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Advanced Higher Maths

Advanced Higher - Unit 2.1 Binomial Expansion & Complex Numbers Solutions

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Ex1 Binomial Expansion

1. $(x-y)^5 = x^5\left(1-\frac{y}{x}\right)^5 = x^5\left[1+5\left(-\frac{y}{x}\right)+10\left(-\frac{y}{x}\right)^2+10\left(-\frac{y}{x}\right)^3+5\left(-\frac{y}{x}\right)^4+\left(-\frac{y}{x}\right)^5\right]$

$$= x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$$

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2. $(x+1)^5 = x^5\left(1+\frac{1}{x}\right)^5 = x^5\left[1+5\left(\frac{1}{x}\right)+10\left(\frac{1}{x}\right)^2+10\left(\frac{1}{x}\right)^3+5\left(\frac{1}{x}\right)^4+\left(\frac{1}{x}\right)^5\right]$

$$= x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$$

(b) $(1-x)^4 = \big(1+(-x)\big)^4 = 1+4(-x)+6(-x)^2+4(-x)^3+1(-x)^4$

$$= 1 - 4x + 6x^2 - 4x^3 + x^4$$

(c) $(a+2b)^3 = a^3\left(1+\left(\frac{2b}{a}\right)\right)^3 = a^3\left[1+3\left(\left(\frac{2b}{a}\right)\right)+3\left(\left(\frac{2b}{a}\right)\right)^2+1\left(\left(\frac{2b}{a}\right)\right)^3\right]$

$$= a^3 + 6a^2b + 12ab^2 + 8b^3$$

(d) $(2a-b)^5 = 32a^5\left(1+\left(\frac{-b}{2a}\right)\right)^5 = 32a^5\left[1+5\left(\frac{-b}{2a}\right)+10\left(\frac{-b}{2a}\right)^2+10\left(\frac{-b}{2a}\right)^3+5\left(\frac{-b}{2a}\right)^4+1\left(\frac{-b}{2a}\right)^5\right]$

$$= 32a^5 - 80a^4b + 80a^3b^2 - 40a^2b^3 + 10ab^4 - b^5$$

(e) $(2x-3y)^5 = 32x^5\left(1+\left(\frac{-3y}{2x}\right)\right)^5 = 32x^5\left[1+5\left(\frac{-3y}{2x}\right)+10\left(\frac{-3y}{2x}\right)^2+10\left(\frac{-3y}{2x}\right)^3+5\left(\frac{-3y}{2x}\right)^4+1\left(\frac{-3y}{2x}\right)^5\right]$

$$= 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 \ + 810ab^4 - 243b^5$$

(f) $\left(x+\frac{1}{x}\right)^5 = x^5\left(1+\left(\frac{1}{x^2}\right)\right)^5 = x^5\left[1+5\left(\frac{1}{x^2}\right)+10\left(\frac{1}{x^2}\right)^2+10\left(\frac{1}{x^2}\right)^3+5\left(\frac{1}{x^2}\right)^4+1\left(\frac{1}{x^2}\right)^5\right]$

$$= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$$

(g) $(a-2b)^6 = a^6\left(1+\left(\frac{-2b}{a}\right)\right)^6 = a^6\left[1+6\left(\frac{-2b}{a}\right)+15\left(\frac{-2b}{a}\right)^2+20\left(\frac{-2b}{a}\right)^3+15\left(\frac{-2b}{a}\right)^4+6\left(\frac{-2b}{a}\right)^5+1\left(\frac{-2b}{a}\right)^6\right]$

$$= a^6 - 12a^5b + 60a^4b^2 - 160a^3b^3 \ + 240a^2b^4 - 192ab^5 + 64b^6$$

(h) $(2a+b)^7 = 128a^6\left(1+\left(\frac{b}{2a}\right)\right)^7 = 128a^7\left[1+7\left(\frac{b}{2a}\right)+21\left(\frac{b}{2a}\right)^2+35\left(\frac{b}{2a}\right)^3+35\left(\frac{b}{2a}\right)^4+21\left(\frac{b}{2a}\right)^5+7\left(\frac{b}{2a}\right)^6+1\left(\frac{b}{2a}\right)^7\right]$

$$= 128a^7 + 448a^6b + 672a^5b^2 + 560a^4b^3 \ + 280a^3b^4 + 84a^2b^5 + 14ab^6 + b^7$$

Ex2 Binomial Expansion - $\frac{n!}{r!(n-r)!}$

1. $(3+x)^3 = 3^3 + 3(3^2)(x) + 3(3)(x)^2 + x^3$

$= 27 + 27x + 9x^2 + x^3$

(b) $(5+2x)^3 = 5^3 + 3(5)^2(2x) + 3(5)(2x)^2 + (2x)^3$

$= 125 + 150x + 60x^2 + 8x^3$

(c) $(3+x)^4 = 3^4 + 4(3)^3(x) + 6(3)^2(x)^2 + 4(3)(x)^3 + (3)(x)^4$

$= 81 + 108x + 54x^2 + 12x^3 + 3x^4$

(d) $(2-x)^4 = 2^4 + 4(2)^3(-x) + 6(2)^2(-x)^2 + 4(2)(-x)^3 + (2)(-x)^4$

$= 16 - 32x + 24x^2 - 8x^3 + 2x^4$

(e) $(x+2y)^3 = x^3 + 3(x)^2(2y) + 3(x)(2y)^2 + (2y)^3$

$= x^3 + 6x^2y + 12xy^2 + 8y^3$

(f) $(2x-3y)^3 = (2x)^3 + 3(2x)^2(-3y) + 3(2x)(-3y)^2 + (-3y)^3$

$= 8x^3 - 36x^2y + 54xy^2 - 27y^3$

(g) $(1+3x)^4 = 1^4 + 4(1)^3(3x) + 6(1)^2(3x)^2 + 4(1)(3x)^3 + (1)(3x)^4$

$= 1 + 12x + 54x^2 + 108x^3 + 81x^4$

(h) $(2-3x)^5 = 2^5 + 5(2)^4(-3x) + 10(2)^3(-3x)^2 + 10(2)^2(-3x)^3 + 5(2)(-3x)^4 + (-3x)^5$

$= 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5$

(i) $\left(2x-\frac{3}{x}\right)^5=(2x)^5+5(2x)^4\left(-\frac{3}{x}\right)+10(2x)^3\left(-\frac{3}{x}\right)^2+10(2x)^2\left(-\frac{3}{x}\right)^3+5(2x)\left(-\frac{3}{x}\right)^4+\left(-\frac{3}{x}\right)^5$

$$=32x^5-240x^3+720x-\frac{1080}{x}+\frac{810}{x^3}-\frac{243}{x^5}$$

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2. $(2+x)^5=(2)^5+5(2)^4(x)+10(2)^3(x)^2+10(2)^2(x)^3+5(2)(x)^4+(x)^5$

$$=32+80x+80x^2+40x^3+10x^4+x^5$$

$(2+0.1)^5=32+80(0.1)+80(0.1)^2+40(0.1)^3+10(0.1)^4+(0.1)^5=\mathbf{40.84101}$

(b) $(2-0.1)^5=32+80(-0.1)+80(-0.1)^2+40(-0.1)^3+10(-0.1)^4+(-0.1)^5=24.76099$

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3. $(2+x)^7=(2)^7+7(2)^6(x)+21(2)^5(x)^2+35(2)^4(x)^3\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$=128+448x+672x^2+560x^3\ldots\ldots\ldots\ldots\ldots\ldots$$

$(2+0.1)^7=128+448(0.1)+672(0.1)^2+560(0.1)^3\ldots\ldots\ldots\ldots\ldots\ldots..=\mathbf{180.08}$ *check*!!!!!!!!!!!!

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4. $(1+x)^4=1^4+4(1)^3(x)+6(1)^2(x)^2+4(1)(x)^3+(1)(x)^4$

$$=1+4x+6x^2+4x^3+x^4$$

$(1.01)^4=(1+0.01)^4$

$$=1+4(1)^3(0.01)+6(1)^2(0.01)^2+4(1)(0.01)^3+(0.01)^4=\mathbf{1.04060}$$

(b) $(1+x)^5=(1)^5+5(1)^4(x)+10(1)^3(x)^2+10(1)^2(x)^3+5(1)(x)^4+(x)^5$

$$=1+5x+10x^2+10x^3+5x^4+x^5$$

$(0.998)^5=(1+(-0.002)^5$

$$=1+5(-0.002)+10(-0.002)^2+10(-0.002)^3+5(-0.002)^4+(-0.002)^5=\mathbf{0.9900399}$$

(c) $(1+x)^{10} = 1 + 10(x) + 45(x)^2 + 120(x)^3 + 210(x)^4 + 252(x)^5 + 210(x)^6 \ldots\ldots\ldots\ldots\ldots\ldots$

$(0.99)^{10} = \left(1 + (-0.01)\right)^{10}$

$= 1 + 10(-0.01) + 45(-0.01)^2 + 120(-0.01)^3 + 210(-0.01)^4 + 252(-0.01)^5 + 210(0.01)^6 \ldots\ldots = \mathbf{0.9044}$

(d) $(1+x)^{10} = 1 + 10(x) + 45(x)^2 + 120(x)^3 + 210(x)^4 + 252(x)^5 + 210(x)^6 + 120(x)^7 + 45(x)^8 + 10(x)^9 + (x)^{10}$

$(1.99)^{10} = \left(1 + (0.99)\right)^{10}$

$= 1 + 10(0.99) + 45(0.99)^2 + 120(0.99)^3 + 210(0.99)^4 + 252(0.99)^5 + 210(0.99)^6 \ldots\ldots = \mathbf{973.9}$

Ex Finding Coefficients of Binomial Expansions

$$(x+y)^n = \binom{n}{r}(x)^{n-r}y^r \qquad \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

1. $The\ coefficient\ for\ term\ x^7$

$$(1+x)^{10} = \binom{10}{r}(1)^{10-r}x^r \qquad r=7$$

$$= \binom{10}{7}(1)^{10-7}x^7$$

$$= \frac{n!}{r!\,(n-r)!}(1)^3x^7$$

$$= \frac{10!}{7!\,(3)!}(1)^3x^7$$

$$= 120x^7$$

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b. $The\ coefficient\ for\ term\ y^3$

$$\left(y-\tfrac{5}{y}\right)^7 = \binom{7}{r}(y)^{7-r}\left(-\tfrac{5}{y}\right)^r$$

$$= \binom{7}{r}(y)^{7-r}(-5)^r y^{-r} \qquad ; \qquad 7-r-r=3\ ;\ 7-2r=3\ :r=2$$

$$= \binom{7}{2}(y)^3(-5)^2$$

$$= 525y^3$$

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2. $The\ Independent\ coefficient\ (x^0!!)$

$$\left(x-\tfrac{2}{x}\right)^{10} = \binom{10}{r}(x)^{7-r}\left(-\tfrac{2}{x}\right)^r$$

$$= \binom{10}{r}(x)^{10-r}(-2)^r(x)^{-r} \qquad ; \qquad 10-r-r=0\ ;\ 10-2r=0\ :r=5$$

$$= \binom{10}{5}(x)^0(-2)^5$$

$$= -8064 \quad as\ required$$

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3. $(1+x+x^2)^3 = \big(1+(x+x^2)\big)^3$

$(1+x)^3 = 1+3x+3x^2+x^3$

$\big(1+(x+x^2)\big)^3 = 1+3(x+x^2)+3\big((x+x^2)\big)^2+\big((x+x^2)\big)^3$

$\big(1+(x+x^2)\big)^3 = 1+(3x+3x^2)+3(x^2+2x^3+x^4)+(x^3+2x^4+x^5+x^4+2x^5+x^6)$

$\big(1+(x+x^2)\big)^3 = 1+3x+6x^2+7x^3+6x^4+3x^5+x^6$

Ex1 Complex Numbers

1. $(a)\ (2i)^2 = 4i^2 = -4$ $(b)\ (3i)^2 = 9i^2 = -9$ $(c)\ (4i)^2 = 16i^2 = -16$

   $(d)\ (-2i)^2 = 4i^2 = -4$ $(e)\ (-3i)^2 = 9i^2 = -9$

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2. $(a)\ x^2 + 4 = 0 \ ;\ x^2 = -4\ ;\ x = \pm 2i$
   $(b)\ x^2 + 9 = 0 \ ;\ x^2 = -9\ ;\ x = \pm 3i$
   $(c)\ x^2 + 3 = 0 \ ;\ x^2 = -3\ ;\ x = \pm\sqrt{3}i$

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3. $(a)\ x^2 - 2x + 2 = 0\ ;\ x = \frac{4\pm\sqrt{16-20}}{2}\ ;\ x = 1 \pm i$
   $(b)\ x^2 - 4x + 5 = 0\ ;\ x = \frac{4\pm\sqrt{16-20}}{2}\ ;\ x = 2 \pm i$
   $(c)\ x^2 - 4x + 13 = 0\ ;\ x = \frac{4\pm\sqrt{16-52}}{2}\ ;\ x = 2 \pm 3i$
   $(d)\ x^2 + 2x + 2 = 0\ ;\ x = \frac{-2\pm\sqrt{4-8}}{2}\ ;\ x = -1 \pm i$
   $(e)\ 4x^2 - 4x + 5 = 0\ ;\ x = \frac{4\pm\sqrt{16-80}}{8}\ ;\ x = \frac{1}{2} \pm i$
   $(f)\ x^2 + 6x + 10 = 0\ ;\ x = \frac{-6\pm\sqrt{36-40}}{2}\ ;\ x = -3 \pm i$
   $(g)\ 2x^2 - 2x + 1 = 0\ ;\ x = \frac{2\pm\sqrt{4-8}}{4}\ ;\ x = \frac{1}{2} \pm \frac{1}{2}i$
   $(h)\ 9x^2 - 6x + 2 = 0\ ;\ x = \frac{6\pm\sqrt{36-72}}{18}\ ;\ x = \frac{1}{3} \pm \frac{1}{3}i$

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4. Miss out

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5. Sum

   $(a)\ (1+i) + (1-i) = 2$ $(b)\ (2+i) + (2-i) = 4$ $(c)\ (2+3i) + (2-3i) = 4$

   $(d)\ (-1+i) + (-1-i) = -2$ $(e)\left(\frac{1}{2}+i\right) + \left(\frac{1}{2}-i\right) = 1$ $(f)\ (-3+i) + (-3-i) = -6$

   (g) $\left(\frac{1}{2}+\frac{1}{2}i\right) + \left(\frac{1}{2}-\frac{1}{2}i\right) = 1$ $(h)\ \left(\frac{1}{3}+\frac{1}{3}i\right) + \left(\frac{1}{3}-\frac{1}{3}i\right) = \frac{2}{3}$

   Product

   $(a)\ (1+i) \times (1-i) = 2$ $(b)\ (2+i) \times (2-i) = 5$ $(c)\ (2+3i) + (2-3i) = 13$

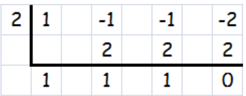
   $(d)\ (-1+i) + (-1-i) = 2$ $(e)\ \left(\frac{1}{2}+i\right) + \left(\frac{1}{2}-i\right) = 1\frac{1}{4}$ $(f)\ (-3+i) + (-3-i) = 10$

   (g) $\left(\frac{1}{2}+\frac{1}{2}i\right) + \left(\frac{1}{2}-\frac{1}{2}i\right) = \frac{1}{2}$ $(h)\left(\frac{1}{3}+\frac{1}{3}i\right) + \left(\frac{1}{3}-\frac{1}{3}i\right) = \frac{2}{9}$ $sum = -\frac{b}{a}$ $product = \frac{c}{a}$

6. $(a)\ x^3 - 1 = 0 \quad ; \quad x^3 = 1 \quad ; \quad x = 1$

$(b)\ x^4 - 1 = 0 \quad ; \quad x^4 = 1 \quad ; \quad x = \pm 1\ and\ \pm i$

$(c)\ x^3 - x^2 - x - 2 = 0 \quad ; \quad (x-2)(\ x^2 + x + 1) = 0$



2	1		-1		-1		-2
			2		2		2
	1		1		1		0

$$(x-2)(\ x^2 + x + 1) = 0 \quad ; \quad x = 2\ and \quad x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$(d)\ (x^2 + 4)((x^2 + 9) = 0 \quad ; \quad x^2 + 4 = 0 \quad ; \quad x = \pm 2i \quad and\ x^2 + 9 = 0 \quad ; \quad x = \pm 3i$

Ex2 Complex Number Operations

1. $(a)\ (3+7i)+(2+i)=5+8i$ $\quad (b)\ (9-2i)-(3+i)=6-3i$

$(c)\ (-2+i)+(7-4i)=5-3i$ $\quad (d)\ (3+2i)+(3-2i)=6$

$(e)\ (-2+i)-(-2-i)=2i$ $\quad (f)\ (a+bi)+(a-bi)=2a$

$(g)\ (a+bi)-(a-bi)=2bi$

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2. $(a)\ i^3=-i$ $\quad (b)\ i^4=i^2\times i^2=1$ $\quad (c)\ i^5=i$ $\quad (d)\ i^6=-1$

$(e)\ i^7=-i$ $\quad (f)\ i^8=1$ $\quad (g)\ i^9=i$ $\quad (h)\ i^{10}=-1$

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3. $(a)\ 2i\times 4i=-8$ $\quad (b)-2\,i^2=2$ $\quad (c)\ i(3+2i)=3i-2=-2+3i$

$(d)\ -i(1-4i)=-4-i$ $\quad (e)(2+i)(3+i)=5+5i$ $\quad (f)\ (6-5i)(2+3i)=27+8i$

$(g)\ (2+3i)(2-3i)=13$ $\quad (h)(a+bi)(a-bi)=a^2+b^2$

$(i)\ \ (\boldsymbol{a}+\boldsymbol{bi})(\boldsymbol{c}+\boldsymbol{di})=(ac-bd)+(bc+ad)i$ $\quad (j)\ (a+bi)(c-di)=(ac+bd)+(bc-ad)i$

$(k)\ \ (1+i)^3=1+3i+3i^2+i^3=-2+3i-i=-2+2i$

$(l)\ \ (1+i)^4=1+4i+6i^2+4i^3+i^4=1-6+1+4i-4i=-4$

$(m)\ \ (1+i)^4(1-i)^5=(1+i)(1-i)(1+i)(1-i)(1+i)(1-i)(1+i)(1-i)(1-i)$

$=(2)(2)(2)(2)(1-i)=16(1-i)=16-16i$

$(n)\ \ (3+i)^2+(3-i)^2=(8+6i)+(8-6i)=16$

$(o)\ \ (cost+isint)^2=cos^2t-sin^2t+i2costsint=cos2t+isin2t$

$(p)\ \ (cosA+isinA)(cosB+isinB)=cosAcosB-sinAcosB+isinAcosB+cosAsinB$

$=\cos(A+B)\,AcosB+isin(A+B)$

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4. $(a)\ \ (2+i)(2-i)=5$ $\quad (b)\ \ (1-2i)(1+2i)=5$ $\quad (c)\ \ (5-i)(5+i)=26$

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5. $(a)\ \frac{(4+i)}{i}\times\frac{-i}{-i}=1-4i$ $\qquad (b)\ \frac{1}{(2+i)}\times\frac{(2-i)}{(2-i)}=\frac{2}{5}-\frac{i}{5}$

$(c)\ \frac{(2-i)}{(1-2i)}\times\frac{(1+2i)}{(1+2i)}=\frac{4+3i}{5}=\frac{4}{5}+\frac{3}{5}i$ $\qquad (d)\ \frac{(5+i)}{(5-i)}\times\frac{(5+i)}{(5+i)}=\frac{24+10i}{26}=\frac{12}{13}+\frac{5}{13}i$

$(e)\ \frac{(a+bi)}{(a-bi)}\times\frac{(a+bi)}{(a+bi)}=\frac{(a^2-b^2)+2abi}{a^2+b^2}=\frac{(a^2-b^2)}{a^2+b^2}+\frac{2abi}{a^2+b^2}$

$(f)\ \frac{(a+bi)}{(c+di)}\times\frac{(c-di)}{(c-di)}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}=\frac{(ac+bd)}{c^2+d^2}+\frac{(bc-ad)i}{c^2+d^2}$

$(g)\ \frac{(10+5i)}{(2-i)}\times\frac{(2+i)}{(2+i)}=\frac{15+20i}{5}=3+4i$

$(h)\ \frac{1}{(cosA+isinA)}\times\frac{(cosA-isinA)}{(cosA-isinA)}=\frac{(cosA-isinA)}{(cos^2A+sin^2A)}=cosA-isinA$

$(h)\ \frac{(cosA+isinA)}{(cosA-isinA)}\times\frac{(cosA+isinA)}{(cosA+isinA)}=\frac{(cos^2A-sin^2A)+i2sinAcosA}{(cos^2A+sin^2A)}=cos2A+isin2A$

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6. $\big((x-1)-i\big)\big((x-1)+i\big)=(x^2-2x+1+1)+(1-x+x-1)i=x^2-2x+2$

For an equation to have roots of $(1+i)$ *and* $(1-i)$ *we need*

$(x-(1+i)(x-(1-i)=0$; $\big((x-1)+i\big)\big((x-1)-i\big)=0$; $x^2-2x+2=0$

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7. $\sqrt{3-4i}=x+yi$; $(3-4i)=x^2-y^2+2xyi$; $x^2-y^2=3$; $2xy=-4$

$x^2-y^2=3$; $2xy=-4$; $\boldsymbol{y=-\frac{2}{x}}$; $x^2-\left(-\frac{2}{x}\right)^2=3$; $x^4-4=3x^2$

$x^4-3x^2-4=0$; $(x^2-4)(x^2+1)=0$

$(x^2+1)=0$; *no solution since x is real* $\qquad (x^2-4)=0 \qquad x=2$ *and* $x=-2$

$x=2\ \ y=-1$ *and* $x=-2\ \ y=1$; $\boldsymbol{\sqrt{3-4i}=\pm(2-i)}$

b. $\sqrt{21-20i} = x + yi \quad ; \quad (21-20i) = x^2 - y^2 + 2xyi \quad ; \quad x^2 - y^2 = 21 \quad ; \quad 2xy = -20$

$x^2 - y^2 = 21 \quad ; \quad 2xy = -20 \quad ; \quad \boldsymbol{y = -\frac{10}{x}} \quad ; \quad x^2 - \left(-\frac{10}{x}\right)^2 = 21 \quad ; \quad x^4 - 100 = 21x^2$

$x^4 - 21x^2 - 100 = 0 \quad ; \quad (x^2-25)(x^2+4) = 0$

$(x^2+4) = 0$; *no solution since x is real* $\quad (x^2 - 25) = 0 \quad x = 5 \ \text{and} \ x = -5$

$x = 5 \ y = -2 \ \text{and} \ x = -2 \ y = 2 \quad ; \quad \boldsymbol{\sqrt{21-20i} = \pm(5-2i)}$

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c. $\sqrt{2i} = x + yi \quad ; \quad (2i) = x^2 - y^2 + 2xyi \quad ; \quad x^2 - y^2 = 0 \quad ; \quad 2xy = 2$

$x^2 - y^2 = 0 \quad ; \quad 2xy = 2 \quad ; \quad \boldsymbol{y = \frac{1}{x}} \quad ; \quad x^2 - \left(\frac{1}{x}\right)^2 = 0 \quad ; \quad x^4 - 1 = 0$

$x^4 - 1 = 0 \quad ; \quad (x^2-1)(x^2+1) = 0$

$(x^2+1) = 0$; *no solution since x is real* $\quad (x^2 - 1) = 0 \quad x = 1 \ \text{and} \ x = -1$

$x = 1 \ y = 1 \ \text{and} \ x = -1 \ y = -1 \quad ; \quad \boldsymbol{\sqrt{2i} = \pm(1+i)}$

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d. $\sqrt{-24+10i} = x + yi \quad ; \quad (-24+10i) = x^2 - y^2 + 2xyi \quad ; \quad x^2 - y^2 = -24 \quad ; \quad 2xy = 10$

$x^2 - y^2 = -24 \quad ; \quad 2xy = 10 \quad ; \quad \boldsymbol{y = \frac{5}{x}} \quad ; \quad x^2 - \left(\frac{5}{x}\right)^2 = -24 \quad ; \quad x^4 - 25 = -24x^2$

$x^4 + 24x^2 - 25 = 0 \quad ; \quad (x^2+25)(x^2-1) = 0$

$(x^2+25) = 0$; *no solution since x is real* $\quad (x^2 - 1) = 0 \quad x = 1 \ \text{and} \ x = -1$

$x = 1 \ y = 5 \ \text{and} \ x = -1 \ y = -5 \quad ; \quad \boldsymbol{\sqrt{-24+10i} = \pm(1+5i)}$

Ex3 Complex Number Modules & Argument $\left(|\boldsymbol{z}| = \sqrt{\boldsymbol{x^2} + \boldsymbol{y^2}} \quad ; \quad \angle(\boldsymbol{z}) = \boldsymbol{tan^{-1}}\left(\frac{y}{x}\right)\right)$

1. $z = (1 + \sqrt{3}i)$; $|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$; $\angle(z) = tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

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b. $z = (2 - 2i)$; $|z| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$; $\angle(z) = tan^{-1}\left(-\frac{2}{2}\right) = -\frac{\pi}{4}$

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c. $z = (-\sqrt{2} - \sqrt{2}i)$; $|z| = \sqrt{(-\sqrt{2})^2 + (-\sqrt{2})^2} = 2$; $\angle(z) = tan^{-1}\left(\frac{-\sqrt{2}}{-\sqrt{2}}\right) = -\frac{3\pi}{4}$

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d. $z = (0 + 2i)$; $|z| = \sqrt{(0)^2 + (2)^2} = 2$; $\angle(z) = tan^{-1}\left(\frac{2}{0}\right) = \frac{\pi}{2}$

---

e. $z = (3 + 0i)$; $|z| = \sqrt{(3)^2 + (0)^2} = 3$; $\angle(z) = tan^{-1}\left(\frac{0}{3}\right) = 0$

---

f. $z = (-\sqrt{3} + i)$; $|z| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$; $\angle(z) = tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}$

---

g. $z = (0 - 3i)$; $|z| = \sqrt{(0)^2 + (-3)^2} = 3$; $\angle(z) = tan^{-1}\left(\frac{-3}{0}\right) = -\frac{\pi}{2}$

---

h. $z = (-5 + 0i)$; $|z| = \sqrt{(5)^2 + (0)^2} = 5$; $\angle(z) = tan^{-1}\left(\frac{0}{-5}\right) = \pi$

---

i. $z = (-3 - 3i)$; $|z| = \sqrt{(-3)^2 + (-3)^2} = 3\sqrt{2}$; $\angle(z) = tan^{-1}\left(\frac{-3}{-3}\right) = \frac{-3\pi}{4}$

---

2. $z_1 = \left(-3 + 3\sqrt{3}i\right)$ and $z_2 = \left(\sqrt{3} + i\right)$

$$|z_1| = \sqrt{(-3)^2 + \left(3\sqrt{3}\right)^2} = 6 \quad and \ \angle z_1 = tan^{-1}\left(\frac{3\sqrt{3}}{-3}\right) = \frac{2\pi}{3}$$

$$|z_2| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2 \quad and \ \angle z_2 = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$z_1 z_2 = \left(-3 + 3\sqrt{3}i\right) \times \left(\sqrt{3} + i\right) = \left(-3\sqrt{3} - 3\sqrt{3}\right) + 6i = -6\sqrt{3} + 6i$$

$$|z_1 z_2| = \sqrt{\left(-6\sqrt{3}\right)^2 + (6)^2} = 12 \quad and \ \angle z_2 = tan^{-1}\left(\frac{6}{-6\sqrt{3}}\right) = \frac{5\pi}{6}$$

---

b. $z_1 = (0 + 3i)$ and $z_2 = \left(\sqrt{2} - \sqrt{2}i\right)$

$$|z_1| = \sqrt{(0)^2 + (3)^2} = 3 \quad and \ \angle z_1 = tan^{-1}\left(\frac{3}{0}\right) = \frac{\pi}{2}$$

$$|z_2| = \sqrt{\left(\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2} = 2 \quad and \ \angle z_2 = tan^{-1}\left(\frac{-\sqrt{2}}{\sqrt{2}}\right) = -\frac{\pi}{4}$$

$$z_1 z_2 = (0 + 3i) \times \left(\sqrt{2} - \sqrt{2}i\right) = \left(3\sqrt{2} + 3\sqrt{2}i\right)$$

$$|z_1 z_2| = \sqrt{\left(3\sqrt{2}\right)^2 + \left(3\sqrt{2}\right)^2} = 6 \quad and \ \angle z_2 = tan^{-1}\left(\frac{3\sqrt{2}}{3\sqrt{2}}\right) = \frac{\pi}{4}$$

---

3. $z_1 = \left(-3 + 3\sqrt{3}i\right) \qquad and\ z_2 = \left(\sqrt{3} + i\right)$

$$|z_1| = \sqrt{(-3)^2 + \left(3\sqrt{3}\right)^2} = 6 \qquad and\ \angle z_1 = tan^{-1}\left(\frac{3\sqrt{3}}{-3}\right) = \frac{2\pi}{3}$$

$$|z_2| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2 \qquad and\ \angle z_2 = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\frac{z_1}{z_2} = \frac{\left(-3 + 3\sqrt{3}i\right)}{\left(\sqrt{3} + i\right)} \times \frac{\left(\sqrt{3} - i\right)}{\left(\sqrt{3} - i\right)} = \frac{\left(-3\sqrt{3} + 3\sqrt{3}\right) + 12i}{4} = 0 + 3i$$

$$\left|\frac{z_1}{z_2}\right| = \sqrt{(0)^2 + (3)^2} = 3 \qquad and\ \angle z_2 = tan^{-1}\left(\frac{3}{0}\right) = \frac{\pi}{2}$$

---

b. $z_1 = (0 + 3i) \qquad and\ z_2 = \left(\sqrt{2} - \sqrt{2}i\right)$

$$|z_1| = \sqrt{(0)^2 + (3)^2} = 3 \qquad and\ \angle z_1 = tan^{-1}\left(\frac{3}{0}\right) = \frac{\pi}{2}$$

$$|z_2| = \sqrt{\left(\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2} = 2 \qquad and\ \angle z_2 = tan^{-1}\left(\frac{-\sqrt{2}}{\sqrt{2}}\right) = -\frac{\pi}{4}$$

$$\frac{z_1}{z_2} = \frac{(0 + 3i)}{\left(\sqrt{2} - \sqrt{2}i\right)} \times \frac{\left(\sqrt{2} + \sqrt{2}i\right)}{\left(\sqrt{2} + \sqrt{2}i\right)} = \frac{\left(-3\sqrt{2} + 3\sqrt{2}i\right)}{4}$$

$$|z_1 z_2| = \sqrt{\left(\frac{-3\sqrt{2}}{4}\right)^2 + \left(\frac{3\sqrt{2}}{4}\right)^2} = \frac{3}{2} \qquad and\ \angle z_2 = tan^{-1}\left(-\frac{\frac{3\sqrt{2}}{4}}{\frac{3\sqrt{2}}{4}}\right) = \frac{3\pi}{4}$$

---

4. $z = (1 + i) \quad ; \quad iz = (-1 + i)$

$|z| = \sqrt{(1)^2 + (1)^2} = \sqrt{2} \quad ; \quad |iz| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$\angle z = tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4} \quad ; \quad \angle iz = tan^{-1}\left(\frac{1}{-1}\right) = \frac{3\pi}{4}$

---

b. $z = \left(-\sqrt{3} - i\right) \quad ; \quad iz = (1 - \sqrt{3}i)$

$|z| = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = 4 \quad ; \quad |iz| = \sqrt{(1)^2 + \left(1 - \sqrt{3}\right)^2} = 2$

$\angle z = tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = -\frac{5\pi}{6} \quad ; \quad \angle iz = tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$

---

5. $z = (1 + i) \quad ; \quad \bar{z} = (1 - i)$

$|z| = \sqrt{(1)^2 + (1)^2} = \sqrt{2} \quad ; \quad |\bar{z}| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$

$\angle z = tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4} \quad ; \quad \angle\, \bar{z} = tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$

---

b. $z = \left(-\sqrt{3} - i\right) \quad ; \quad \bar{z} = (-\sqrt{3} + i)$

$|z| = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = 4 \quad ; \quad |\,\bar{z}| = \sqrt{\left(-\sqrt{3}\right)^2 + (1)^2} = 4$

$\angle z = tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = -\frac{5\pi}{6} \quad ; \quad \angle iz = tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}$

---

6. $z = (1+i)$; $\bar{z} = (1-i)$; $z\bar{z} = (1+i)(1-i) = 2 + 0i$

$|z\bar{z}| = 2$; $\angle\, \overline{zz} = tan^{-1}\left(\frac{0}{2}\right) = 0$

---

b. $z = \left(-\sqrt{3} - i\right)$; $\bar{z} = \left(-\sqrt{3} + i\right)$; $z\bar{z} = \left(-\sqrt{3} - i\right)\left(-\sqrt{3} + i\right) = 4 + 0i$

$|z\bar{z}| = 4$; $\angle z\, \bar{z} = tan^{-1}\left(\frac{0}{4}\right) = 0$

---

7. Summarising Rules

$|z_1 z_2| = |z_1| \times |z_2|$ $arg(z_1 z_2) = arg(z_1) + arg(z_2)$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

$|iz| = |z|$ $arg(iz) = arg(z) + \frac{\pi}{2}$ $about\ the\ orign$

$|\bar{z}| = |z|$ $arg(\bar{z}) = -arg(z)$ $or\ relfection\ in\ real\ axis$

$z|\bar{z}|$ is $\boldsymbol{ALWAYS}$ $real\ and\ arg(z\bar{z}) = 0$

Ex 6 – Complex Roots

1. $Find\ all\ the\ roots\ of\ f(z) = z^3 - 11z + 20 = 0$

$z = -4 \quad f(-4) = (-4)^3 - 11 \times (-4) + 20 = -64 + 44 + 20 = 0$; $hence\ (z+4)\ is\ a\ factor$

$$z+4\overset{\displaystyle z^2 - 4z + 5}{\overline{\big)\,z^3 + 0 - 11z + 20}}$$

$z^2 - 4z + 5 = 0$; $a = 1 \quad b = -4 \quad c = 5$; $z = \dfrac{4 \pm \sqrt{16-20}}{2} = (2 \pm i)$

$Roots\ are\ z = -4\ \ and\ (2 \pm i)$

---

2. $Verify\ z = (1+i)\ is\ a\ factor\ of\ \ f(z) = z^4 + 3z^2 - 6z + 10 = 0$

$f(z) = (1+i)^4 + 3(1+i)^2 - 6(1+i) + 10 = (1 + 4i - 6 - 4i + 1) + 6i - 6 - 6i + 10 = 0$

$Since\ (1+i)\ is\ a\ factor\ then\ (1-i)\ is\ also\ a\ factor.$

$Quadratic\ from\ roots\ is\ \ (z-(1+i))(z-(1-i)) \ = ((z-1)-i)((z-1)+i)$

$Difference\ of\ two\ squares \qquad = (z^2 - 2z + 1) + 1) = (z^2 - 2z + 2)$

$$z^2 - 2z + 2\overset{\displaystyle z^2 + 2z + 5}{\overline{\big)\,z^4 + 0 + 3z^2 - 6z + 10}}$$

$z^2 + 2z + 5 = 0$; $a = 1 \quad b = 2 \quad c = 5$; $z = \dfrac{-2 \pm \sqrt{4-20}}{2} = (-1 \pm 2i)$

$Roots\ are\ z = (1 \pm i)\ \ and\ (-1 \pm 2i)$

---

3. $Verify\ z = (-2+3i)\ is\ a\ factor\ of\ \ f(z) = \ z^4 + 7z^2 - 12z + 130 = 0$

$f(z) = \ (-2+3i)^4 + 7(-2+3i)^2 - 12(-2+3i) + 130$

$= (16 - 96i - 216 + 216i + 81) - 35 - 84i + 24 - 36i + 130 = 0 \quad hence\ (-2+3i) is\ a\ factor$

$Since\ (-2+3i)\ is\ a\ factor\ then\ (-2-3i)\ is\ also\ a\ factor.$

$Quadratic\ from\ roots\ is\ \ (z-(-2+3i))(z-(-2-3i)) \ = \big((z+2)-3i\big)((z+2)+3i)$

$Difference\ of\ two\ squares \qquad = (z^2+4z+4)+9) = (z^2+4z+13)$

$$\begin{array}{r} z^2 - 4z + 10 \\ z^2+4z+13\overline{\big)\,z^4+0+7z^2-12z+130} \end{array}$$

$z^2 - 4z + 10 = 0 \qquad ; \qquad a = 1 \ \ b = -4 \ \ c = 10 \qquad ; \ \ z = \dfrac{4 \pm \sqrt{16-40}}{2} = (2 \pm \sqrt{6}i)$

$Roots\ are\ z = (-2 \pm 3i)\ \ and\ \big(2 \pm \sqrt{6}i\big)$

---

4. $z = Given\ (2-i)\ is\ a\ factor\ of\ \ f(z) = \ 3z^3 - 10z^2 + 7z + 10 = 0$

$Since\ (2-i)\ is\ a\ factor\ then\ (2+i)\ is\ also\ a\ factor.$

$Quadratic\ from\ roots\ is\ \ (z-(2-i))(z-(2+i))\ \ = \big((z-2)-i\big)((z-2)+i)$

$Difference\ of\ two\ squares \qquad = (z^2-4z+4)+1) = (z^2-4z+5)$

$$\begin{array}{r} 3z\ +\ 2 \\ z^2-4z+5\overline{\smash{)}3z^3-10z^2+7z+10} \end{array}$$

$3z+2=0 \quad ; \quad z = -\dfrac{2}{3}$

$Roots\ are\ z = (2 \pm i)\ and\ z = -\dfrac{2}{3}$

---

5. $z = Given\ (1-2i)\ is\ a\ factor\ of\ \ f(z) = \ z^3 + z + 10 = 0$

$Since\ (1-2i)\ is\ a\ factor\ then\ (1+2i)\ is\ also\ a\ factor.$

$Quadratic\ from\ roots\ is\ \ (z-(1-2i))(z-(1+2i))\ \ = \big((z-1)-2i\big)((z-1)+2i)$

$Difference\ of\ two\ squares \qquad = (z^2-2z+1)+4) = (z^2-2z+5)$

$$\begin{array}{r} z+\ 2 \\ z^2-2z+5\overline{\smash{)}z^3+0+z+10} \end{array}$$

$z+2=0 \quad ; \quad z=-2$

$Roots\ are\ z = (1 \pm 2i)\ and\ z = -2$

---

6. $z = Given\ (3+i)\ is\ a\ factor\ of\ \ f(z) = z^3 - 3z^2 - 8z + 30 = 0$

$Since\ (3+i)\ is\ a\ factor\ then\ (3-i)\ is\ also\ a\ factor.$

$Quadratic\ from\ roots\ is\ \ (z-(3+i))(z-(3-i)\ \ = \big((z-3)+i\big)\big((z-3)-i\big)$

$Difference\ of\ two\ squares \qquad = (z^2 - 6z + 9) + 1) = (z^2 - 6z + 10)$

$$\begin{array}{r} z + 3 \\ z^2 - 6z + 10 \big) \overline{z^3 - 3z^2 - 8z + 30} \end{array}$$

$z + 3 = 0 \quad ; \quad z = -3$

$Roots\ are\ z = (3 \pm i)\ and\ z = -3$

---

7. $Show\ z = (-1+i)\ is\ a\ factor\ of\ \ f(z) = z^4 - 2z^3 - z^2 + 2z + 10 = 0$

$Suppose\ (-1+i)\ is\ a\ factor\ then\ (-1-i)\ is\ also\ a\ factor.$

$Quadratic\ from\ roots\ is\ \ (z-(-1+i))(z-(-1-i))\ \ = ((z+1)-i)((z+1)+i)$

$Difference\ of\ two\ squares \qquad = (z^2 + 2z + 1) + 1) = (z^2 + 2z + 2)$

$$\begin{array}{r} z^2 - 4z + 5 \\ z^2 + 2z + 2 \big) \overline{z^4 - 2z^3 - z^2 + 2z + 10} \end{array}$$

$Since\ remainder\ 0\ then\ (-1 \pm i)\ are\ factorsof\ f(z)$

$z^2 - 4z + 5 = 0 \quad ; \quad a = 1 \ \ b = -4 \ \ c = 5 \quad ; \ z = \dfrac{4 \pm \sqrt{16 - 20}}{2} = (2 \pm i)$

$Roots\ are\ z = (-1 \pm i)\ \ and\ (2 \pm i)$