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Advanced Higher Maths

Advanced Higher - Unit 2.2 Sequence & Series Solutions

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Ex1 Arithmetic Sequences and Series - $u_n = a + (n-1)d$; $S_n = \frac{n}{2}[2a + (n-1)d]$

1. $3, 11, 19, \dots$; $a = 3 \quad d = 8$; $u_n = 3 + 8(n-1) = \mathbf{8n - 5}$; $u_n = 8(19) - 5 = 147$

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b. $8, 5, 2, \dots$; $a = 8 \quad d = 3$; $u_n = 8 - 3(n-1) = \mathbf{3n + 11}$; $u_{15} = -3(15) + 11 = -34$

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c. $7, 6.5, 6, \dots$; $a = 7 \quad d = -0.5$; $u_n = 7 - 0.5(n-1) = \mathbf{-0.5n + 7.5}$; $u_{12} = -0.5(12) + 7.5 = 1.5$

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2. $2,4,6, \dots 46$; $a = 2 \quad d = 2$; $u_n = 2 + 2(n-1)$; $46 = 2 + 2(n-1)$; $n = \frac{44}{2} + 1 = 23$

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b. $50,47,44, \dots 14$; $a = 50 \quad d = -3$; $u_n = 50 - 3(n-1)$; $14 = 50 - 3(n-1)$; $n = \frac{-36}{-3} + 1 = 13$

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c. $2, -9, -20 \dots -130$; $a = 2 \quad d = -11$; $u_n = 2 - 11(n-1)$; $-130 = 2 - 11(n-1)$; $n = \frac{-132}{-11} + 1 = 13$

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3. $4 + 10 + 16 + \cdots$; $a = 4 \quad d = 6$; $S_{12} = \frac{12}{2}[2(4) + 6(12-1)] = 444$

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b. $15 + 13 + 11 + \cdots$; $a = 15 \quad d = -2$; $S_{20} = \frac{20}{2}[2(15) - 2(20-1)] = -80$

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c. $20 + 13 + 6 + \cdots$; $a = 20 \quad d = -7$; $S_{16} = \frac{16}{2}[2(20) - 7(16-1)] = -520$

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4. $u_2 = 15 \quad u_5 = 21 \;;\; 15 = a + (2-1)d \;;\; a + d = 15 \qquad 21 = a + (5-1)d \;;\; a + 4d = 21$

$a + d = 15 \;;\; a + 4d = 21 \quad sim.eqn \;\; d = 2 \quad a = 13 \;;\; S_{10} = \frac{10}{2}[2(13) + 2(10-1)] = 220$

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b. $u_4 = 18 \quad d = -5 \;;\; a + (4-1)d \;;\; a + 3(-5) = 18 \quad a = 33$

$S_{16} = \frac{16}{2}[2(33) - 5(16-1)] = -72$

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c. $u_3 = 7 \quad u_{12} = 61 \;;\; a + (3-1)d = 7 \;;\; a + 2d = 7 \;;\; a + (12-1)d = 61 \;;\; a + 11d = 61$

$a + 2d = 7 \;;\; a + 11d = 61 \quad sim.eqn \;\; d = 6 \quad a = -5 \;;\; S_{15} = \frac{15}{2}[2(-5) + 6(15-1)] = 555$

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5. $S_{10} = 120 \;;\; 120 = \frac{10}{2}[2a + 9d] \;;\; 10a + 45d = 120 \;;\; \mathbf{2a + 9d = 24}$

$S_{20} = 840 \;;\; 840 = \frac{20}{2}[2a + 19d] \;;\; 20a + 190d = 840 \;;\; \mathbf{2a + 19d = 84}$

$2a + 9d = 24 \;;\; 2a + 19d = 84 \;;\; sim.eqn \;\; d = 6 \quad a = -15 \;;\; S_{30} = \frac{30}{2}[2(-15) + 6(30-1)] = 2160$

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6. $u_{15} = 7 \;;\; 7 = a + d(15-1) \;;\; 7 = a + 14d \;;\; \mathbf{a + 14d = 7}$

$S_9 = 18 \;;\; 18 = \frac{9}{2}[2a + (9-1)d] \;;\; 9a + 36d = 18 \;;\; \mathbf{a + 4d = 2}$

$a + 14d = 7 \;;\; a + 4d = 2 \;;\; sim.eqn \;\; d = 0.5 \quad a = 0 \;;\; u_{20} = 0 + 0.5(20-1) = 9.5$

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7. $28+24+20+\cdots$; $a=28$ $d=-4$; $0=\frac{n}{2}[2a+(n-1)d]$; $\frac{n}{2}[56-4(n-1)]=0$

$n[56-4(n-1)]=0$; $-4n^2+56n-4n=0$; $n^2-16n+n=0$; $n^2-15n=0$

$n^2-15n=0$; $n(n-15)=0$; $n=0$ and $n=15$

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8. $u_3=x$ $u_6=2x$ $u_1=3$; $3=a+(1-1)d$; $a=3$

$u_3=3+(n-1)d=x$; $3+\boldsymbol{2d}=\boldsymbol{x}$ $u_6=a+(n-1)d=x$; $3+\boldsymbol{5d}=\boldsymbol{2x}$

$3+2d=x$; $3+5d=2x$; $3d=x$; $3d=3+2d$; $d=3$

$u_{10}=3+3(n-1)=3+3(10-1)=30$

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9. $1+3+5+\cdots$ $u_1=a+(n-1)d=1$; $u_1=a+(1-1)d=1$; $a=1$

$1+3+5+\cdots$ $u_2=1+(2-1)d=3$; $d=2$

$S_n=\frac{n}{2}[2(a)+d(n-1)]$; $a=1$ $d=2$; $1521=\frac{n}{2}[2(1)+2(n-1)]$

$n+n^2-n=1521$; $n^2=1521$; $n=39$

Ex2A Geometric Sequences and Series - $u_n = ar^{(n-1)}$; $S_n = \frac{a(1-r^n)}{(1-r)}$ or $S_n = \frac{a(r^n-1)}{(r-1)}$

1. (a) $1,3,9,27,\ldots$ $r = \frac{3}{1} = 3$ (b) $12,6,3,1.5,\ldots$ $r = \frac{6}{12} = \frac{1}{2}$

   (c) $7,0.7,0.07,\ldots$ $r = \frac{0.7}{7} = \frac{1}{10}$ (d) $18,54,162,\ldots$ $r = \frac{54}{18} = 3$

   (e) $2.25,1.5,1,\ldots$ $r = \frac{2}{2.25} = \frac{2}{3}$ (f) $\frac{1}{4},\frac{1}{8},\frac{1}{16},\ldots$ $r = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}$

   (g) $1,-1,1,-1,\ldots$ $r = \frac{-1}{1} = -1$ (h) $1,-2,4,-8\ldots$ $r = \frac{-2}{1} = -2$

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2. (a) $u_n, = 3^{(n-1)} = 1,3,9,27\ldots$ (b) $u_n, = 3(-2)^{(n-1)} = 3,-6,12,-24\ldots$

   (c) $u_n, = 6\left(\frac{1}{2}\right)^{(n-1)} = 6,3,\frac{3}{2},\frac{3}{4}\ldots$

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3. (a) $1,2,4\ldots$; $r = \frac{2}{1} = 2$; $u_n = 1(2)^{(n-1)}$; $u_5 = (2)^{(5-1)} = 16$

   (b) $2,6,18\ldots$; $r = \frac{6}{2} = 3$; $u_n = 2(3)^{(n-1)}$; $u_6 = 2(3)^{(6-1)} = 486$

   (c) $4,12,36\ldots$; $r = \frac{12}{4} = 3$; $u_n = 4(3)^{(n-1)}$; $u_6 = 4(3)^{(6-1)} = 972$

   (d) $2,20,200\ldots$; $r = \frac{20}{2} = 10$; $u_n = 2(10)^{(n-1)}$; $u_5 = 2(10)^{(5-1)} = 20000$

   (e) $1,-2,4\ldots$; $r = -\frac{2}{1} = -2$; $u_n = 1(-2)^{(n-1)}$; $u_6 = 1(-2)^{(6-1)} = -32$

   (f) $6,3,\frac{3}{2}\ldots$; $r = \frac{3}{6} = \frac{1}{2}$; $u_n = 6\left(\frac{1}{2}\right)^{(n-1)}$; $u_6 = 6\left(\frac{1}{2}\right)^{(7-1)} = \frac{3}{32}$

4. (a) $1,2,4\ldots$; $r=\frac{2}{1}=2$; $u_n=(2)^{(n-1)}$

(b) $3,6,12\ldots$; $r=\frac{6}{3}=2$; $u_n=3(2)^{(n-1)}$

(c) $2,-6,18\ldots$; $r=-\frac{6}{2}=-3$; $u_n=2(-3)^{(n-1)}$

(d) $9,3,1\ldots$; $r=\frac{1}{3}$; $u_n=9\left(\frac{1}{3}\right)^{(n-1)}=3^2(3^{-1})^{(n-1)}=3^2 3^{-n+1}=3^{-n+3}=(3^{-1})^{(n-3)}=\left(\frac{1}{3}\right)^{(n-3)}$

(e) $4,2,1\ldots$; $r=\frac{1}{2}$; $u_n=4\left(\frac{1}{2}\right)^{(n-1)}=2^2 2^{-n+1}=2^{-n+3}=(2^{-1})^{(n-3)}=\left(\frac{1}{2}\right)^{(n-3)}$

(e) $\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16}\ldots$; $r=\frac{\frac{1}{4}}{\frac{1}{2}}=\frac{1}{2}$; $u_n=\frac{1}{2}\left(\frac{1}{2}\right)^{(n-1)}=\left(\frac{1}{2}\right)^{n}$

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5. (a) $a=6$; $u_3=ar^{(n-1)}$; $6r^2=24$; $r=\pm 2$; $u_5=6(2)^{(5-1)}=96$

(b) $a=50$; $u_4=50r^{(4-1)}$; $50r^3=400$; $r=2$; $u_5=50(2)^{(5-1)}=800$

(c) $a=36$; $u_2=36r^{(2-1)}$; $36r=-12$; $r=-\frac{1}{3}$; $u_5=36\left(-\frac{1}{3}\right)^{(5-1)}=\frac{4}{9}$

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6. (a) $1+2+4\ldots\ to\ 8\ terms$; $a=1$; $r=\frac{2}{1}=2$; $S_n=\frac{1(2^8-1)}{(2-1)}=255$

(b) $2+6+18\ldots\ to\ 6\ terms$; $a=2$; $r=\frac{6}{2}=3$; $S_n=\frac{2(3^6-1)}{(3-1)}=728$

(c) $2-4+8\ldots\ to\ 5\ terms$; $a=2$; $r=-\frac{4}{2}=-2$; $S_n=\frac{2((-2)^5-1)}{(-2-1)}=22$

(d) $2-6+18\ldots\ to\ 5\ terms$; $a=2$; $r=-\frac{6}{2}=-3$; $S_n=\frac{2((-3)^5-1)}{(-3-1)}=122$

(e) $1+\frac{1}{2}+\frac{1}{4}\ldots\ to\ 6\ terms$; $a=1$; $r=\frac{\frac{1}{2}}{1}=\frac{1}{2}$; $S_n=\frac{1\left(\left(\frac{1}{2}\right)^6-1\right)}{\left(\frac{1}{2}-1\right)}=\frac{63}{32}=1\frac{31}{32}$

(f) $1+\frac{1}{3}+\frac{1}{9}\ldots\ to\ 5\ terms$; $a=1$; $r=\frac{\frac{1}{3}}{1}=\frac{1}{3}$; $S_n=\frac{1\left(\left(\frac{1}{3}\right)^5-1\right)}{\left(\frac{1}{3}-1\right)}=\frac{121}{81}=1\frac{40}{81}$

(g) $1+x+x^2\ldots\ to\ n\ terms$; $a=1$; $r=\frac{x}{1}=x$; $S_n=\frac{1((x)^n-1)}{(x-1)}=\frac{x^n-1}{x-1}=\frac{1-x^n}{1-x}$

(h) $1-y+y^2\ldots\ to\ n\ terms$; $a=1$; $r=-\frac{y}{1}=-y$; $S_n=\frac{1((-yx)^n-1)}{((-y)-1)}=\frac{1-(-y^n)}{1+y}$

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7. $3+3^2+3^3\ldots\ 3^n=363$ $a=3$ $r=3$; $S_n=\frac{3(3^n-1)}{(3-1)}=363$; $3^n=242+1$; $n=log_3 243=5$

(b) $2+2^2\ldots 2^n=510$ $a=2$ $r=2$; $S_n=\frac{2(2^n-1)}{(2-1)}=510$; $2^n=255+1$; $n=log_2 256=8$

Ex2B – Sum to Infinity - $S_\infty = \frac{a}{1-r}$ *exists if and only if* $-1 < r < 1$

1. $(a)\ \ 1 + \frac{1}{3} + \frac{1}{9} \ldots \ ; \quad limit\ exist\ since\ -1 < r < 1\ ;\ a = 1\ ;\ r = \frac{\frac{1}{3}}{1} = \frac{1}{3}\ ;\ S_\infty = \frac{1}{1-\frac{1}{3}} = \mathbf{\frac{3}{2}}$

$(b)\ \ 1 + 2 + 4 \ldots\ ;\ a = 1\ ;\ r = \frac{2}{1} = 2\ ;\ S_\infty\ does\ not\ exist\ since\ r = 2$

$(c)\ \ 4 + 1 + \frac{1}{4} \ldots\ ;\ limit\ exist\ since\ -1 < r < 1\ ;\ a = 4\ ;\ r = \frac{1}{4}\ ;\ S_\infty = \frac{4}{1 - \frac{1}{4}} = \mathbf{\frac{16}{3}}$

$(d)\ \ 8 + 4 + 2 \ldots\ ;\ limit\ exist\ since\ -1 < r < 1\ ;\ a = 8\ ;\ r = \frac{1}{2}\ ;\ S_\infty = \frac{8}{1 - \frac{1}{2}} = \mathbf{16}$

$(e)\ \ 1 - 5 + 25 \ldots\ ;\ a = 1\ ;\ r = -\frac{5}{1}\ ;\ S_\infty\ does\ not\ exist\ since\ r = -5$

$(f)\ \ 10 - 9 + 8.1 \ldots\ limit\ exist\ since\ -1 < r < 1\ ;\ a = 10\ ;\ r = -\frac{9}{10}\ ;\ S_\infty = \frac{10}{1 + \frac{9}{10}} = \mathbf{\frac{100}{19}}$

$(g)\ \ 1 - \frac{1}{2} + \frac{1}{4} \ldots\ limit\ exist\ since\ -1 < r < 1\ ;\ a = 1\ ;\ r = -\frac{\frac{1}{2}}{1} = \frac{1}{2}\ ;\ S_\infty = \frac{1}{1 + \frac{1}{2}} = \mathbf{\frac{2}{3}}$

$(h)\quad 2+\frac{4}{3}+\frac{8}{9}\ldots\ limit\ exist\ since\ -1<r<1\quad;\quad a=1\quad;\quad r=\frac{\frac{4}{3}}{2}=\frac{2}{3}\quad;\quad S_\infty=\frac{2}{1-\frac{2}{3}}=\mathbf{6}$

Ex3 – Expansion of Geometric Series $\frac{1}{a+b} = \frac{1}{a}\left(1 - \frac{b}{a} + \frac{b^2}{a^2} - \frac{b^3}{a^3}\right) \ldots \ldots \ldots$

1. (a) $\frac{1}{1+2x} = \frac{1}{1}\left(1 - \frac{2x}{1} + \frac{(2x)^2}{1} - \frac{(2x)^3}{1}\right) = 1 - 2x + 4x^2 - 8x^3$

(b) $\dfrac{1}{1-3x} = \dfrac{1}{1+(-3x)} = \dfrac{1}{1}\left(1 - \dfrac{(-3x)}{1} + \dfrac{(-3x)^2}{1} - \dfrac{(-3x)^3}{1}\right) = 1 + 3x + 9x^2 + 27x^3$

(c) $\dfrac{1}{1+\frac{x}{2}} == \dfrac{1}{1}\left(1 - \dfrac{\left(\frac{x}{2}\right)}{1} + \dfrac{\left(\frac{x}{2}\right)^2}{1} - \dfrac{\left(\frac{x}{2}\right)^3}{1}\right) = 1 - \dfrac{x}{2} + \dfrac{x^2}{4} - \dfrac{x^3}{8}$

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2. (a) $\frac{1}{2+4x} = \frac{1}{2(1+2x)} = \frac{1}{2}\left(1 - \frac{2x}{1} + \frac{(2x)^2}{1} - \frac{(2x)^3}{1}\right) = 1 - x + 2x^2 - 4x^3$

(b) $\dfrac{1}{3-x} = \dfrac{1}{3(1+(-\frac{x}{3})} = \dfrac{1}{3}\left(1 - \dfrac{\left(-\frac{x}{3}\right)}{1} + \dfrac{\left(-\frac{x}{3}\right)^2}{1} - \dfrac{\left(-\frac{x}{3}\right)^3}{1}\right) = \dfrac{1}{3} + \dfrac{x}{9} + \dfrac{x^2}{27} + \dfrac{x^3}{81}$

(c) $\dfrac{1}{2-3x} = \dfrac{1}{2\left(1+\left(\frac{-3x}{2}\right)\right)} = \dfrac{1}{2}\left(1 - \dfrac{\left(\frac{-3x}{2}\right)}{1} + \dfrac{\left(\frac{-3x}{2}\right)^2}{1} - \dfrac{\left(\frac{-3x}{2}\right)^3}{1}\right) = \dfrac{1}{2} + \dfrac{3x}{4} + \dfrac{9x^2}{8} + \dfrac{27x^3}{16}$

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3. (a) $\sqrt{(0.9)} = (1-0.1)^{\frac{1}{2}} = 1^{\frac{1}{2}} + \frac{1}{2}(-0.1) - \frac{1}{8}(-0.1)^2 - \frac{3}{48}(-0.1)^3 = 0.949$

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4. $\frac{1}{1+4x} = \frac{1}{1}\left(1 - \frac{4x}{1} + \frac{(4x)^2}{1} - \frac{(4x)^3}{1}\right) = 1 - 4x + 16x^2 - 64x^3$

$\dfrac{1-3x}{1+4x} = (1-3x)(1 - 4x + 16x^2 - 64x^3 - 3x + 12x^2 - 48x^3 \ldots . = 1 - 7x + 28x^2 - 112x^3$

Ex1 – Maclaurin's Theorem $f(x) = f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + \cdots,$

1. $f(x) = cosx \;\; f(0) = 1 \;\; ; \;\; f^{'}(x) = -sinx \;\; f^{'}(0) = 0 \;\; ; \;\; f^{''}(x) = -cosx \;\; f^{''}(0) = -1$

$f^3(x) = sinx \;\; f^3(0) = 0 \;\; ; \;\; f^4(x) = cosx \;\; f^4(0) = 1 \;\; ; \;\; f^5(x) = -sinx \;\; f^5(0) = 0$

$f^6(x) = -cosx \;\; f^6(0) = -1$

$$f(x) = f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + \cdots, = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - - \frac{x^6}{6!}$$

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2. $f(x) = tanx \;\; f(0) = 0 \;\; ; \;\; f^{'}(x) = sec^2x \;\; f^{'}(0) = 1 \;\; ; \;\; f^{''}(x) = 2sec^2xtanx \;\; f^{''}(0) = 0$

$f(x) = 2sec^2xtanx \;\; ; \;\; u = 2tanx \;\; \frac{du}{dx} = 2sec^2x \quad v = sec^2x \;\; \frac{dv}{dx} = sec^2xtanx$

$f^3(x) = sec^2x \times 2sec^2x + 2tanx \times sec^2xtanx = 2sec^4x + 2sec^2xtan^2x \;\; ; \;\; f^3(0) = 2$

$$f(x) = f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + \cdots, = x + \frac{x^3}{3} \ldots.$$

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3. $f(x) = sin^{-1}x \;\; f(0) = 0 \;\; ; \;\; f^{'}(x) = \frac{1}{\sqrt{(1-x^2)}} \;\; f^{'}(0) = 1 \;\; ; \;\; f^{''}(x) = \frac{x}{(1-x^2)^{\frac{3}{2}}} \;\; f^{''}(0) = 0$

$f''(x) = \frac{x}{(1-x^2)^{\frac{3}{2}}} \;\; ; \;\; u = x \;\; \frac{du}{dx} = 1 \quad v = (1-x^2)^{-\frac{3}{2}} \;\; \frac{dv}{dx} = 3x(1-x^2)^{-\frac{5}{2}}$

$f^3(x) = (1-x^2)^{-\frac{3}{2}} \times 1 + x \times 3x(1-x^2)^{-\frac{5}{2}} \;\; ; \;\; f^3(0) = 1$

$$f(x) = f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + \cdots, = x + \frac{x^3}{6}$$

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4. $f(x) = ln(1-x)$ $f(0) = 0$; $f^{'}(x) = \frac{-1}{1-x}$ $f^{'}(0) = -1$; $f^{''}(x) = \frac{-1}{(1-x)^2}$ $f^{''}(0) = -1$

$f^3(x) = \dfrac{2}{(1-x)^3}$; $f^3(0) = -2$; $f^4(x) = \dfrac{6}{(1-x)^4}$; $f^4(0) = -6$

$$f(x) = f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + \cdots, = -x - x^2 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$$

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5. $f(x) = e^{3x}$ $f(0) = 1$; $f^{'}(x) = 3e^{3x}$ $f^{'}(0) = 3$; $f^{''}(x) = 9e^{3x}$ $f^{''}(0) = 9$

$f^3(x) = 27e^{3x}$; $f^3(0) = 27$; $f^4(x) = 81e^{3x}$; $f^4(0) = 81$

$$f(x) = f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + \cdots, = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$$

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6. $f(x) = ln(1+2x)$ $f(0) = 0$; $f^{'}(x) = 2(1+2x)^{-1}$ $f^{'}(0) = 2$;

$f^{''}(x) = -4(1+2x)^{-2}$ $f^{''}(0) = -4$; $f^3(x) = 16(1+2x)^{-3}$; $f^3(0) = 16$

$f^4(x) = -96(1+2x)^{-4}$; $f^4(0) = -96$; $f^5(x) = 768(1+2x)^{-5}$; $f^5(0) = 768$

$$f(x) = f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + \cdots, = 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \frac{32}{5}x^5$$

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7. $f(x) = sin3x$ $f(0) = 0$; $f^{'}(x) = 3cos3x$ $f^{'}(0) = 3$;

$f^{''}(x) = -9sin3x$ $f^{''}(0) = 0$; $f^3(x) = -27cos3x$; $f^3(0) = -27$

$f^4(x) = 81sin3x$; $f^4(0) = 0$; $f^5(x) = 243cos3x$; $f^5(0) = 243$

$$f(x) = f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + \cdots, = 3x - \frac{9}{2}x^3 + \frac{81}{40}x^5$$

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8. $f(x) = tan2x \quad f(0) = 0 \quad ; \quad f^{'}(x) = 3cos3x \quad f^{'}(0) = 3 \quad ;$

From Q2

$$tanx = f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + \cdots, = x + \frac{x^3}{3}$$

$$\tan(2x) = f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + \cdots, = 2x + \frac{8}{3}x^3 \ldots\ldots$$

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9. $f(x) = ln(2+x) \quad f(0) = ln2 \quad ; \quad f^{'}(x) = (2+x)^{-1} \quad f^{'}(0) = \frac{1}{2} \quad ;$

$f^{''(x)} = -(2+x)^{-2} \quad f(0) = -\frac{1}{4} \quad ; \quad f^3(x) = 2(2+x)^{-3} \quad f^{'}(0) = \frac{1}{4} \quad ;$

$$f(x) = f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + \cdots, = ln2 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{24}x^3$$

Ex2 – Combining Expansions

1. $f(x) = e^{sinx}$ $using\ substitution$ $(answer\ upto\ x^4)$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots, = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \qquad sinx = x - \frac{x^3}{6}$$

$$f(x) = e^{sinx} = 1 + \left(x - \frac{x^3}{6}\right) + \frac{1}{2}\left(x - \frac{x^3}{6}\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{6}\right)^3 + \frac{1}{24}\left(x - \frac{x^3}{6}\right)^4$$

$$f(x) = e^{sinx} = 1 + x - \frac{x^3}{6} + \frac{1}{2}\left(x^2 - \frac{x^4}{3}\right) + \frac{1}{6}(x^3) + \frac{1}{24}x^4 = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{8}x^4$$

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2. $f(x) = ln(1 + sinx)$ $using\ substitution$ $(answer\ upto\ x^4)$

$$ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots, \qquad sinx = x - \frac{x^3}{6}$$

$$f(x) = ln(1 + sinx) = \left(x - \frac{x^3}{6}\right) - \frac{1}{2}\left(x - \frac{x^3}{6}\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{6}\right)^3 - \frac{1}{4}\left(x - \frac{x^3}{6}\right)^4$$

$$f(x) = ln(1 + sinx) = x - \frac{x^3}{6} - \frac{1}{2}\left(x^2 - \frac{x^4}{3}\right) + \frac{1}{3}(x^3) - \frac{1}{4}x^4$$

$$f(x) = ln(1 + sinx) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4$$

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3. $f(x) = e^x(sinx)\ multiplication \quad (answer\ upto\ x^5)$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots, = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \qquad sinx = x - \frac{x^3}{6} + \frac{x^5}{120}$$

$$f(x) = e^x(sinx) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)$$

$$= \left(x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24}\right) + \left(-\frac{x^3}{6} - \frac{x^4}{6} - \frac{x^5}{12}\right) + \left(\frac{x^5}{120}\right)$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}$$

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4. $f(x) = ln(1 + e^x)\ differentiate \quad (answer\ upto\ x^3)$

$$f(x) = ln(1 + e^x)\ f(0) = ln2 \quad ; \quad f'(x) = e^x(1 + e^x)^{-1}\ f'(0) = \frac{1}{2} \quad ;$$

$$f''(x) = e^x(1 + e^x)^{-1} - e^{2x}(1 + e^x)^{-2} \qquad f''(0) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \quad ;$$

$$f^3(x) = [e^x(1 + e^x)^{-1} - e^{2x}(1 + e^x)^{-2}] - [2e^{2x}(1 + e^x)^{-2} - 2e^{3x}(1 + e^x)^{-3}]$$

$$f^3(0) = \left[\frac{1}{2} - \frac{1}{4}\right] - \left[\frac{1}{2} - \frac{1}{4}\right] = 0$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots, = ln2 + \frac{1}{2}x + \frac{1}{8}x^2$$
