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Advanced Higher Maths

Advanced Higher - Unit 2.3 Summation & Mathematical Proof Solutions

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Ex4 Summation Notation - $\sum_{k=1}^{n} f(k)$

1. $\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25$

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b. $\sum_{k=1}^{9}(2k-1) = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$

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c. $\sum_{k=1}^{10} \frac{2520}{k} = 2520 + 1260 + 840 + 630 + 504 + 420 + 360 + 315 + 280 + 252$

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2. $1 + 2 + 3 + 4 \ldots + 50 = \sum_{k=1}^{50} k \qquad Arithmetc\ Series\ 50 = 1 + 1(n-1) \quad ; \quad n = 50 \qquad k^{th} term = k$

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b. $5 + 10 + 15 + \cdots + 30 = \sum_{k=1}^{6} 5k \qquad Arithmetc\ Series\ 30 = 5 + 5(n-1) \quad ; \quad n = 6 \qquad k^{th} term = 5k$

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c. $3 + 5 + 7 + 9 + 11 + 13 = \sum_{k=1}^{6} 2k + 1 \quad Arithmetc\ Series\ 13 = 3 + 2(n-1)\ ;\ n = 6 \quad k^{th} term = 2k + 1$

Ex5 – Summation of a Series - $\sum_{k=1}^{n} k = 1 + 2 + 3 \ldots\ldots n = \frac{n(n+1)}{2}$

1. $\sum_{k=1}^{10} k = \frac{10(11)}{2} = 55$

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2. $\sum_{k=1}^{20} 2k = 2\sum_{k=1}^{20} k = 2\left[\frac{20(21)}{2}\right] = 420$

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3. $\sum_{k=1}^{8}(2k+3) = 2\sum_{k=1}^{8} k + \sum_{k=1}^{8} 3 = 2\left[\frac{8(9)}{2}\right] + (8 \times 3) = 72 + 24 = 96$

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4. $\sum_{k=1}^{20}(4k+5) = 4\sum_{k=1}^{20} k + \sum_{k=1}^{20} 5 = 4\left[\frac{20(21)}{2}\right] + (20 \times 5) = 420 + 100 = 520$

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5. $\sum_{k=1}^{10}(3k-1) = 3\sum_{k=1}^{10} k - \sum_{k=1}^{10} 1 = 3\left[\frac{10(11)}{2}\right] - (10 \times 5) = 165 - 10 = 155$

Ex 5 – Prove by Induction

Steps : 1. Prove for n = 1

2. Assume true for n = k

3. Show true for n = k + 1

All about detail

1. $Prove\ by\ Induction \sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

Part A : $n = 1$ $\sum_{r=1}^{1} \frac{1}{1(1+1)} = \frac{1}{2}$; $\frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$; $Hence\ true\ for\ n = 1$

Part B : Assume true for n = k, where k ≥ 1

$$n = k \qquad \sum_{r=1}^{k} \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Prove true for n = k + 1

$$n = k + 1 \qquad \sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{(k+1)}{(k+2)} = \frac{(k+1)}{(k+1)+1}$$

Hence true also for n = k + 1

True for n = 1 ⟹ True for n = 2 since k ≥ 1

True for n = 2 ⟹ True for n = 3 and so on for all values of n.

2. $Prove\ by\ Induction\ \sum_{r=3}^{n}(2r-1)=(n-2)(n+2) \quad n\geq 3$

Part A : $n=3$ $\quad \sum_{r=3}^{3}(2\times 3-1)=5 \quad ; \quad (3-2)(3+2)=5 \quad ; \quad Hence\ true\ for\ n=3$

Part B : Assume true for n = k, where k ≥ 3

$$n=k \qquad \sum_{r=1}^{k}(2r-1)=(k-2)(k+2)$$

Prove true for n = k + 1

$$n=k+1 \qquad \sum_{r=1}^{k+1}(2r-1)=(k-2)(k+2)+(2(k+1)-1)$$

$$=(k-2)(k+2)+(2k+1)=k^2+2k-3=(k-1)(k+3)=((k+1)-2)\big((k+1)+2\big)$$

Hence true also for n = k + 1

True for n = 3 ⟹ True for n = 4 since k ≥ 3

True for n = 5 ⟹ True for n = 6 and so on for all values of n.

4. $Prove\ by\ Induction\ \ (cos\theta + isin\theta)^n = cos(n\theta) + isin(n\theta) \quad for\ all\ n$

Part A : $n = 1 \quad (cos\theta + isin\theta)^1 = cos(1\theta) + isin(1\theta) \quad ; \ Hence\ true\ for\ n = 1$

Part B : Assume true for n = k, where k ≥ 1

$$n = k \qquad\qquad (cos\theta + isin\theta)^n = cos(n\theta) + isin(n\theta)$$

Prove true for n = k + 1

$n = k + 1 \quad = \ (cos\theta + isin\theta)^{k+1} = \ (cos\theta + isin\theta)^k(cos\theta + isin\theta)$

$= cos(k\theta) + isin(k\theta)(cos\theta + isin\theta)$

$= cos(k\theta)cos\theta - sin(k\theta)sin\theta + isin(k\theta)cos\theta + cos(k\theta)isin\theta$

$= cos(k+1)\theta + isin(k+1)\theta$

Hence true also for n = k + 1

True for n = 1 ⟹ True for n = 2 since k ≥ 1

True for n = 3 ⟹ True for n = 4 and so on for all values of n.

5. $Prove\ by\ Induction \sum_{k=1}^{n}(2k-1) = n^2 \quad for\ all\ n$

Part A : $n = 1 \quad \sum_{k=1}^{1}(2k-1) = 2 \times 1 - = 1 \quad ; \quad n^2 = 1^2 = 1 \quad ; \ Hence\ true\ for\ n = 1$

Part B : Assume true for n = k, where k ≥ 1

$$n = k \qquad \sum_{k=1}^{n}(2k-1) = k^2$$

Prove true for n = k + 1

$$n = k + 1 \qquad \sum_{k=1}^{n}(2k-1) = k^2 + 2(k+1) - 1 = k^2 + 2k + 1 = (k+1)^2$$

Hence true also for n = k + 1

True for n = 1 ⟹ True for n = 2 since k ≥ 1

True for n = 3 ⟹ True for n = 4 and so on for all values of n.

Ex 5 (2) – Prove by Induction

1. $Prove\ by\ Induction\ \sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2) \quad for\ all\ n$

Part A : $n = 1 \quad \sum_{r=1}^{1} r(r+1) = 1(1+1) = 2 \quad ;$

$$\frac{1}{3}n(n+1)(n+2) = \frac{1}{3}1(1+1)(1+2) = 2 \ \ ; \ Hence\ true\ for\ n = 1$$

Part B : Assume true for n = k, where k ≥ 1

$$n = k \qquad \sum_{k=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

Prove true for n = k + 1

$$n = k+1 \qquad \sum_{k=1}^{n} r(r+1) = \frac{1}{3}k(k+1)(k+2) + (k+1)(k+1+1)$$

$$\frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) = (k+1)(k+2)\left[\frac{1}{3}k+1\right] = (k+1)(k+2)\left[\frac{k+3}{3}\right]$$

$$= \frac{1}{3}(k+1)(k+2)(k+3)$$

Hence true also for n = k + 1

True for n = 1 ⟹ True for n = 2 since k ≥ 1

True for n = 3 ⟹ True for n = 4 and so on for all values of n.

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2. $Prove\ by\ Induction\ \sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3) \quad for\ all\ n$

Part A : $n = 1 \quad \sum_{r=1}^{1} r(r+1)(r+2) = 1(1+1)(1+2) = 6 \quad ;$

$$\frac{1}{3}n(n+1)(n+2)(n+3) = \frac{1}{4}1(1+1)(1+2)(1+3) = 6 \ ;\ Hence\ true\ for\ n = 1$$

Part B : Assume true for n = k, where k ≥ 1

$$n = k \qquad \sum_{k=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$$

Prove true for n = k + 1

$$n = k+1 \quad \sum_{k=1}^{n} r(r+1)(r+2) = \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3)$$

$$\frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3)$$

$$= (k+1)(k+2)(k+3)\left[\frac{1}{4}k + 1\right] = (k+1)(k+2)(k+3)\left[\frac{k+4}{4}\right]$$

$$= \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$$

Hence true also for n = k + 1

True for n = 1 ⟹ True for n = 2 since k ≥ 1

True for n = 3 ⟹ True for n = 4 and so on for all values of n.

3. $Prove\ by\ Induction\ \ that\ \ \frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots\frac{n}{2n+1}\quad for\ all\ n$

$$Prove\ by\ Induction\ S_n = \sum_{r=1}^{n}\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}\quad for\ all\ n$$

Part A : $n=1\quad \sum_{r=1}^{n}\frac{1}{(2(1)-1)(2(1)+1)}=\frac{1}{1\times3}=\frac{1}{3}\quad ;\ \frac{1}{2(1)+1}=\frac{1}{3}\quad ;\ Hence\ true\ for\ n=1$

Part B : Assume true for n = k, where k ≥ 1

$$n=k\qquad \sum_{r=1}^{n}\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Prove true for n = k + 1

$$n=k+1\qquad \sum_{r=1}^{n}\frac{1}{(2n-1)(2n+1)}=\frac{k}{(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$\frac{k}{(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{1}{(2k+1)}\left[\frac{k}{1}+\frac{1}{(2k+3)}\right]$$

$$=\frac{1}{2k+1}\left[\frac{k(2k+3)+1}{(2k+3)}\right]=\frac{1}{2k+1}\left[\frac{2k^2+3k+1}{(2k+3)}\right]$$

$$=\frac{1}{(2k+1)}\left[\frac{(2k+1)(k+1)}{(2k+3)}\right]=\frac{(k+1)}{(2k+3)}=\frac{(k+1)}{(2k+2+1)}=\frac{(k+1)}{2(k+1)+1)}$$

Hence true also for n = k + 1

True for n = 1 ⟹ True for n = 2 since k ≥ 1

True for n = 3 ⟹ True for n = 4 and so on for all values of n.

4. $Prove\ by\ Induction\ \ that\ \frac{1}{1\times 2x3}+\frac{1}{2\times 3\times 4}+\frac{1}{3\times 4\times 5}+\cdots\frac{1}{4}-\frac{1}{2(n+1)(n+2)}\quad for\ all\ n$

$$Prove\ by\ Induction\ S_n=\sum_{r=1}^{n}\frac{1}{n(n+1)(n+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}\quad for\ all\ n$$

Part A : $n=1\quad \sum_{r=1}^{n}\frac{1}{(1)(2)(3)}=\mathbf{\frac{1}{6}}$;

$\frac{1}{4}-\frac{1}{2(n+1)(n+2)}=\frac{1}{4}-\frac{1}{2(1+1)(1+2)}=\frac{1}{4}-\frac{1}{12}=\frac{3}{12}-\frac{1}{12}=\frac{2}{12}=\mathbf{\frac{1}{6}}$; $Hence\ true\ for\ n=1$

Part B : Assume true for n = k, where k ≥ 1

$$n=k\qquad \sum_{r=1}^{n}\frac{1}{n(n+1)(n+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$$

Prove true for n = k + 1

$$n=k+1\quad \sum_{r=1}^{n}\frac{1}{n(n+1)(n+2)}=\frac{1}{4}-\frac{1}{2(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

$$\frac{1}{4}-\frac{1}{(k+1)(k+2)}\left[\frac{1}{2}-\frac{1}{(k+3)}\right]=\frac{1}{4}-\frac{1}{(k+1)(k+2)}\left[\frac{(k+3)-2}{2(k+3)}\right]$$

$$=\frac{1}{4}-\frac{1}{(k+1)(k+2)}\left[\frac{(k+1)}{2(k+3)}\right]=\frac{1}{4}-\frac{1}{2(k+2)(k+3)}$$

Hence true also for n = k + 1

True for n = 1 ⟹ True for n = 2 since k ≥ 1

True for n = 3 ⟹ True for n = 4 and so on for all values of n.

5. (a) $Give\ \sum_{k=1}^{n} r = \frac{n(n+1)}{2}$; $\sum_{k=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$; $\sum_{k=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$;

$$\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = \frac{1}{6}n(n+1)(2n+1) + \frac{n(n+1)}{2}$$

$$= \frac{1}{6}n(n+1)(2n+1) + \frac{n(n+1)}{2} = n(n+1)\left[\frac{1}{6}(2n+1) + \frac{1}{2}\right] = n(n+1)\left[\frac{1}{3}n + \frac{1}{6} + \frac{1}{2}\right]$$

$$= n(n+1)\left[\frac{1}{3}n + \frac{2}{3}\right]$$

$$= \frac{1}{3}n(n+1)(n+2)$$

(b). $\sum_{r=1}^{n} r(r+1)(r+2) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = \frac{1}{4}n(n+1)(n+2)(n+3)$

$$\sum_{r=1}^{n} r(r+1)(r+2) = \sum_{r=1}^{n} r(r^2+3r+2) = \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$$

$$= \frac{1}{4}n^2(n+1)^2 + 3\left(\frac{1}{6}n(n+1)(2n+1)\right) + 2\frac{n(n+1)}{2}$$

$$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{2}n(n+1)(2n+1) + n(n+1)$$

$$= n(n+1)\left[\frac{1}{4}n(n+1) + \frac{1}{2}(2n+1) + 1\right]$$

$$= n(n+1)\left[\frac{1}{4}n^2 + \frac{1}{4}n + n + \frac{1}{2} + 1\right] = n(n+1)\left[\frac{1}{4}n^2 + \frac{5}{4}n + \frac{3}{2}\right]$$

$$= \frac{1}{4}n(n+1)[n^2+5n+6]$$

$$= \frac{1}{4}n(n+1)[n^2+5n+6] = \frac{1}{4}n(n+1)(n+2)(n+3)$$

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6. $Prove\ by\ Induction\ that\ n^3 + 3n^2 - 10n\ is\ divisible\ by\ 3$

Part A : $n = 1$; $n^3 + 3n^2 - 10n = (1)^3 + 3(1)^2 - 10(1) = -6 = -2(3)$; $Hence\ true\ for\ n = 1$

Part B : Assume true for n = k, where k ≥ 1

Prove true for n = k + 1

$n = k + 1 \quad (k+1)^3 + 3(k+1)^2 - 10(k+1)$

$= (k^3 + 3k^2 + 3k + 1) + 3k^2 + 6k + 3 - 10k - 10$

$= k^3 + 6k^2 - k - 6 = k^3 - k + 6k^2 - 6 = k^3 - k + 6k^2 - 6$

$= k(k^2 - 1) + 6(k^2 - 1)$

$= (k^2 - 1)(k + 6)$

$= (k - 1)(k + 1)(k + 6)$ **(Multiple of three terms hence divisible by 3!)**

Hence true also for n = k + 1

True for n = 1 ⟹ True for n = 2 since k ≥ 1

True for n = 3 ⟹ True for n = 4 and so on for all values of n.

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7. $Prove\ by\ Induction\ that\ 7^n + 4^n + 1^n\ is\ divisible\ by\ 6$

Part A : $n = 1$; $7^n + 4^n + 1^n = 7^1 + 4^1 + 1^1 = 12 = 6(2)$; $Hence\ true\ for\ n = 1$

Part B : Assume true for n = k, where k ≥ 1

Prove true for n = k + 1

$n = k + 1 \quad 7^{(k+1)} + 4^{(k+1)} + 1^{(k+1)}$

$= 7^k \times 7 + 4^k \times 4 + 1^k \times 1$

$From\ assumption\ n = k\ true$; $7^k + 4^k + 1^k = 6p$; $7^k = (6p - 4^k - 1^k)$

$= 7(6p - 4^k - 1^k) + 4(4^k) + 1^k$

$= 42p - 7(4^k) - 7(1^k) + 4(4^k) + 1^k$

$= 42p - 3(4^k) - 6(1^k)$

$= 42p - 3 \times 4(4^{k-1}) - 6(1^k)$

$= 42p - 12(4^{k-1}) - 6(1^k)$

$= 6(7p - 2(4^{k-1}) - 1^k)$

Hence true also for n = k + 1

True for n = 1 ⟹ True for n = 2 since k ≥ 1

True for n = 3 ⟹ True for n = 4 and so on for all values of n.

8. $Prove\ by\ Induction\ that\ for\ all\ integers\ n > 2\ ,\quad 2^n > 2n$

Part A : $n = 3$; $2^3 > 2 \times 3$; $8 > 6$; $Hence\ true\ for\ n = 3$

Part B : Assume true for n = k, where k ≥ 3

$$2^k > 2k$$

Prove true for n = k + 1

$n = k + 1 \quad 2^{(k+1)} > 2(k + 1)$

$= 2(2^k) > 2(k + 1)$

$= 2(2^k) > 2k + 2$

$= (2^k) > 2k = k + k + 2 > k + 1 \quad for\ k$

Hence true also for n = k + 1

True for n = 3 ⟹ True for n = 4 since k ≥ 3

True for n = 5 ⟹ True for n = 6 and so on for all values of n.