = -7\frac{dP}{dt}$$

$$\frac{dp}{dt} = -50$$

$$\frac{dm}{dt} = \frac{25}{-7}\times 14 = \ -50$$

---

3. $A = \pi r^2$ $\quad \frac{dA}{dt}$ $when\, r = 10,\ given\ that$ $\frac{dr}{dt} = 0.2$

$using\ implicit\ differentiation\ technique$: $\qquad \frac{dA}{dt} = 2\pi r \times \frac{dr}{dt}$

$r = 10 \qquad \frac{dA}{dt} = 2\pi(10) \times 0.2$

$$\frac{dA}{dt} = 20\pi \times 0.2 = 4\pi$$

4a. $xy = 40\ ;\ y = \frac{40}{x}$ $\qquad using\ implicit\ \ differentiation\ technique$:

$$\frac{dy}{dt} = -40x^{-2} \times \frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{40}{x^2} \times \frac{dx}{dt}$$

4b. $Find\ \frac{dy}{dt}\ when\ x = 8,\ given\ that\ \frac{dx}{dt} = 0.2$

$$\frac{dy}{dt} = -\frac{40}{(8)^2} \times 0.2 = -\frac{1}{8}\ cm/sec$$

5a. $50 = \pi r^2 h \quad ; \; h = \frac{50}{\pi r^2} \qquad using\ implicit\ differentiation\ technique$

$$\frac{dh}{dt} = \frac{-100r^{-3}}{\pi} \times \frac{dr}{dt}$$

$$\frac{dh}{dt} = \frac{-100}{\pi r^3} \times \frac{dr}{dt}$$

5b. $\frac{dr}{dt}\ when\ r = 8,\ given\ that \quad \frac{dh}{dt} = 3$

$$\frac{dh}{dt} = \frac{-100}{\pi r^3} \times \frac{dr}{dt}$$

$$3 = \frac{-100}{\pi (5)^3} \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{3\pi(5)^3}{-100} = -11.8\ cm/sec$$

---

6. $v = s^2 + 3$ $\quad$ *using implicit differentiation technique*

$$\frac{dv}{dt} = 2s \times \frac{ds}{dt} \qquad \left(v = \frac{ds}{dt} = s^2 + 3\right)$$

$$\frac{dv}{dt} = 2s(s^2 + 3) \qquad a = \frac{dv}{dt}$$

$$a = 2s(s^2 + 3)\ cm/sec^2$$

---

7. $V = \frac{4}{3}\pi r^3$ $\quad \frac{dV}{dr} = 4\pi r^2 \quad \frac{dV}{dt} = 200 \quad r = 100$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 200 = 4\pi r^2 \frac{dr}{dt} \qquad \Rightarrow \frac{dr}{dt} = \frac{50}{\pi r^2}$$

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt} = 8\pi r\left(\frac{50}{\pi r^2}\right)$$

$$\frac{dS}{dt} = 8\pi r\left(\frac{50}{\pi r^2}\right) = \frac{400}{r} = \frac{400}{100} = 4cm^2/sec$$

Ex4 Log Differentiation

1. $y = 10^x$ $\qquad lny = xln10$

$$\frac{1}{y}\frac{dy}{dx} = ln10 \quad \Rightarrow \quad \frac{dy}{dx} = yln10 \quad \Rightarrow \quad \frac{dy}{dx} = 10^x ln10$$

---

2. $y = 2^{x^2}$ $\qquad lny = x^2 ln2$

$$\frac{1}{y}\frac{dy}{dx} = 2xln2 \quad \Rightarrow \quad \frac{dy}{dx} = y2xln2 \quad \Rightarrow \quad \frac{dy}{dx} = y = 2^{x^2} \times 2xln2$$

---

3. $y = x^{-x}$ $\qquad lny = -xlnx$

$$\frac{1}{y}\frac{dy}{dx} = lnx \times (-1) + (-x) \times \frac{1}{x} \quad \Rightarrow \quad \frac{dy}{dx} = y(-lnx - 1\,) \quad \Rightarrow \quad \frac{dy}{dx} = -x^{-x}(lnx + 1\,)$$

---

4. $y = x^{sinx}$ $\qquad lny = sinxlnx$

$$\frac{1}{y}\frac{dy}{dx} = lnx \times cosx + sinx \times \frac{1}{x} \quad \Rightarrow \quad \frac{dy}{dx} = y\left(lnxcosx + \frac{sinx}{x}\right) \quad \Rightarrow \quad \frac{dy}{dx} = x^{sinx}\left(lnxcosx + \frac{sinx}{x}\right)$$

---

5. $y = x^{\frac{1}{x}} = x^{x^{-1}}$ $\qquad lny = x^{-1}lnx$

$\frac{1}{y}\frac{dy}{dx} = lnx \times (-x^{-2}) + (x^{-1}) \times \frac{1}{x} \quad \Rightarrow \frac{dy}{dx} = y\left(-\frac{1}{x^2}lnx + \frac{1}{x^2}\right) \quad \Rightarrow \frac{dy}{dx} = \frac{x^{\frac{1}{x}}}{x^2}(1 - lnx)$

$$\frac{dy}{dx} = x^{\frac{1}{x}-2}(1 - lnx)$$

---

6. $y = x^{lnx}$ $\qquad lny = lnx \times lnx$

$\frac{1}{y}\frac{dy}{dx} = lnx \times \frac{1}{x} + lnx \times \frac{1}{x} \quad \Rightarrow \frac{dy}{dx} = y\left(\frac{1}{x}lnx + \frac{1}{x}lnx\right) \quad \Rightarrow \frac{dy}{dx} = x^{lnx}\left(\frac{1}{x}lnx + \frac{1}{x}lnx\right)$

$$\frac{dy}{dx} = x^{lnx}\,(2x^{-1}lnx)$$

$$\frac{dy}{dx} = 2x^{lnx-1}(lnx)$$

---

7. $y = (lnx)^x$ $\qquad lny = x \times ln(l\,n(x))$

$\frac{1}{y}\frac{dy}{dx} = ln(lnx) \times (1) + x \times \frac{1}{lnx} \times \frac{1}{x} \quad \Rightarrow \frac{dy}{dx} = y\left(ln(lnx) + \frac{x}{xlnx}\right)$

$$\Rightarrow \quad \frac{dy}{dx} = (lnx)^x\left(ln(lnx) + \frac{x1}{lnx}\right)$$

---

8. $y = x^{sinx}$ $\qquad lny = sinxlnx$

$\frac{1}{y}\frac{dy}{dx} = lnx \times cosx + sinx \times \frac{1}{x} \quad \Rightarrow \frac{dy}{dx} = y\left(lnxcosx + \frac{sinx}{x}\right) \quad \Rightarrow \frac{dy}{dx} = x^{sinx}\left(lnxcosx + \frac{sinx}{x}\right)$

---

9. $y = \frac{x^5}{\sqrt{(3x+5)}}$ $u = x^5$ $\frac{du}{dx} = 5x^4$ $v = (3x+5)^{\frac{1}{2}}$ $\frac{dv}{dx} = \frac{3}{2}(3x+5)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \left(\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right) = \frac{(3x+5)^{\frac{1}{2}} \times 5x^4 - x^5 \times \frac{3}{2}(3x+5)^{-\frac{1}{2}}}{(3x+5)} = \frac{5x^4(3x+5) - \frac{3}{2}x^5}{(x+1)^{\frac{3}{2}}}$$

$$= \frac{15x^5 + 25x^4 - \frac{3}{2}x^5}{(x+1)^{\frac{3}{2}}}$$

$$= \frac{30x^5 + 50x^4 - 3x^5}{2(x+1)^{\frac{3}{2}}}$$

$$= \frac{30x^5 + 50x^4 - 3x^5}{2(x+1)^{\frac{3}{2}}}$$

$$= \frac{x^4(27x+50)}{2(x+1)^{\frac{3}{2}}}$$

Log method !!!. $y = \frac{x^5}{\sqrt{(3x+5)}}$ $\Rightarrow lny = ln\left(\frac{x^5}{\sqrt{(3x+5)}}\right) = ln\, x^5 - \frac{1}{2}ln(3x+5)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{5x^4}{x^5} - \frac{1}{2}\frac{3}{(3x+5)}$$

$$\frac{dy}{dx} = \frac{x^5}{(3x+5)^{\frac{1}{2}}}\left(\frac{5x^4}{x^5} - \frac{1}{2}\frac{3}{(3x+5)}\right)$$

$$= \frac{5x^4}{(3x+5)^{\frac{1}{2}}} - \frac{3x^5}{2(3x+5)^{\frac{3}{2}}}$$

$$= \frac{10x^4(3x+5)^{\frac{3}{2}} - 3x^5(3x+5)^{\frac{1}{2}}}{(3x+5)^{\frac{1}{2}} \times 2(3x+5)^{\frac{3}{2}}} = \frac{(3x+5)^{\frac{1}{2}}(10x^4(3x+5) - 3x^5)}{(3x+5)^{\frac{1}{2}} \times 2(3x+5)^{\frac{3}{2}}}$$

$$= \frac{(10x^4(3x+5)+3x^5)}{(3x+5)^{\frac{3}{2}}} = \frac{30x^5+50x^4-3x^5}{(3x+5)^{\frac{3}{2}}} = \frac{x^4(27x+50)}{2(x+1)^{\frac{3}{2}}}$$

---

10. $y = \frac{x^3(2x-1)^5}{(x+1)^2}$

$$u = \boldsymbol{x^3(2x-1)^5} \quad \frac{du}{dx} = (2x-1)^5 \times 3x^2 + x^3 \times 10(2x-1)^4 = \boldsymbol{3x^2(2x-1)^5 + 10x^3(2x-1)^4}$$

$$v = (\boldsymbol{x+1})^{\boldsymbol{2}} \qquad \frac{dv}{dx} = 2(x+1)$$

$$\frac{dy}{dx} = \left(\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right) = \frac{(x+1)^2 \times (3x^2(2x-1)^5 + 10x^3(2x-1)^4) - x^3(2x-1)^5 \times 2(x+1)}{(x+1)^4}$$

$$= \frac{(x+1)(2x-1)^4[(3x^2(x+1)(2x-1) + 10x^3(x+1) - 2x^3(2x-1)]}{(x+1)^4}$$

$$= \frac{(2x-1)^4[6x^4 + 3x^3 - 3x^2 + 10x^4 + 10x^3 - 4x^4 + 2x^3]}{(x+1)^3}$$

$$= \frac{(2x-1)^4[12x^4 + 15x^3 - 3x^2]}{(x+1)^3}$$

$$= \frac{3x^2(2x-1)^4(4x^2+5x-1)}{(x+1)^3}$$

Log method !!!. $y = \frac{x^3(2x-1)^5}{(x+1)^2}$ $\quad \Rightarrow lny = ln\left(\frac{x^3(2x-1)^5}{(x+1)^2}\right) = 3ln\, x + 5ln(2x-1) - 2ln(x+1)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} + \frac{10}{(2x-1)} - \frac{2}{(x+1)}$$

$$\frac{dy}{dx} = y\left(\frac{3(2x-1)(x+1) + 10x(x+1) - 2x(2x-1)}{x(2x-1)(x+1)}\right)$$

$$\frac{dy}{dx} = y\left(\frac{6x^2 + 3x - 3 + 10x^2 + 10x - 4x^2 + 2x)}{x(2x-1)(x+1)}\right)$$

$$\frac{dy}{dx} = y\left(\frac{12x^2 + 15x - 3}{x(2x-1)(x+1)}\right) = \frac{x^3(2x-1)^5}{(x+1)^2} \times \left(\frac{12x^2 + 15x - 3}{x(2x-1)(x+1)}\right)$$

$$\frac{dy}{dx} = \frac{3x^3(2x-1)^5}{(x+1)^2}\left(\frac{4x^2+5x-1}{x(2x-1)(x+1)}\right) = \frac{3x^2(2x-1)^4(4x^2+5x-1)}{(x+1)^3}$$

Ex 5 Parametric Differentiation

1. $x = t^3 + t^2$ $y = t^2 + t$ $\frac{dx}{dt} = 3t^2 + 2t$ $\frac{dy}{dt} = 2t + 1$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = (2t+1) \times \frac{1}{3t^2 + 2t} = \frac{(2t+1)}{t(3t+2)}$$

---

b. $x = 4cos\theta$ $y = 3sin\theta$ $\frac{dx}{dt} = -4sin\theta$ $\frac{dy}{dt} = 3cos\theta$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 3cos\theta \times \frac{1}{-4sin\theta} = -\frac{3}{4}cot\theta$$

---

c. $x = \frac{1}{(1+t)}$ $y = \frac{t}{(1-t)}$ $\frac{dx}{dt} = \frac{-1}{(1+t)^2}$ $\frac{dy}{dt} = \frac{(1-t)\times 1 - t\times(-1)}{(1-t)^2} = \frac{1-t+t}{(1-t)^2} = \frac{1}{(1-t)^2}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1}{(1-t)^2} \times \frac{(1+t)^2}{-1} = -\frac{(1+t)^2}{(1-t)^2}$$

---

d. $x = \frac{(t-1)}{(t+1)}$ $y = \frac{(2t-1)}{(t-2)}$

$$\frac{dx}{dt} = \frac{(t+1)\times 1 - (t-1)\times 1}{(t+1)^2} = \frac{2}{(t+1)^2}$$

$$\frac{dy}{dt} = \frac{(t-2)\times 2 - (2t-1)\times 1}{(t-2)^2} = \frac{-3}{(t-2)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-3}{(t-2)^2} \times \frac{(t+1)^2}{2} = -\frac{3}{2}\frac{(t+1)^2}{(t-2)^2}$$

e. $x = (t+1)^2$ $y = (t^2 - 1)$ $\frac{dx}{dt} = 2(t+1)$ $\frac{dy}{dt} = 2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \times \frac{1}{2(t+1)} = \frac{t}{(t+1)}$$

---

f. $x = \frac{t}{(1-t)}$ $y = \frac{t^2}{(t+3)}$

$$\frac{dx}{dt} = \frac{(1-t)\times 1 - t \times (-1)}{(1-t)^2} = \frac{1}{(1-t)^2}$$

$$\frac{dy}{dt} = \frac{(t+3)\times 2t - t^2 \times 1}{(t+3)^2} = \frac{t^2 + 6t}{(t+3)^2} = \frac{t(t+6)}{(t+3)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{t(t+6)}{(t+3)^2} \times \frac{(1-t)^2}{1} = \frac{t(t+6)(1-t)^2}{(t+3)^2}$$

---

g. $x = cos2\theta$ $y = 4sin\theta$ $\frac{dx}{dt} = -2sin2\theta$ $\frac{dy}{dt} = 4cos\theta$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 4cos\theta \times \frac{1}{-2sin2\theta} = \frac{4cos\theta}{-4sin\theta cos\theta} = -cosec\theta$$

---

h. $x = acos^2\theta \qquad y = asin^3\theta$

$$\frac{dx}{d\theta} = 2acos\theta \times -sin\theta = -asin2\theta \qquad \frac{dy}{d\theta} = 3asin^2\theta cos\theta = \frac{3}{2}asin2\theta sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{3}{2a}asin2\theta sin\theta \times \frac{1}{-asin2\theta} = -\frac{\mathbf{3}}{\mathbf{2}}\boldsymbol{sin\theta}$$

---

i. $x = e^t cost \qquad y = e^t sint$ Product

$$\frac{dx}{dt} = cost \times e^t - e^t \times sint = e^t(cost - sint) \qquad \frac{dy}{dt} = sint \times e^t + e^t \times cost = e^t(cost + sint)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = e^t(cost + sint) \times \frac{1}{e^t(cost - sint)} = \frac{(cost + sint)}{(cost - sint)}$$

---

j. $x = a(t - cost) \qquad y = a(1 + sint)$

$$\frac{dx}{dt} = a + asint \qquad \frac{dy}{dt} = acost$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = acost \times \frac{1}{a + asint} = \frac{acost}{a(1 + sint)} = \frac{cost}{(1 + sint)}$$

2a. $x = ct \quad y = \frac{c}{t} = ct^{-1} \quad \textit{Find tangent equation at } P(x, y) = \left(ct, \frac{c}{t}\right)$

$$\frac{dx}{dt} = c \qquad \frac{dy}{dt} = -ct^{-2} = -\frac{c}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{c}{t^2} \times \frac{1}{c} = -\frac{1}{t^2}$$

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$x + t^2 y = 2ct$$

---

b. $x = at^2 \quad y = at(t^2 - 1) \quad \textit{Find tangent equation at } P(x, y) = (at^2, at(t^2 - 1))$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = a(t^2 - 1) + 2at^2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{a(t^2 - 1) + 2at^2}{2at} = \frac{(t^2 - 1) + 2t^2}{2t} = \frac{3t^2 - 1}{2t}$$

$$y - at(t^2 - 1) = \frac{3t^2 - 1}{2t}(x - at^2)$$

$$2ty - (3t^2 - 1)x = 2at^2(t^2 - 1) - at^2(3t^2 - 1)$$

$$2ty - (3t^2 - 1)x = 2at^4 - 2at^2 - 3at^4 + at^2$$

$$2ty - (3t^2 - 1)x = -at^2 - at^4$$

$$2ty - (3t^2 - 1)x = -at^2(1 + t^2)$$

$$(3t^2 - 1)x - 2ty = at^2(1 + t^2)$$

c. $x = \frac{a}{2}(t + t^{-1}) \quad y = \frac{a}{2}(t - t^{-1}) \quad \textit{Find tangent equation at } P(x,y) = \left(\frac{a}{2}\left(t + \frac{1}{t}\right), \frac{a}{2}\left(t - \frac{1}{t}\right)\right)$

$$\frac{dx}{dt} = \frac{a}{2} - \frac{a}{2t^2} = \frac{a}{2}\left(1 - \frac{1}{t^2}\right) \qquad \frac{dy}{dt} = \frac{a}{2} + \frac{a}{2t^2} = \frac{a}{2}\left(1 + \frac{1}{t^2}\right)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\frac{a}{2}\left(1 + \frac{1}{t^2}\right)}{\frac{a}{2}\left(1 - \frac{1}{t^2}\right)} = \frac{\left(1 + \frac{1}{t^2}\right)}{\left(1 - \frac{1}{t^2}\right)} = \frac{\frac{(t^2+1)}{t^2}}{\frac{(t^2-1)}{t^2}} = \frac{(t^2+1)}{(t^2-1)}$$

$$y - \frac{a}{2}\left(t - \frac{1}{t}\right) = \frac{(t^2+1)}{(t^2-1)}\left(x - \frac{a}{2}\left(t + \frac{1}{t}\right)\right)$$

$$(t^2-1)y - \frac{a}{2}(t^2-1)\left(t - \frac{1}{t}\right) = (t^2+1)\left(x - \frac{a}{2}\left(t + \frac{1}{t}\right)\right)$$

$$(t^2-1)y - \frac{a}{2}(t^2-1)\left(t - \frac{1}{t}\right) = (t^2+1)x - \frac{a}{2}(t^2+1)\left(t + \frac{1}{t}\right)$$

$$(t^2+1)x - (t^2-1)y = \frac{a}{2}(t^2+1)\left(t + \frac{1}{t}\right) - \frac{a}{2}(t^2-1)\left(t - \frac{1}{t}\right)$$

$$(t^2+1)x - (t^2-1)y = \frac{a}{2}\left[\left(t^3 + t + t + \frac{1}{t}\right) - \left(t^3 - t - t + \frac{1}{t}\right)\right]$$

$$(t^2+1)x - (t^2-1)y = \frac{a}{2}(4t)$$

$$(t^2+1)x - (t^2-1)y = 2at$$

---

d. $x = sec\theta \quad y = tan\theta \quad Find\ tangent\ equation\ at\ P(x,y) = (sec\theta, tan\theta)$

$$\frac{dx}{dt} = sec\theta tan\theta \qquad \frac{dy}{dt} = sec^2\theta$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{sec^2\theta}{sec\theta tan\theta} = \frac{sec\theta}{tan\theta} = \frac{\frac{1}{cos\theta}}{\frac{sin\theta}{cos\theta}} = \frac{1}{sin\theta}$$

$$y - tan\theta = \frac{1}{sin\theta}(x - sec\theta)$$

$$sin\theta y - sin\theta tan\theta = x - sec\theta$$

$$x - sin\theta y = sin\theta tan\theta - sec\theta$$

$$x - sin\theta y = \frac{sin^2\theta}{cos\theta} - \frac{1}{cos\theta}$$

$$x - ysin\theta = \frac{sin^2\theta - 1}{cos\theta}$$

$$x - ysin\theta = \frac{cos^2\theta}{cos\theta}$$

$$x - sin\theta y = cos\theta$$

$$x\frac{1}{cos\theta} - ytan\theta = 1$$

$$xsec\theta - ytan\theta = 1$$

Ex6 Second Order Differentiation Parametric Equations

1a. $x = \frac{1}{t^2} = t^{-2} \quad y = 1 + t \quad Find\ \frac{dy}{dx}\ and\ \frac{d^2y}{dx^2}$

$$\frac{dx}{dt} = -2t^{-3} = \frac{-2}{t^3} \qquad \frac{dy}{dt} = 1$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 1 \times -\frac{t^3}{2} = -\boldsymbol{\frac{t^3}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{d}{dt}\left(-\frac{t^3}{2}\right) \times -\frac{t^3}{2}$$

$$= -\frac{3t^2}{2} \times -\frac{t^3}{2}$$

$$= \frac{3t^5}{4}$$

---

b. $x = (t+1)^2 \quad y = t^2 - 1 \quad Find\ \frac{dy}{dx}\ and\ \frac{d^2y}{dx^2}$

$$\frac{dx}{dt} = 2(t+1) \qquad \frac{dy}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \times \frac{1}{2(t+1)} = \frac{\boldsymbol{t}}{(t+1)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{d}{dt}\left(\frac{\boldsymbol{t}}{(t+1)}\right) \times \frac{1}{2(t+1)}$$

$$= \left[\frac{(t+1)\times 1 - t \times 1}{(t+1)^2}\right] \times \frac{1}{2(t+1)}$$

$$= \left[\frac{(t+1) - t}{(t+1)^2}\right] \times \frac{1}{2(t+1)}$$

$$= \frac{1}{2(t+1)^3}$$

---

c. $x = 4cost \quad y = 3sint \quad Find\ \frac{dy}{dx}\ and\ \frac{d^2y}{dx^2}$

$$\frac{dx}{dt} = -4sint \qquad \frac{dy}{dt} = 3cost$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 3cost \times \frac{-1}{4sint} = -\boldsymbol{\frac{3}{4}cot\ t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{d}{dt}\left(-\frac{3}{4}cot\ t\right) \times \frac{-1}{4sint}$$

$$= \left(\frac{3}{4}cosec^2t\right) \times \frac{-1}{4sint}$$

$$= -\boldsymbol{\frac{3}{16}cosec^3t}$$

---

d. $x = cos^2t \quad y = sint \quad Find\ \frac{dy}{dx}\ and\ \frac{d^2y}{dx^2}$

$$\frac{dx}{dt} = -2cost \times sint \qquad \frac{dy}{dt} = cost$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = cost \times \frac{1}{-2cost \times sint} = -\boldsymbol{\frac{1}{2}cosect}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{d}{dt}\left(-\boldsymbol{\frac{1}{2}cosec\ t}\right) \times \frac{1}{-2cost \times sint}$$

$$= \left(\frac{1}{2}cosect \times cott\right) \times \frac{1}{-2cost \times sint}$$

$$= \left(\boldsymbol{\frac{1}{2}}\frac{1}{sint} \times \frac{cost}{sint}\right) \times \frac{1}{-2cost \times sint}$$

$$= -\frac{1}{4}\boldsymbol{cosec^3t}$$

---

e. Miss out !

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2. $x = t - cost \quad y = sint \quad Find\ Stationary\ Points$

$$\frac{dx}{dt} = 1 + sint \qquad \frac{dy}{dt} = cost$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = cost \times \frac{1}{1 + sint} = \frac{cost}{1 + sint}$$

$For\ stationary\ point\ \frac{dy}{dx} = 0 \qquad \frac{cost}{1 + sint} = 0$

$$cost = 0$$

$$t = \frac{\pi}{2}\ \ and\ \ \frac{3\pi}{2}$$

$$t = \frac{\pi}{2}\ \ ;\ \ x = t - cost = \frac{\pi}{2} - cos\frac{\pi}{2} = \frac{\pi}{2} \qquad y = sint = sin\frac{\pi}{2} = 1 \qquad \left(\frac{\pi}{2}, 1\right)$$

$$t = \frac{3\pi}{2}\ \ ;\ \ x = t - cost = \frac{3\pi}{2} - cos\frac{3\pi}{2} = \frac{3\pi}{2} \qquad y = sint = sin\frac{3\pi}{2} = -1 \qquad \left(\frac{3\pi}{2}, -1\right)$$

3a. $x = 4 - t^2 \quad y = 4t - t^3 \quad \textit{Find Stationary Points and Nature}$

$$\frac{dx}{dt} = -2t \qquad \frac{dy}{dt} = 4 - 3t^2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = (4 - 3t^2) \times \frac{-1}{2t} = \frac{(3t^2 - 4)}{2t}$$

$$\textit{For stationary point } \frac{dy}{dx} = 0 \qquad \frac{(3t^2 - 4)}{2t} = 0$$

$$(3t^2 - 4) = 0$$

$$t = \sqrt{\frac{4}{3}} \quad and \quad -\sqrt{\frac{4}{3}}$$

$$t = \sqrt{\frac{4}{3}} \ ; \ x = 4 - t^2 = 4 - \frac{4}{3} = \frac{8}{3} \qquad y = 4t - t^3 = 4\left(\sqrt{\frac{4}{3}}\right) - \left(\sqrt{\frac{4}{3}}\right)^3 = \frac{16}{9}\sqrt{3} \qquad \left(\mathbf{\frac{8}{3}, \frac{16}{9}\sqrt{3}}\right)$$

$$t = -\sqrt{\frac{4}{3}} \ ; \ x = 4 - t^2 = 4 - \frac{4}{3} = \frac{8}{3} \qquad y = 4t - t^3 = 4\left(-\sqrt{\frac{4}{3}}\right) - \left(-\sqrt{\frac{4}{3}}\right)^3 = \frac{16}{9}\sqrt{3} \qquad \left(\mathbf{\frac{8}{3}, -\frac{16}{9}\sqrt{3}}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{d}{dt}\left(\frac{(3t^2 - 4)}{2t}\right) \times \frac{1}{-2t}$$

$$= \left(\frac{2t \times 6t - 3t^2 \times 2}{4t^2}\right) \times \frac{1}{-2t} == -\left(\mathbf{\frac{3}{4t}}\right)$$

$$t = \sqrt{\frac{4}{3}} \ ; \ \frac{d^2y}{dx^2} < 0 \qquad Max.TP \ at \qquad \left(\mathbf{\frac{8}{3}, \frac{16}{9}\sqrt{3}}\right)$$

$$t = \sqrt{\frac{4}{3}} \ ; \ \frac{d^2y}{dx^2} > 0 \qquad Mini.TP \ at \qquad \left(\mathbf{\frac{8}{3}, -\frac{16}{9}\sqrt{3}}\right)$$

---

b. $x = (5-3t)^2 \quad y = 6t - t^2 \quad \textit{Find Stationary Points and Nature}$

$$\frac{dx}{dt} = -6(5-3t) = 6(3t-5) \qquad \frac{dy}{dt} = (6-2t)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = (6-2t) \times \frac{1}{6(3t-5)} = \frac{(6-2t)}{6(3t-5)} = \frac{(3-t)}{3(3t-5)}$$

$$\textit{For stationary point } \frac{dy}{dx} = 0 \qquad \frac{(3-t)}{3(3t-5)} = 0$$

$$(3-t) = 0$$

$$\boldsymbol{t = 3}$$

$$t = 3 \;\; ; \;\; x = (5-3t)^2 = (5-3(3))^2 = 16 \qquad y = 6t - t^2 = 6(3) - (3)^2 = 9 \quad (\mathbf{16}, \mathbf{9})$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{d}{dt}\left(\frac{(3-t)}{3(3t-5)}\right) \times \frac{1}{6(3t-5)}$$

$$= \left(\frac{3(3t-5)\times(-1) - (3-t)\times 9}{9(3t-5)^2}\right) \times \frac{1}{6(3t-5)} = \left(\frac{\mathbf{-12}}{\mathbf{54(3t-5)^3}}\right)$$

$$t = 3 \;\; ; \;\; \frac{d^2y}{dx^2} = \left(\frac{18(3) - 12}{9(3(3)-5)^3}\right) < 0 \qquad \boldsymbol{Max.TP\ at} \quad (\mathbf{16}, \mathbf{9})$$

---

c. $x = (t^2 + 1) \quad y = t(t-3)^2 \quad$ *Find Stationary Points and Nature*

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = (t-3)^2 \times 1 + t \times 2(t-3) = (t-3)^2 + 2t(t-3) = (t-3)[t-3+2t] = 3(t-3)(t-1)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 3(t-3)(t-1) \times \frac{1}{2t} = \frac{3(t-3)(t-1)}{2t}$$

$\text{For stationary point } \frac{dy}{dx} = 0 \qquad \frac{3(t-3)(t-1)}{2t} = 0 \qquad \text{For later } \frac{3(t^2-4t+3)}{2t}$

$$(t-3)(t-1) = 0$$

$$\boldsymbol{t = 1 \text{ and } 3}$$

$t = 1 \;;\; x = (t^2+1) = ((1)^2+1) = 2 \qquad y = t(t-3)^2 = (1)((1)-3)^2 = 9 \qquad (\mathbf{2}, \mathbf{4})$

$t = 3 \;;\; x = (t^2+1) = ((3)^2+1) = 10 \qquad y = t(t-3)^2 = (3)((3)-3)^2 = 9 \qquad (\mathbf{10}, \mathbf{0})$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{d}{dt}\left(\frac{3(t^2-4t+3)}{2t}\right) \times \frac{1}{2t}$$

$$= \left(\frac{2t \times 3(3t-4) - 3(t^2-4t+3) \times 2}{4t^2}\right) \times \frac{1}{2t} = \left(\frac{12t^2-18}{8t^3}\right) = \boldsymbol{\frac{6t^2-9}{4t^3}}$$

$t = 1 \;;\; \frac{d^2y}{dx^2} = \left(\frac{6(1)^2-9}{4(1)^3}\right) < 0 \qquad \boldsymbol{Max. TP\ at} \qquad (\mathbf{2}, \mathbf{4})$

$t = 3 \;;\; \frac{d^2y}{dx^2} = \left(\frac{6(3)^2-9}{4(3)^3}\right) > 0 \qquad \boldsymbol{Mini. TP\ at} \qquad (\mathbf{10}, \mathbf{0})$

---

2. $x = t - sint \quad y = 1 - cost \quad Show\ y^2\frac{d^2y}{dx^2} + 1 = 0$

$$\frac{dx}{dt} = 1 - cost \qquad \frac{dy}{dt} = sint$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = sint \times \frac{1}{(1 - cost)} = \frac{sint}{(1 - cost)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{d}{dt}\left(\frac{sint}{(1 - cost)}\right) \times \frac{1}{(1 - cost)}$$

$$= \left(\frac{(1 - cost) \times cost - sint \times sint}{(1 - cost)^2}\right) \times \frac{1}{(1 - cost)} = \left(\frac{cost - cos^2t - sin^2t}{(1 - cost)^3}\right) = \frac{cost - 1}{(1 - cost)^3}$$

$$\frac{cost - 1}{(1 - cost)^3} = -\frac{(1 - cost)}{(1 - cost)^3} = \frac{-1}{(1 - cost)^2}$$

$$LHS = y^2\frac{d^2y}{dx^2} + 1 = (1 - cost\,)^2 \times \frac{-1}{(1 - cost)^2} + 1 = 0 = RHS$$

Ex 7 Parametric Differentiation – Motion in a Plane

. 1. $x = t(2-t) \quad y = t(3-t) \quad$ *Find the speed when* $t = 0$ *and when* $t = 2$

$$\frac{dx}{dt} = (2-t)\times 1 + t\times(-1) = 2(1-t) \qquad \frac{dy}{dt} = (3-t)\times 1 + t\times(-1) = (3-2t)$$

$$t = 0 \ ; \ \frac{dx}{dt} = 2(1-t) = 2(1-(0)) = \mathbf{2} \qquad \frac{dy}{dt} = (3-2t) = (3-2(0)) = \mathbf{3}$$

$$|\underline{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{4+9} = \sqrt{13}$$

$$t = 2 \ ; \ \frac{dx}{dt} = 2(1-t) = 2(1-(2)) = -\mathbf{2} \qquad \frac{dy}{dt} = (3-2t) = (3-2(2)) = -1$$

$$|\underline{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(4+1} = \sqrt{5}$$

---

2. $x = t + 1 \quad y = t^2 \quad$ *Find the speed when* $t = 2$

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 2t$$

$$t = 2 \ ; \ \frac{dx}{dt} = \mathbf{1} \qquad \frac{dy}{dt} = 2t = 2\times 2 = \mathbf{4}$$

$$|\underline{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1+16} = \sqrt{\mathbf{17}}$$

---

3. $x = cos2t \qquad y = 2sint \quad Find\ the\ speed\ when\ t = 0$

$$\frac{dx}{dt} = -2sin2t \qquad \frac{dy}{dt} = 2cost$$

$$t = 0 \ ; \ \frac{dx}{dt} = -2sin2t = -2sin2(0) = \mathbf{0} \qquad \frac{dy}{dt} = 2cost = 2\cos(0) = \mathbf{2}$$

$$|\underline{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{0+4} = \mathbf{2}$$

---

4. $x = e^t \qquad y = e^{-2t} \quad Find\ the\ speed\ when\ t = ln3$

$$\frac{dx}{dt} = e^t \qquad \frac{dy}{dt} = -2e^{-2t}$$

$$t = ln3 \ ; \ \frac{dx}{dt} = e^t = e^{ln3} = \mathbf{3} \qquad \frac{dy}{dt} = -2e^{-2t} = -2e^{-2ln3} = -\mathbf{\frac{2}{9}}$$

$$|\underline{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3)^2 + \left(-\frac{2}{9}\right)^2} = \mathbf{\frac{\sqrt{733}}{9}}$$

---

5. $x = sect \qquad y = tant \qquad Find\ the\ speed\ when\ t = \frac{\pi}{6}$

$$\frac{dx}{dt} = sect\ tant \qquad \frac{dy}{dt} = sec^2 t$$

$$t = \frac{\pi}{6}\ ;\ \frac{dx}{dt} = sect\ tant = sec\frac{\pi}{6}\ tan\frac{\pi}{6} = \mathbf{\frac{2}{3}} \qquad \frac{dy}{dt} = sec^2 t = sec^2\frac{\pi}{6} = \mathbf{\frac{4}{3}}$$

$$|\underline{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2} = \frac{\sqrt{20}}{3} = \mathbf{\frac{2\sqrt{5}}{3}}$$

---

6. $x = ln(t+1) \qquad y = t^2 \qquad Find\ the\ speed\ when\ t = 1$

$$\frac{dx}{dt} = \frac{1}{(t+1)} \qquad \frac{dy}{dt} = 2t$$

$$t = 1\ ;\ \frac{dx}{dt} = \frac{1}{(t+1)} = \frac{1}{((1)+1)} = \mathbf{\frac{1}{2}} \qquad \frac{dy}{dt} = 2t = 2t = \mathbf{2}$$

$$|\underline{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(\frac{1}{2}\right)^2 + (2)^2} = \frac{17}{4} = \mathbf{\frac{\sqrt{17}}{2}}$$

---

7. $x = 4cost \quad y = 3sint \quad Find\ Max\ and\ Mini\ speed\ and\ corresponding\ postions.$

$$\frac{dx}{dt} = -4sint \qquad \frac{dy}{dt} = 3cost$$

$$|\underline{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-4sint)^2 + (3cost)^2} = \sqrt{16sin^2t + 9cos^2t}$$

$$= \sqrt{9sin^2t + 9cos^2t + 7sin^2t}$$

$$= \sqrt{9(sin^2t + cos^2t) + 7sin^2t}$$

$$= \sqrt{\mathbf{9+7}\boldsymbol{sin^2t}} = \boldsymbol{as\ required}$$

$$For\ stationay\ points\ \frac{dx}{dt} = 0 \qquad \frac{dx}{dt} = -4sint = 0 \qquad t = 0\ \ and\ \pi$$

$$For\ stationay\ points\ \frac{dy}{dt} = 0 \qquad \frac{dy}{dt} = 3cost = 0 \qquad t = \frac{\pi}{2}\ \ and\ \frac{3\pi}{2}$$

$$For\ \ t = 0\ ; x = 4cost = 4cos(0) = 0 \qquad y = 3sint = 3sin(0) = 0 \qquad (\mathbf{4}, \mathbf{0})$$

$$For\ \ t = 0\ ; x = 4cost = 4cos(\pi) = -4 \qquad y = 3sint = 3sin(0) = 0 \qquad (-\mathbf{4}, \mathbf{0})$$

$$|\underline{v}| = \sqrt{9+7sin^2t} = \sqrt{9+7sin^2(0)} = \sqrt{9} = 3$$

$$|\boldsymbol{\underline{v}}| = \sqrt{\mathbf{9+7}\boldsymbol{sin^2t}} = \sqrt{\mathbf{9+7}\boldsymbol{sin^2(\pi)}} = \sqrt{\mathbf{9}} = \mathbf{3}$$

$$For\ \ t = \frac{\pi}{2}\ ; x = 4cost = 4cos\left(\frac{\pi}{2}\right) = 0 \qquad y = 3sint = 3sin\left(\frac{\pi}{2}\right) = 3 \qquad (\mathbf{0}, \mathbf{3})$$

$$For\ \ t = \frac{3\pi}{2}\ ; x = 4cost = 4cos\left(\frac{3\pi}{2}\right) = 0 \qquad y = 3sint = 3sin\left(\frac{3\pi}{2}\right) = 3 \qquad (\mathbf{0}, -\mathbf{3})$$

$$|\underline{v}| = \sqrt{9+7sin^2t} = \sqrt{9+7sin^2\left(\frac{\pi}{2}\right)} = \sqrt{16} = 4$$

$$|\boldsymbol{\underline{v}}| = \sqrt{\mathbf{9+7}\boldsymbol{sin^2t}} = \sqrt{\mathbf{9+7}\boldsymbol{sin^2}\left(\frac{\mathbf{3}\boldsymbol{\pi}}{\mathbf{2}}\right)} = \sqrt{\mathbf{16}} = \mathbf{4}$$
