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Advanced Higher Maths

Unit 1.3 Integration Solutions Part 2

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Ex1 Substitution

1. $\int \frac{1}{\sqrt{49-x^2}}dx = \int \frac{1}{\sqrt{7^2-x^2}}dx = sin^{-1}\left(\frac{x}{7}\right) + c$

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2. $\int \frac{1}{49+x^2}dx = \int \frac{1}{7^2+x^2}dx = \frac{1}{7}tan^{-1}\left(\frac{x}{7}\right) + c$

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3. $\int \frac{1}{\sqrt{9-x^2}}dx = \int \frac{1}{\sqrt{3^2-x^2}}dx = sin^{-1}\left(\frac{x}{3}\right) + c$

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4. $\int \frac{1}{100+x^2}dx = \int \frac{1}{10^2+x^2}dx = \frac{1}{10}tan^{-1}\left(\frac{x}{10}\right) + c$

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5. $\int \frac{1}{\sqrt{(36-25x^2)}}dx = \int \frac{1}{\sqrt{25\left(\frac{36}{25}-x^2\right)}}dx = \int \frac{1}{5\sqrt{\left(\left(\frac{6}{5}\right)^2-x^2\right)}}dx = \frac{1}{5}sin^{-1}\left(\frac{5x}{6}\right) + c$

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6. $\int \frac{1}{(36+25x^2)}dx = \int \frac{1}{25\left(\frac{36}{25}+x^2\right)}dx = \int \frac{1}{25\left(\left(\frac{6}{5}\right)^2+x^2\right)}dx = \frac{1}{25}\times\frac{5}{6}tan^{-1}\left(\frac{5x}{6}\right) + c = \frac{1}{30}tan^{-1}\left(\frac{5x}{6}\right) + c$

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7. $\int \frac{2}{(25+4x^2)}dx = 2\int \frac{1}{4\left(\frac{25}{4}+x^2\right)}dx = \frac{1}{2}\int \frac{1}{4\left(\left(\frac{5}{2}\right)^2+x^2\right)}dx = \frac{1}{2}\times\frac{2}{5}tan^{-1}\left(\frac{2x}{5}\right) + c = \frac{1}{5}tan^{-1}\left(\frac{2x}{5}\right) + c$

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8. $\int \frac{3}{\sqrt{(36-9x^2)}}dx = 3\int \frac{1}{\sqrt{9\left(\frac{36}{9}-x^2\right)}}dx = \int \frac{1}{\sqrt{((2)^2-x^2)}}dx = sin^{-1}\left(\frac{x}{2}\right) + c$

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9. $\int_1^{\sqrt{3}} \frac{2}{(1+x^2)}dx = 2[tan^{-1}(x)]_1^{\sqrt{3}} = 2\left[[tan^{-1}(\sqrt{3})] - [tan^{-1}(1)]\right] = 2\left[\frac{\pi}{3}-\frac{\pi}{4}\right] = \frac{\pi}{6}$

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10. $\int_0^{\sqrt{2}} \frac{1}{\sqrt{(4-x^2)}}dx = \int_0^{\sqrt{2}} \frac{1}{\sqrt{((2)^2-x^2)}}dx = \left[sin^{-1}\left(\frac{x}{2}\right)\right]_0^{\sqrt{2}} = \left[sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right] - [sin^{-1}(0)] = \left[\frac{\pi}{4} - 0\right] = \frac{\pi}{4}$

11. $\int_{\frac{1}{2}}^{1} \frac{3}{\sqrt{(1-x^2)}}\,dx = 3[sin^{-1}(x)]_{\frac{1}{2}}^{1} = 3\left[[sin^{-1}(1)] - \left[sin^{-1}\left(\frac{1}{2}\right)\right]\right] = 3\left[\frac{\pi}{2} - \frac{\pi}{6}\right] = \pi$

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12. $\int_0^3 \frac{1}{(9+x^2)}\,dx = \int_0^3 \frac{1}{(3^2+x^2)}\,dx = \frac{1}{3}\left[tan^{-1}\left(\frac{x}{3}\right)\right]_0^3 = \frac{1}{3}\left[[tan^{-1}(1)] - [tan^{-1}(0)]\right] = \frac{1}{3}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{12}$

Ex 2 Integration by Partial Fractions

1. $\int \frac{(x+8)}{(x+2)(x+4)} dx$

$$\frac{x+8}{(x+2)(x+4)} = \frac{A}{(x+2)} + \frac{B}{(x+4)} \qquad x+8 = A(x+4) + B(x+2)$$

$x = -2 \quad 6 = 2A \qquad A = 3$

$x = -4 \quad 4 = -2B \qquad B = -2$

$$\int \frac{(x+8)}{(x+2)(x+4)} dx = \int \frac{3}{(x+2)} - \frac{2}{(x+4)} dx$$

$$= 3ln|x+2| - 2\ln|x+2| + c$$

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2. $\int \frac{x^2}{x^2-4} dx = \int 1 + \frac{4}{x^2-4} dx = \int \frac{x^2}{(x+2)(x-2)} dx$

$$\frac{4}{(x+2)(x-2)} = \frac{A}{(x+2)} + \frac{B}{(x-2)} \qquad 4 = A(x-2) + B(x+2)$$

$x = -2 \quad 4 = -4A \qquad A = -1$

$x = 2 \quad 4 = 4B \qquad B = 1$

$$\int \frac{x^2}{x^2-4} dx = \int 1 + \frac{-1}{(x+2)} + \frac{1}{(x-2)} dx$$

$$= x + ln|x-2| - \ln|x+2| + c$$

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3. $\int \frac{x^2-6x-7}{(x-1)(x-2)(x+3)}dx$

$$\frac{x^2-6x-7}{(x-1)(x-2)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x+3)}$$

$$x^2-6x-7 = A(x-2)(x+3) + B(x-1)(x+3) + C((x-1)(x-2)$$

$x=1 \quad -12 = -4A \qquad A = 3$

$x=2 \quad -15 = 5B \qquad B = -3$

$x=0 \quad -7 = -6A - 3B + 2C \qquad ; \qquad -7 = -18 + 9 + 2C \quad ; \qquad C = 1$

$$\int \frac{x^2-6x-7}{(x-1)(x-2)(x+3)}dx = \int \frac{3}{(x-1)} - \frac{3}{(x-2)} + \frac{1}{(x+3)}dx$$

$$= 3ln|x-1| - 3\,ln|x-2| + ln|x+3| + c$$

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4. $\int \frac{x^3-2x-13}{x^2-2x-3}dx = \int x + 2 + \frac{5x-7}{x^2-2x-3}dx = \int x + 2 + \frac{5x-7}{(x+1)(x-3)}dx$

$$\frac{5x-7}{(x+1)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-3)} \qquad 5x-7 = A(x-3) + B(x+1)$$

$x=-1 \quad -12 = -4A \qquad A = 3$

$x=3 \qquad 8 = 4B \qquad B = 2$

$$\int \frac{x^3-2x-13}{x^2-2x-3}dx = \int x + 2 + \frac{3}{(x+1)} + \frac{2}{(x-3)}dx$$

$$= \frac{1}{2}x^2 + 2x + 3ln|x+1| - 2\ln|x-3| + c$$

Ex 3 Denominator has linear and repeated factor

1. $\int \frac{3x^2+x+1}{x(x+1)^2}dx$

$$\frac{3x^2+x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} \qquad 3x^2+x+1 = A(x+1)^2 + Bx(x+1) + Cx$$

$x=-1 \quad 3=-C \quad C=-3$

$x=0 \quad 1=A \quad A=1$

$x=1 \quad 5=4A+2B+C \quad ; \quad 5=4+2B-3 \quad B=2$

$$\int \frac{3x^2+x+1}{x(x+1)^2}dx = \int \frac{1}{x} + \frac{2}{(x+1)} - \frac{3}{(x+1)^2}dx$$

$$= ln|x| + 2ln|x+1| + \frac{3}{(x+1)} + c$$

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2. $\int \frac{x^2-2x+10}{(x+2)(x-1)^2}dx$

$$\frac{x^2-2x+10}{(x+2)(x-1)^2} = \frac{A}{(x+2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$x^2-2x+10 = A(x-1)^2 + B(x-1)(x+2) + C(x+2)$$

$x=1 \quad 9=3C \quad C=3$

$x=-2 \quad 18=9A \quad A=2$

$x=0 \quad 10=A-2B+2C \quad ; \quad 10=2-2B+6 \quad B=-1$

$$\int \frac{x^2-2x+10}{(x+2)(x-1)^2}dx = \int \frac{2}{(x+2)} - \frac{1}{(x-1)} + \frac{3}{(x-1)^2}dx$$

$$= 2ln|x+2| - ln|x-1| - \frac{3}{(x-1)} + c$$

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3. $\int \frac{25}{(x+2)(2x-1)^2}\,dx$

$$\frac{25}{(x+2)(2x-1)^2} = \frac{A}{(x+2)} + \frac{B}{(2x-1)} + \frac{C}{(2x-1)^2}$$

$$25 = A(2x-1)^2 + B(2x-1)(x+2) + C(x+2)$$

$x = -2 \qquad 25 = 25A \qquad A = 1$

$x = \frac{1}{2} \qquad 25 = \frac{5}{2}C \qquad C = 10$

$x = 0 \qquad 25 = A - 2B + 2C \qquad ; \qquad 25 = 1 - 2B + 20 \qquad B = -2$

$$\int \frac{25}{(x+2)(2x-1)^2}\,dx = \int \frac{1}{(x+2)} - \frac{2}{(2x-1)} + \frac{10}{(2x-1)^2}\,dx$$

$$= ln|x+2| - ln|2x-1| - \frac{5}{(2x-1)} + c$$

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4. $\int \frac{5x+2}{(x-2)^2(x+1)}\,dx$

$$\frac{5x+2}{(x-2)^2(x+1)} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$5x + 2 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

$x = -1 \qquad -3 = 9A \qquad A = -\frac{1}{3}$

$x = 2 \qquad 12 = 3C \qquad C = 4$

$x = 0 \qquad 2 = 4A - 2B + C \qquad ; \qquad 2 = -\frac{4}{3} - 2B + 4 \qquad B = \frac{1}{3}$

$$\int \frac{5x+2}{(x-2)^2(x+1)}\,dx = \int \frac{-1}{3(x+1)} + \frac{1}{3(x-2)} + \frac{4}{(x-2)^2}\,dx$$

$$= \frac{1}{3}ln|x-2| - \frac{1}{3}ln|x+1| - \frac{4}{(x-2)^2} + c$$

Ex 4 Denominator has irreducible quadratic factor

1. $\int \frac{3x+1}{(x-1)(x^2+1)} dx$ ***check*** !!!!!!

$$\frac{3x+1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{(Bx+C)}{(x^2+1)}$$

$$3x+1 = A(x^2+1) + (Bx+C)(x-1)$$

$x = 1$ $\quad 4 = 2A \quad A = 2$

$x = 0$ $\quad 1 = A - C \quad C = 2 - 1 = 1$

$x = 2$ $\quad 7 = 10 + 4B + 2 - 1 \quad ; \quad -4 = 4B \quad B = -1$

$$\int \frac{3x+1}{(x-1)(x^2+1)} dx = \int \frac{2}{(x-1)} - \frac{(x-1)}{(x^2+1)} dx = \int \frac{2}{(x-1)} - \frac{x}{(x^2+1)} + \frac{1}{(x^2+1)} dx$$

$\int \frac{f'(x)}{f(x)} dx = ln|f(x)|$

$$= 2ln|x-1| - \frac{1}{2} ln|x^2+1| + tan^{-1}x + c$$

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2. $\int \frac{3x^2+92x}{(x+6)(x^2+1)} dx$

$$\frac{3x^2+92x}{(x+6)(x^2+1)} = \frac{A}{(x+6)} + \frac{(Bx+C)}{(x^2+1)}$$

$$3x^2+92x = A(x^2+1) + (Bx+C)(x+6)$$

$x = -6 \qquad -44\,4 = 37A \qquad A = -12$

$x = 0 \qquad 0 = A + 6C \qquad C = 12 \div 6 = 2$

$x = 1 \qquad 95 = 2A + B + 6B + C + 6C \qquad ; \qquad 95 = -24 + 7B + 14 \qquad B = 15$

$$\int \frac{3x^2+92x}{(x+6)(x^2+1)} dx = \int \frac{-12}{(x+6)} + \frac{(15x+2)}{(x^2+1)} dx = \int \frac{-12}{(x+6)} + \frac{15x}{(x^2+1)} + \frac{2}{(x^2+1)} dx$$

$\int \frac{f'(x)}{f(x)} dx = ln|f(x)|$

$$= \frac{15}{2} ln|x^2+1| - 12ln|x+6| + 2tan^{-1}x + c$$

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3. $\int \frac{x}{(x^4-1)}\,dx = \int \frac{x}{(x+1)(x-1)(x^2+1)}\,dx$

$$\frac{x}{(x+1)(x-1)(x^2+1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{(Cx+D)}{(x^2+1)}$$

$$x = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x^2-1)$$

$x = -1 \qquad -1 = -4A \qquad A = \frac{1}{4}$

$x = 1 \qquad 1 = 4B \qquad B = \frac{1}{4}$

$x = 0 \qquad 0 = -A + B - D \qquad ; \quad 0 = -\frac{1}{4} + \frac{1}{4} - D \qquad D = 0$

$x = 2 \qquad 2 = 5A + 15B + 6C \qquad ; \quad 2 = \frac{5}{4} + \frac{15}{4} + 6C \qquad C = -\frac{1}{2}$

$$\int \frac{x}{(x^4-1)}\,dx = \int \frac{1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{x}{2(x^2+1)}\,dx$$

$$\int \frac{f'(x)}{f(x)}\,dx = ln|f(x)|$$

$$= \frac{1}{4} ln|x+1| + \frac{1}{4} ln|x-1| - \frac{1}{4} ln|x^2+1| + c$$

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4. $\int \frac{x}{(x+1)(x^2+4)} dx$

$$\frac{x}{(x+1)(x^2+4)} = \frac{A}{(x+1)} + \frac{(Bx+C)}{(x^2+4)}$$

$$x = A(x^2+4) + (Bx+C)(x+1)$$

$x = -1 \qquad -1 = 5A \qquad A = -\frac{1}{5}$

$x = 0 \qquad 0 = 4A + C \qquad C = \frac{4}{5}$

$x = 1 \qquad 1 = 5A + 2B + 2C \qquad ; \quad 1 = -1 + 2B + \frac{8}{5} \qquad B = \frac{1}{5}$

$$\int \frac{x}{(x+1)(x^2+4)} dx = \int \frac{-1}{5(x+1)} + \frac{x}{5(x^2+4)} + \frac{4}{5(x^2+4)} dx$$

$$\int \frac{f'(x)}{f(x)} dx = ln|f(x)|$$

$$= \frac{2}{5} tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{5} ln|x+1| + \frac{1}{10} ln|x^2+4| + c$$

Ex 5 Integration by Parts

1. $\int xsinx\,dx$

$$u = x \quad ; \quad du = dx \quad ; \quad dv = sinx\,dx \quad ; \quad v = -cosx$$

$$\int udv = uv - \int vdu$$

$$\int udv = -xcosx - \int -cosx\ dx = -\,xcosx + \int cosx\ dx$$

$$= -xcosx + sinx + c$$

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2. $\int xsin3x\,dx$

$$u = x \quad ; \quad du = dx \quad ; \quad dv = sin3x\,dx \quad ; \quad v = -\frac{1}{3}cos3x$$

$$\int udv = uv - \int vdu$$

$$\int udv = -\frac{1}{3}xcos3x - \int -\frac{1}{3}cos3x\ dx = -\frac{1}{3}xcos3x + \frac{1}{3}\int cos3x\ dx$$

$$= -\frac{1}{3}xcos3x + \frac{1}{9}sin3x + c$$

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3. $\int \sqrt{x}lnx\,dx$

$$u = lnx \quad ; \quad du = \frac{1}{x}dx \quad ; \quad dv = x^{\frac{1}{2}}\,dx \quad ; \quad v = \frac{2}{3}x^{\frac{3}{2}}$$

$$\int udv = uv - \int vdu$$

$$\int udv = \frac{2}{3}x^{\frac{3}{2}}lnx - \int \frac{2}{3}x^{\frac{3}{2}}\ \frac{1}{x}dx = \frac{2}{3}x^{\frac{3}{2}}lnx - \frac{2}{3}\int x^{\frac{1}{2}}\ dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}lnx - \frac{4}{9}x^{\frac{3}{2}} + c$$

---

4. $\int \frac{1}{x^3} lnx \, dx$

$$u = lnx \quad ; \quad du = \frac{1}{x} dx \quad ; \quad dv = x^{-3} \, dx \quad ; \quad v = -\frac{1}{2} x^{-2}$$

$$\int udv = uv - \int vdu$$

$$\int udv = -\frac{1}{2} x^{-2} lnx - \int -\frac{1}{2} x^{-2} \, \frac{1}{x} dx = -\frac{1}{2} x^{-2} lnx + \frac{1}{2} \int x^{-3} \, dx$$

$$= -\frac{1}{2x^2} lnx - \frac{1}{4x^2} + c$$

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5. $\int xe^x \, dx$

$$u = x \quad ; \quad du = dx \quad ; \quad dv = e^x \, dx \quad ; \quad v = e^x$$

$$\int udv = uv - \int vdu$$

$$\int udv = xe^x - \int e^x \, dx = xe^x - e^x + c$$

$$= e^x (x - 1) + c$$

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6. $\int xcos4x \, dx$

$$u = x \quad ; \quad du = dx \quad ; \quad dv = cos4x \, dx \quad ; \quad v = \frac{1}{4} sin4x$$

$$\int udv = uv - \int vdu$$

$$\int udv = \frac{1}{4} xsin4x - \frac{1}{4} \int sin4x \, dx$$

$$= \frac{1}{4} xsin4x + \frac{1}{16} cos4x + c$$

Ex 6 Repeated Integration by Parts

1. $\int x^2 cosx\, dx$

$$u = x^2 \quad ; \quad du = 2xdx \quad ; \quad dv = cosx\, dx \quad ; \quad v = sinx$$

$$\int udv = uv - \int vdu$$

$$\int udv = x^2 sinx - \int 2xsinx\ dx = x^2 sinx - 2\int xsinx\ dx$$

$$u = x \quad ; \quad du = dx \quad ; \quad dv = sinx\, dx \quad ; \quad v = -cosx$$

$$= x^2 sinx - 2\left[-xcosx - \int -cosx\ dx\right]$$

$$= x^2 sinx + 2xcosx - 2sinx + c$$

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2. $\int x^2 sin3x\, dx$

$$u = x^2 \quad ; \quad du = 2xdx \quad ; \quad dv = sin3x\, dx \quad ; \quad v = -\frac{1}{3}cos3x$$

$$\int udv = uv - \int vdu$$

$$\int udv = -\frac{1}{3}x^2 cos3x + \frac{2}{3}\int xcos3x\ dx$$

$$u = x \quad ; \quad du = dx \quad ; \quad dv = cos3x\, dx \quad ; \quad v = \frac{1}{3}sin3x$$

$$= -\frac{1}{3}x^2 cos3x + \frac{2}{3}\left[\frac{1}{3}xsin3x - \frac{1}{3}\int sin3x\ dx\right]$$

$$= -\frac{1}{3}x^2 cos3x + \frac{2}{9}xsin3x + \frac{2}{27}cos3x + c$$

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3. $\int x^2e^{2x}\,dx$

$$u = x^2 \quad ; \quad du = 2xdx \quad ; \quad dv = e^{2x}\,dx \quad ; \quad v = \frac{1}{2}e^{2x}$$

$$\int udv = uv - \int vdu$$

$$\int udv = \frac{1}{2}x^2e^{2x} - \frac{1}{2}\int 2xe^{2x}\ dx = \frac{1}{2}x^2e^{2x} - \int xe^{2x}\ dx$$

$$u = x \quad ; \quad du = dx \quad ; \quad dv = e^{2x}\,dx \quad ; \quad v = \frac{1}{2}e^{2x}$$

$$= \frac{1}{2}x^2e^{2x} - \left[\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\ dx\right]$$

$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c = \frac{1}{2}e^{2x}\left(x^2 - x + \frac{1}{4}\right) + c$$

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4. $\int x^2cos2x\,dx$

$$u = x^2 \quad ; \quad du = 2xdx \quad ; \quad dv = cos2x\,dx \quad ; \quad v = \frac{1}{2}sin2x$$

$$\int udv = uv - \int vdu$$

$$\int udv = \frac{1}{2}x^2sin2x - \int xsin2x\ dx$$

$$u = x \quad ; \quad du = dx \quad ; \quad dv = sin2x\,dx \quad ; \quad v = -\frac{1}{2}cos2x$$

$$= \frac{1}{2}x^2sin2x - \left[-\frac{1}{2}xcos2x + \frac{1}{2}\int cos2x\ dx\right]$$

$$= \frac{1}{2}x^2sin2x + \frac{1}{2}xcos2x - \frac{1}{4}sin2x + c$$

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5. $\int x^2 e^{-x}\, dx$

$$u = x^2 \quad ; \quad du = 2xdx \quad ; \quad dv = e^{-x}\, dx \quad ; \quad v = -e^{-x}$$

$$\int udv = uv - \int vdu$$

$$\int udv = -x^2 e^{-x} + 2\int xe^{-x}\ dx$$

$$u = x \quad ; \quad du = dx \quad ; \quad dv = e^{-x}\, dx \quad ; \quad v = -e^{-x}$$

$$= -x^2 e^{-x} + 2\left[-xe^{-x} + \int e^{-x}\ dx\right]$$

$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c = -e^{-x}(x^2 + 2x + 2) + c$$

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6. $\int x^3 e^{x}\, dx$

$$u = x^3 \quad ; \quad du = 3x^2 dx \quad ; \quad dv = e^{x}\, dx \quad ; \quad v = e^{x}$$

$$\int udv = uv - \int vdu$$

$$\int udv = x^3 e^{x} - 3\int x^2 e^{x}\ dx$$

$$u = x^2 \quad ; \quad du = 2xdx \quad ; \quad dv = e^{x}\, dx \quad ; \quad v = e^{x}$$

$$= x^3 e^{x} - 3\left[x^2 e^{x} - 2\int xe^{x}\ dx\right]$$

$$u = x \quad ; \quad du = dx \quad ; \quad dv = e^{x}\, dx \quad ; \quad v = e^{x}$$

$$= x^3 e^{x} - 3x^2 e^{x} + 6\int xe^{x}\ dx = x^3 e^{x} - 3x^2 e^{x} + 6\left[xe^{x} - \int e^{x}\ dx\right]$$

$$= x^3 e^{x} - 3x^2 e^{x} + 6xe^{x} - 6e^{x} + c = e^{x}(x^3 - 3x^2 + 6x - 6) + c$$

Ex 7 Integration by Parts with Dummy Function

1. $\int tan^{-1}x\,dx$

$$u = tan^{-1}x \quad ; \quad du = \frac{1}{1+x^2}dx \quad ; \quad dv = 1dx \quad ; \quad v = x$$

$$\int udv = uv - \int vdu$$

$$\int udv = xtan^{-1}x - \int \frac{x}{1+x^2}dx$$

$$= xtan^{-1}x - \frac{1}{2}ln|1+x^2| + c$$

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2. $\int sin^{-1}3x\,dx$ **check !!!!!**

$$u = sin^{-1}3x \quad ; \quad du = \frac{3}{\sqrt{1-9x^2}}dx \quad ; \quad dv = 1dx \quad ; \quad v = x$$

$$\int udv = uv - \int vdu$$

$$\int udv = xsin^{-1}3x - 3\int \frac{x}{\sqrt{1-9x^2}}dx$$

$$u = 1-9x^2 \quad ; \quad du = -18xdx \quad ; \quad -\frac{du}{18} = xdx$$

$$\int \frac{x}{\sqrt{1-9x^2}}dx = \int -\frac{du}{18\sqrt{u}} = -\frac{1}{18}\int u^{-\frac{1}{2}}du = -\frac{1}{18}\left[2u^{\frac{1}{2}}\right] = -\frac{1}{9}(1-9x^2)^{\frac{1}{2}}$$

$$= xsin^{-1}3x + \frac{1}{3}\sqrt{(1-9x^2)} + c$$

---

3. $\int tan^{-1}2x\,dx$

$$u = tan^{-1}2x \quad ; \quad du = \frac{2}{1+4x^2}dx \quad ; \quad dv = 1dx \quad ; \quad v = x$$

$$\int udv = uv - \int vdu$$

$$\int udv = xtan^{-1}2x - 2\int \frac{x}{1+4x^2}dx$$

$$u = 1 + 4x^2 \quad ; \quad du = 8xdx \quad ; \quad \frac{du}{8} = xdx$$

$$\int \frac{x}{1+4x^2}dx = \int \frac{du}{8u} = \frac{1}{8}\int \frac{1}{u}du = \frac{1}{8}[ln|1+4x^2|] = \frac{1}{8}ln|1+4x^2|$$

$$= \ xtan^{-1}2x - \frac{1}{4}ln|1+4x^2| + c$$

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4. $\int sin^{-1}\left(\frac{x}{2}\right)\,dx$

$$u = sin^{-1}\left(\frac{x}{2}\right) \quad ; \quad du = \frac{1}{\sqrt{4-x^2}}dx \quad ; \quad dv = 1dx \quad ; \quad v = x$$

$$\int udv = uv - \int vdu$$

$$\int udv = xsin^{-1}\left(\frac{x}{2}\right) - \int \frac{x}{\sqrt{4-x^2}}dx$$

$$u = 4 - x^2 \quad ; \quad du = -2xdx \quad ; \quad -\frac{du}{2} = xdx$$

$$\int \frac{x}{\sqrt{4-x^2}}dx = \int -\frac{du}{2u^{\frac{1}{2}}} = -\frac{1}{2}\int u^{-\frac{1}{2}}du = -\frac{1}{2}\left[2u^{\frac{1}{2}}\right] = -(4-x^2)^{\frac{1}{2}}$$

$$= \ xsin^{-1}\left(\frac{x}{2}\right) + \sqrt{(4-x^2)} + c$$

---

5. $\int cos^{-1}x\,dx$

$$u = cos^{-1}x \quad ; \quad du = -\frac{1}{\sqrt{1-x^2}}dx \quad ; \quad dv = 1dx \quad ; \quad v = x$$

$$\int udv = uv - \int vdu$$

$$\int udv = xcos^{-1}x + \int \frac{x}{\sqrt{1-x^2}}dx$$

$$u = 1 - x^2 \quad ; \quad du = -2xdx \quad ; \quad -\frac{du}{2} = xdx$$

$$\int \frac{x}{\sqrt{1-x^2}}dx = \int -\frac{du}{2u^{\frac{1}{2}}} = -\frac{1}{2}\int u^{-\frac{1}{2}}du = -\frac{1}{2}\left[2u^{\frac{1}{2}}\right] = -(1-x^2)^{\frac{1}{2}}$$

$$= \; xcos^{-1}x - \sqrt{(1-x^2)} + c$$

---

6. $\int tan^{-1}\left(\frac{x}{2}\right)\,dx$

$$u = tan^{-1}\left(\frac{x}{2}\right) \quad ; \quad du = \frac{2}{4\,+x^2}dx \quad ; \quad dv = 1dx \quad ; \quad v = x$$

$$\int udv = uv - \int vdu$$

$$\int udv = xtan^{-1}\left(\frac{x}{2}\right) - 2\int \frac{x}{4\,+x^2}dx$$

$$u = 4 + x^2 \quad ; \quad du = 2xdx \quad ; \quad \frac{du}{2} = xdx$$

$$\int \frac{x}{4\,+x^2}dx = \int \frac{du}{2u} = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}[ln|4+x^2|] = \frac{1}{2}ln|4+x^2|$$

$$= \; xtan^{-1}\left(\frac{x}{2}\right) - ln|4+x^2| + c$$

---

7. $\int ln2x\, dx$

$$u = ln2x \quad ; \quad du = \frac{1}{x}dx \quad ; \quad dv = 1dx \quad ; \quad v = x$$

$$\int udv = uv - \int vdu$$

$$\int udv = xln2x - \int 1\, dx$$

$$= xln2x - x + c$$

---

8. $\int (lnx)^2\, dx$

$$u = (lnx)^2 \quad ; \quad du = \frac{2lnx}{x}dx \quad ; \quad dv = 1dx \quad ; \quad v = x$$

$$\int udv = uv - \int vdu$$

$$\int udv = x(lnx)^2 - 2\int lnx\, dx$$

$$u = lnx \quad ; \quad du = \frac{1}{x}dx \quad ; \quad dv = 1dx \quad ; \quad v = x$$

$$\int udv = xlnx - \int 1\, dx$$

$$= xlnx - x$$

$$x(lnx)^2 - 2[xlnx - x] + c = x(lnx)^2 - 2xlnx + 2x + c$$

Ex 8 Integration by Parts bck to Original Form

1. $\int e^x sin2x\, dx$

$$u = e^x \quad ; \quad du = e^x dx \quad ; \quad dv = sin2x\, dx \quad ; \quad v = -\frac{1}{2}cos2x$$

$$I = \int udv = uv - \int vdu$$

$$I = \int udv = -\frac{1}{2}e^x cos2x + \frac{1}{2}\int e^x cos2x\, dx$$

$$u = e^x \quad ; \quad du = e^x dx \quad ; \quad dv = cos2x\, dx \quad ; \quad v = \frac{1}{2}sin2x$$

$$\int udv = \frac{1}{2}e^x sin2x - \frac{1}{2}\int e^x sin2x\, dx$$

$$= \frac{1}{2}e^x sin2x - \frac{1}{2}I$$

$$I = -\frac{1}{2}e^x cos2x + \frac{1}{2}\left[\frac{1}{2}e^x sin2x - \frac{1}{2}I\right] = -\frac{1}{2}e^x cos2x + \frac{1}{4}e^x sin2x - \frac{1}{4}I$$

$$I + \frac{1}{4}I = -\frac{1}{2}e^x cos2x + \frac{1}{4}e^x sin2x$$

$$\frac{5}{4}I = -\frac{1}{2}e^x cos2x + \frac{1}{4}e^x sin2x$$

$$I = \frac{1}{5}e^x sin2x - \frac{2}{5}e^x cos2x + c$$

2. $\int e^{-x} sinx\, dx$

$$u = e^{-x} \quad ; \quad du = -e^{x} dx \quad ; \quad dv = sinx\, dx \quad ; \quad v = -cosx$$

$$I = \int udv = uv - \int vdu$$

$$I = \int udv = -e^{-x} cosx - \int e^{-x} cosx\, dx$$

$$u = e^{-x} \quad ; \quad du = -e^{-x} dx \quad ; \quad dv = cosx\, dx \quad ; \quad v = sinx$$

$$\int udv = e^{-x} sinx + \int e^{-x} sinx\, dx$$

$$= e^{-x} sinx + I$$

$$I = -e^{-x} cosx - [e^{-x} sinx + I] = -e^{-x} cosx - e^{-x} sinx - I$$

$$2I = -e^{-x} cosx - e^{-x} sinx$$

$$I = -\frac{1}{2} e^{-x} cosx - \frac{1}{2} e^{-x} sinx + c$$

$$I = -\frac{1}{2} e^{-x} (cosx + sinx) + c$$

---

3. $\int e^{-2x} cos3x\, dx$

$$u = e^{-2x} \quad ; \quad du = -2e^{-2x} dx \quad ; \quad dv = cos3x\, dx \quad ; \quad v = \frac{1}{3} sin3x$$

$$I = \int udv = uv - \int vdu$$

$$I = \int udv = \frac{1}{3} e^{-2x} sin3x + \frac{2}{3} \int e^{-2x} sin3x\, dx$$

$$u = e^{-2x} \quad ; \quad du = -2e^{-2x} dx \quad ; \quad dv = sin3x\, dx \quad ; \quad v = -\frac{1}{3} cos3x$$

$$\int udv = -\frac{1}{3} e^{-2x} cos3x - \frac{2}{3} \int e^{-2x} cos3x\, dx$$

$$= -\frac{1}{3} e^{-2x} cos3x - \frac{2}{3} I$$

$$I = \frac{1}{3} e^{-2x} sin3x + \frac{2}{3} \left[-\frac{1}{3} e^{-2x} cos3x - \frac{2}{3} I \right] = \frac{1}{3} e^{-2x} sin3x - \frac{2}{9} e^{-2x} cos3x - \frac{4}{9} I$$

$$\frac{13}{9} I = \frac{1}{3} e^{-2x} sin3x - \frac{2}{9} e^{-2x} cos3x$$

$$I = \frac{3}{13} e^{-2x} sin3x - \frac{2}{13} e^{-2x} cos3x + c$$

$$I = \frac{1}{13} e^{-2x} (3sin3x - 2cos3x) + c$$

---

4. $\int e^x (cosx)^2\, dx$

$u = (cosx)^2 \quad ; \quad du = -2sinxcosxdx = -sin2x \quad ; \quad dv = e^x\, dx \quad ; \quad v = e^x$

$$I = \int udv = uv - \int vdu$$

$$I_T = \int udv = e^x (cosx)^2 + \int e^x sin2x\, dx$$

$$I_1 = \int e^x sin2x\, dx$$

$u = sin2x \quad ; \quad du = 2cos2xdx \quad ; \quad dv = e^x\, dx \quad ; \quad v = e^x$

$$I_1 = \int udv = e^x sin2x - 2\int e^x cos2x\, dx$$

$u = cos2x \quad ; \quad du = -2sin2xdx \quad ; \quad dv = e^x\, dx \quad ; \quad v = e^x$

$$\int udv = e^x cos2x + 2\int e^x sin2x\, dx$$

$$I_1 = e^x sin2x - 2[e^x cos2x + 2I]$$

$$I_1 = e^x sin2x - 2e^x cos2x - 4I$$

$$5I_1 = e^x sin2x - 2e^x cos2x$$

$$I_1 = \frac{1}{5}e^x sin2x - \frac{2}{5}e^x cos2x$$

Final Solution:

$$I_T = e^x (cosx)^2 + \frac{1}{5}e^x sin2x - \frac{2}{5}e^x cos2x + c$$
