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Advanced Higher Maths

Advanced Higher - Unit 2.4 Properties of Functions - Solutions

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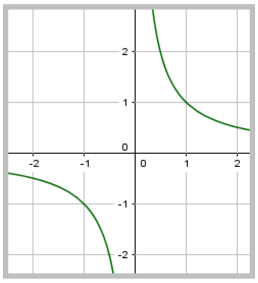
Computer Task -

1. $f(x) = \frac{1}{x}$ $\quad Vertical\ Asymptote\ x = 0$



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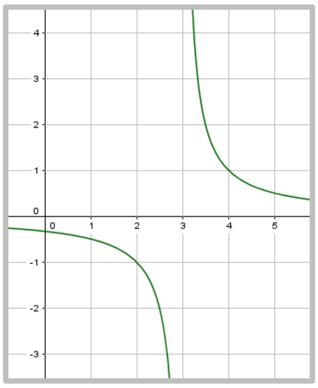
2. $f(x) = \frac{1}{x-3}$ $\quad Vertical\ Asymptote\ x = 3$



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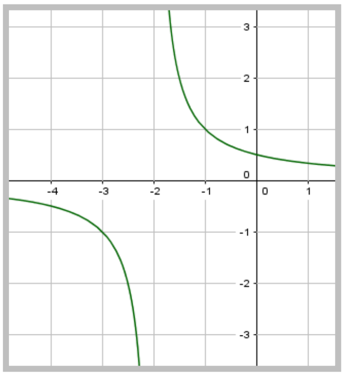
3. $f(x) = \frac{1}{x+2}$ $\quad Vertical\ Asymptote\ x = -2$



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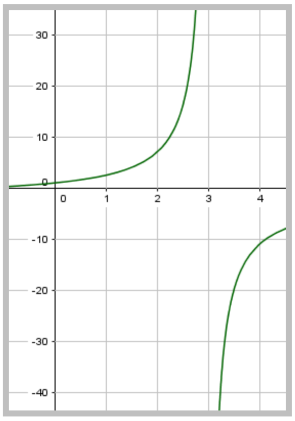
4. $f(x) = \frac{2x+3}{3-x}$ $\quad$ $Vertical\ Asymptote\ x = 3$



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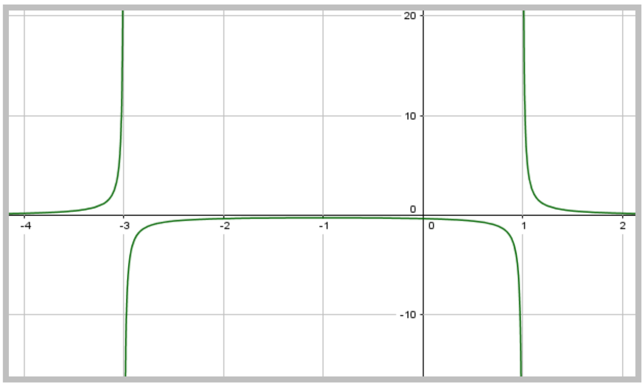
5. $f(x) = \frac{1}{(x-1)(x+3)}$ $\quad$ $Vertical\ Asymptotes\ x = -3\ and\ x = 1$



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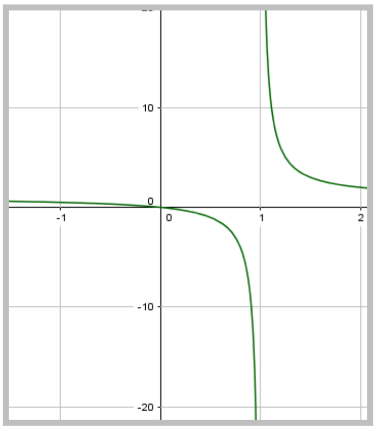
6. $f(x) = \frac{x}{(x-1)}$ $\quad$ $Vertical\ Asymptote\ x = 1$



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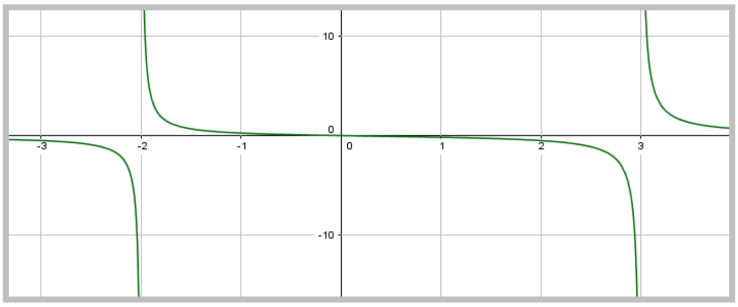
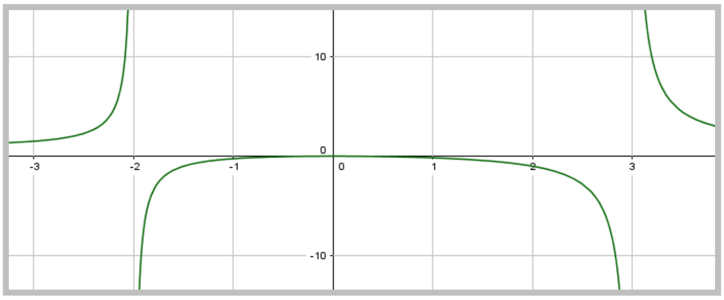
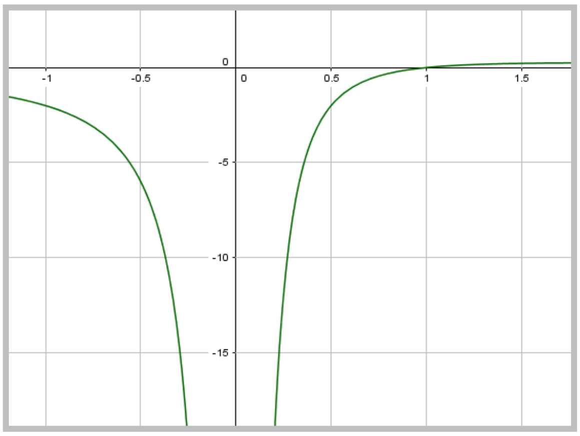
7. $f(x) = \frac{x}{(x+2)(x-3)}$ $\quad$ $Vertical\ Asymptotes\ x = -2\ \ and\ \ x = 3$



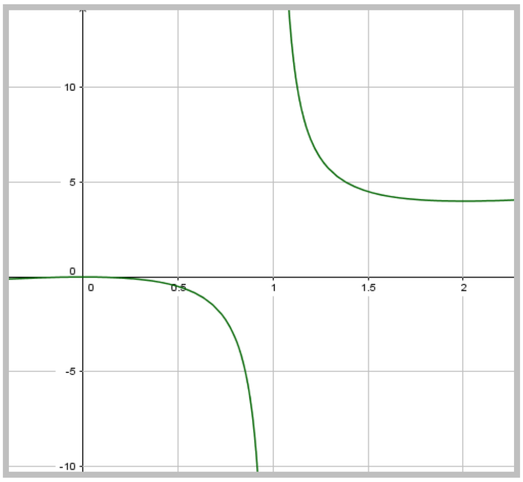
8. $f(x) = \frac{x^2}{(x+2)(x-3)}$ $\quad$ $Vertical\ Asymptotes\ x = -2\ \ and\ \ x = 3$



9. $f(x) = \frac{x-1}{x^2}$ $\quad$ $Vertical\ Asymptote\ x = 0$



10. $f(x) = \frac{x^2}{x-1}$ $\quad$ $Vertical\ Asymptote\ x = 1$



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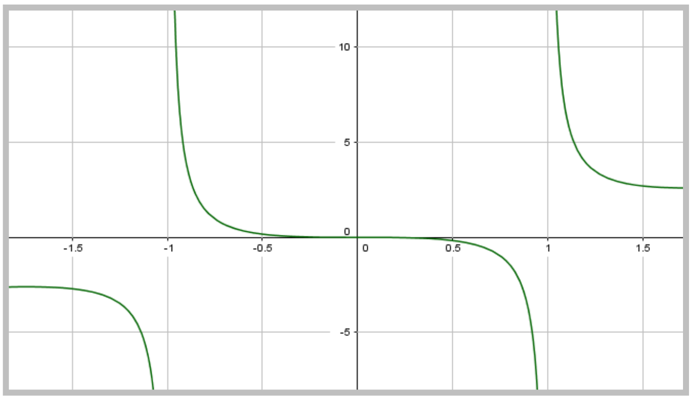
11. $f(x) = \frac{x^3}{x^2-1}$ $\quad$ $Vertical\ Asymptote\ x = -1\ \ and\ \ x = 1$



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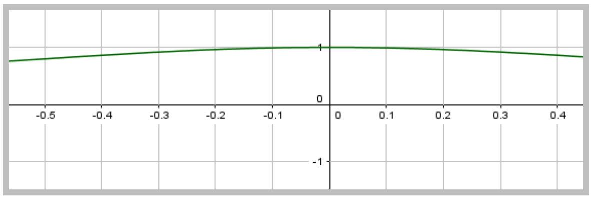
12. $f(x) = \frac{1}{x^2+1}$ $\quad$ $Vertical\ Asymptote =\ none$



Ex1 – Vertical Asymptotes

1. $f(x) = \frac{4}{x-2}$; $Vertical\ Asymptote\ x = 2$;

   $2^- = \left(\frac{+}{-}\right)\ \ y \to -\infty$; $2^+ = \left(\frac{+}{+}\right)\ \ y \to \infty$; →I←

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2. $f(x) = \frac{3x-1}{x^2+2x-3} = \frac{3x-1}{(x+3)(x-1)}$; $Vertical\ Asymptotes\ x = -3\ \ and\ \ x = 1$;

   $-3^- = \left(\frac{(-)}{(-)(-)}\right)\ \ y \to -\infty$; $-3^+ = \left(\frac{(-)}{(+)(-)}\right)\ \ y \to \infty$; $x = -3$ →I←

   $1^- = \left(\frac{(+)}{(+)(-)}\right)\ \ y \to -\infty$; $1^+ = \left(\frac{(+)}{(+)(+)}\right)\ \ y \to \infty$; $x = 1$ →I←

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3. $f(x) = \frac{12}{x^2-2x-3} = \frac{12}{(x-3)(x+1)}$; $Vertical\ Asymptotes\ x = -1\ \ and\ \ x = 3$;

   $-1^- = \left(\frac{(+)}{(-)(-)}\right)\ \ y \to \infty$; $-1^+ = \left(\frac{(+)}{(-)(+)}\right)\ \ y \to -\infty$; $x = -1$ →I←

   $3^- = \left(\frac{(+)}{(-)(+)}\right)\ \ y \to -\infty$; $3^+ = \left(\frac{(+)}{(+)(+)}\right)\ \ y \to \infty$; $x = 1$ →I←

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4. $f(x) = \frac{x+4}{x-2}$; $Vertical\ Asymptote\ x = 2$

   $2^- = \left(\frac{(+)}{(-)}\right)\ \ y \to -\infty$; $2^+ = \left(\frac{(+)}{(+)}\right)\ \ y \to \infty$; $x = 2$ →I←

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5. $f(x) = \frac{x^2}{4-x^2} = \frac{x^2}{(2-x)(2+x)}$; $Vertical\ Asymptotes\ x = -2\ \ and\ \ x = 2$;

$-2^- = \left(\frac{(+)}{(+)(-)}\right)\ \ y \to -\infty$; $-2^+ = \left(\frac{(+)}{(+)(+)}\right)\ \ y \to \infty$; $x = -2$ →I←

$2^- = \left(\frac{(+)}{(+)(+)}\right)\ \ y \to \infty$; $2^+ = \left(\frac{(+)}{(-)(+)}\right)\ \ y \to -\infty$; $x = 1$ →I←

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6. $f(x) = \frac{x(x+1)}{(x-1)(x+2)}$; $Vertical\ Asymptotes\ x = -2\ \ and\ \ x = 1$;

$-2^- = \left(\frac{(-)(-)}{(-)(-)}\right)\ \ y \to \infty$; $-2^+ = \left(\frac{(-)(-)}{(-)(+)}\right)\ \ y \to -\infty$; $x = -2$ →I←

$1^- = \left(\frac{(+)(+)}{(-)(+)}\right)\ \ y \to -\infty$; $1^+ = \left(\frac{(+)(+)}{(+)(+)}\right)\ \ y \to \infty$; $x = 1$ →I←

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7. $f(x) = \frac{(x-1)(x-4)}{x}$; $Vertical\ Asymptote\ x = 0$;

$0^- = \left(\frac{(-)(-)}{(-)}\right)\ \ y \to -\infty$; $0^+ = \left(\frac{(-)(-)}{(+)}\right)\ \ y \to \infty$; $x = 0$ →I←

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8. $f(x) = \frac{x^2+3}{x-1}$; $Vertical\ Asymptote\ x = 1$;

$1^- = \left(\frac{(+)}{(-)}\right)\ \ y \to -\infty$; $1^+ = \left(\frac{(+)}{(+)}\right)\ \ y \to \infty$; $x = 0$ →I←

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9. $f(x) = \frac{x^3}{x^2+3}$; $Vertical\ Asymptote\ none$;

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10. $f(x) = \frac{x}{x^2+4}$; $Vertical\ Asymptote\ none$;

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11. $f(x) = \frac{x^2}{x-1}$; $Vertical\ Asymptote\ x = 1$;

$1^- = \left(\frac{(+)}{(-)}\right)$ $y \rightarrow -\infty$; $1^+ = \left(\frac{(+)}{(+)}\right)$ $y \rightarrow \infty$; $x = 0$ $_{\rightarrow}\mathrm{I}^{\leftarrow}$

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12. $f(x) = \frac{2x^2}{x^2-1} = \frac{2x^2}{(x-1)(x+1)}$; $Vertical\ Asymptotes\ x = -1\ \ and\ \ x = 1$;

$-1^- = \left(\frac{(+)}{(-)(-)}\right)$ $y \rightarrow \infty$; $-1^+ = \left(\frac{(+)}{(-)(+)}\right)$ $y \rightarrow -\infty$; $x = -2$ $^{\rightarrow}\mathrm{I}_{\leftarrow}$

$1^- = \left(\frac{(+)}{(-)(+)}\right)$ $y \rightarrow -\infty$; $1^+ = \left(\frac{(+)}{(+)(+)}\right)$ $y \rightarrow \infty$; $x = 1$ $_{\rightarrow}\mathrm{I}^{\leftarrow}$

Ex2 – Non Vertical Asymptote

1. $f(x) = \frac{4}{x-2} = \frac{\frac{4}{x}}{1-\frac{2}{x}}$; $Non\,Vertical\,Asymptote \quad y = 0$

$x \to -\infty \quad y \to 0^-$; $x \to \infty \quad y \to 0^+$; $y = 0 \quad {}_{\rightarrow}{-}^{\leftarrow}$

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2. $f(x) = \frac{3x-1}{x^2+2x-3} = \frac{\frac{3}{x}-\frac{1}{x^2}}{1+\frac{2}{x}-\frac{3}{x^2}}$; $Non\,Vertical\,Asymptote \quad y = 0$

$x \to -\infty \quad 0^- = \left(\frac{(-)}{(+)}\right) \quad y \to -\infty$; $x \to \infty \quad 0^+ = \left(\frac{(+)}{(+)}\right) \quad y \to \infty$; $y = 0 \quad {}_{\rightarrow}{-}^{\leftarrow}$

---

3. $f(x) = \frac{12}{x^2-2x-3} = \frac{\frac{12}{x^2}}{1-\frac{2}{x}-\frac{3}{x^2}}$; $Non\,Vertical\,Asymptote \quad y = 0$;

$x \to -\infty \quad y \to 0^+$; $x \to \infty \quad y \to 0^+$; $y = 0 \quad {}^{\rightarrow}{-}^{\leftarrow}$

---

4. $f(x) = \frac{x+4}{x-2} = 1 + \frac{6}{x-2} = 1 + \frac{\frac{6}{x}}{1-\frac{2}{x}}$; $Non\,Vertical\,Asymptote \quad y = 1$

$x \to -\infty \quad y \to 1^-$; $x \to \infty \quad y \to 1^+$; $y = 1 \quad {}_{\rightarrow}{-}^{\leftarrow}$

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5. $f(x)=\frac{x^2}{4-x^2}=-1+\frac{4}{4-x^2}=\ -1+\frac{\frac{4}{x^2}}{\frac{4}{x^2}-1}$; $Non\,Vertical\,Asymptotes \quad y=-1$

$x\to-\infty \qquad y\to\ -1^- \quad ;\ x\to\infty \qquad y\to\ -1^- \quad ;\quad y=-1 \qquad {}_{\rightarrow}{}^{-}{}_{\leftarrow}$

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6. $f(x)=\frac{x(x+1)}{(x-1)(x+2)}=\frac{(x^2+x)}{(x^2+x-2)}=1+\frac{(x^2+x)}{(x^2+x-2)}=1+\frac{\frac{2}{x^2}}{(1+\frac{1}{x}-\frac{2}{x^2})}$; $Non\,Vertical\,Asymptotes \quad y=1$

$x\to-\infty \qquad y\to\ 1^+ \quad ;\ x\to\infty \qquad y\to\ 1^+ \qquad ;\quad y=1 \qquad {}^{\rightarrow}{-}^{\leftarrow}$

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7. $f(x)=\frac{(x-1)(x-4)}{(x-2)}=\frac{(x^2-5x+4)}{(x-2)}=(x-3)-\frac{2}{(x-2)}=(x-3)-\frac{\frac{2}{x}}{(1-\frac{2}{x})}$; $Non\,Vertical\,Asymptote \quad y=(x-3)$;

$x\to-\infty \qquad y\to\ (x-3)^+ \quad ;$

$x\to\infty \qquad y\to\ (x-3)^- \ ;\ y=(x-3)\ ;\ {}_{\rightarrow}/^{\leftarrow}$

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8. $f(x)=\frac{x^2+3}{x-1}=(x-1)+\frac{4}{(x-1)}=(x+1)+\frac{\frac{4}{x}}{(1-\frac{1}{x})}$; $Non\,Vertical\,Asymptote\ y=(x+1)$;

$x\to-\infty \qquad y\to\ (x+1)^- \quad ;$

$x\to\infty \qquad y\to\ (x+1)^+ \qquad ;\quad y=(x+1) \qquad {}^{\rightarrow}/_{\leftarrow}$

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9. $f(x) = \frac{x^3}{x^2+3} = x - \frac{3x}{(x^2+3)} = x - \frac{\frac{3}{x}}{(1+\frac{3}{x^2})}$; $Non\,Vertical\,Asymptote\; y = x$;

$x \to -\infty \qquad y \to x^+$;

$x \to \infty \qquad y \to x^-$; $y = x$ $\quad {}_{\rightarrow}/^{\leftarrow}$

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10. $f(x) = \frac{x}{x^2+4} = \frac{\frac{1}{x}}{1+\frac{4}{x^2}}$; $Non\,Vertical\,Asymptote \quad y = 0$;

$x \to -\infty \qquad y \to 0^-$; $x \to \infty \qquad y \to 0^+$; $y = 0$ $\quad {}_{\rightarrow}-^{\leftarrow}$

---

11. $f(x) = \frac{x^2}{x-1} = (x+1) + \frac{1}{(x-1)} = (x+1) + \frac{\frac{1}{x}}{(1-\frac{1}{x})}$; $Non\;Vertical\,Asymptote \quad y = (x+1)$;

$x \to -\infty \qquad y \to (x+1)^-$;

$x \to \infty \qquad y \to (x+1)^+$; $y = (x+1)$ $\quad {}^{\rightarrow}/_{\leftarrow}$

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12. $f(x) = \frac{2x^2}{x^2-1} = 2 + \frac{2}{x^2-1} = 2 + \frac{\frac{2}{x^2}}{1-\frac{1}{x^2}}$; $Non\,Vertical\,Asymptote \quad y = 2$;

$x \to -\infty \qquad y \to 2^{\mp}$; $x \to \infty \qquad y \to 2^+$; $y = 2$ $\quad {}^{\rightarrow}-^{\leftarrow}$

Ex3 – Curve Sketching

Process: 1. Find where curve crosses x and y axis.

2. Find all asymptotes.

3. Find stationary points and their nature.

1.

1. $f(x) = \frac{4}{x-2} = \frac{\frac{4}{x}}{1-\frac{2}{x}}$;

$x - axis\ when\ y = 0$; $None$; $y - axis\ when\ x = 0 \Rightarrow \boldsymbol{y = -2}$

2. $Asymptotes:$ $Vertical:$ $x - 2 = 0$; $x = 2$

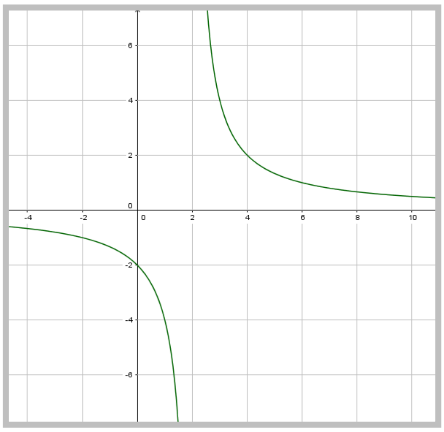
$2^- = \left(\frac{(+)}{(-)}\right)\ y \rightarrow -\infty$; $2^+ = \left(\frac{(+)}{(+)}\right)\ y \rightarrow \infty$; $x = 0$ $_{\rightarrow}\mathrm{I}^{\leftarrow}$

$Non\ Vertical:$ $y = \frac{4}{x-2} = \frac{\frac{4}{x}}{1-\frac{2}{x}}$ $Non\ Vertical\ \ y = 0$

$x \rightarrow -\infty\ \ y \rightarrow 0^-$; $x \rightarrow \infty\ \ y \rightarrow 0^+$; $\boldsymbol{y = 0}$ $_{\rightarrow}-^{\leftarrow}$

3. $Stationary\ points:$ $f'(x) = 0$; $f(x) = 4(x-2)^{-1}$; $f'(x) = -4(x-2)^{-2}$

$f'(x) = \frac{-4}{(x-2)^2}$; $\frac{-4}{(x-2)^2} = 0$; $no\ solution\ so\ no\ stationary\ points$



2.

1. $f(x) = \frac{x-2}{x-1} = 1 - \frac{1}{x-1} = 1 - \frac{\frac{1}{x}}{1-\frac{1}{x}}$

$x - axis\ when\ y = 0\ ;\ x = 2\ ;\ (2,0)\qquad y - axis\ when\ x = 0 \Rightarrow \boldsymbol{y = 2}\ (0,2)$

2. $Asymptotes:\qquad Vertical: x - 1 = 0\ ;\ x = 1$

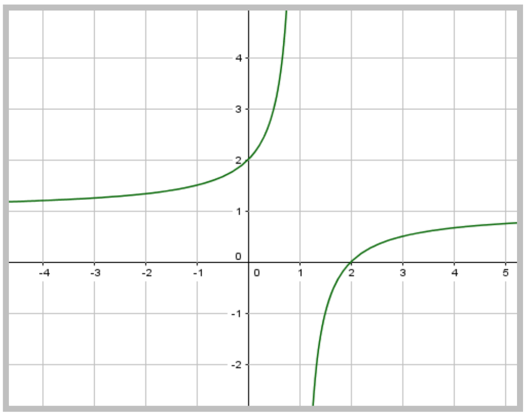
$1^- = \left(\frac{(-)}{(-)}\right)\ y \to \infty\ ;\qquad 1^+ = \left(\frac{(-)}{(+)}\right)\ y \to -\infty\ ;\ \boldsymbol{x = 1}\qquad \overrightarrow{\ }\mathbf{I}\overleftarrow{\ }$

$Non\,Vertical: \qquad y = 1 - \frac{\frac{1}{x}}{1-\frac{1}{x}}\qquad Non\,Vertical\quad y = 1$

$x \to -\infty\quad y \to 1^+\ ;\quad x \to \infty\quad y \to 1^-\ ;\ \boldsymbol{y = 2}\qquad \overrightarrow{\ }\mathbf{-}\overleftarrow{\ }$

3. $Stationary\ points:\ f'(x) = 0\ ;\ f(x) = 1 - \frac{1}{x-1}\ ;\ f(x) = -(x-1)^{-1}$

$f'(x) = \frac{1}{(x-1)^2}\ ;\ \frac{1}{(x-1)^2} = 0\ ;\ no\ solution\ so\ no\ stationary\ points$



3.

1. $f(x) = \frac{x^2+x-2}{x^2+x-6} = \frac{(x+2)(x-1)}{(x+3)(x-2)} = 1 + \frac{4}{(x+3)(x-2)} = 1 + \frac{\frac{4}{x^2}}{1+\frac{1}{x}-\frac{6}{x^2}}$

$x - axis\ when\ y = 0$; $x^2 + x - 2 = 0$; $(x+2)(x-1) = 0$; $x = -2,1$ $(-2,0)$ $(1,0)$

$y - axis\ when\ x = 0 \Rightarrow y = \frac{1}{3}$ $(0, \mathbf{\frac{1}{3}})$

2. $Asymptotes:$ $Vertical:$ $(x+3)(x-2) = 0$; $x = -3\ and\ x = 2$

$-3^- = \left(\frac{(-)(-)}{(-)(-)}\right)$ $y \to \infty$; $-3^+ = \left(\frac{(-)(-)}{(+)(-)}\right)$ $y \to -\infty$; $x = \mathbf{-3}$ $\to \mathbf{I} \leftarrow$

$2^- = \left(\frac{(+)(+)}{(+)(-)}\right)$ $y \to -\infty$; $2^+ = \left(\frac{(+)(+)}{(+)(+)}\right)$ $y \to \infty$; $x = \mathbf{2}$ $\to \mathbf{I} \leftarrow$

$Non\ Vertical:$ $y = 1 + \frac{\frac{4}{x^2}}{1+\frac{1}{x}-\frac{6}{x^2}}$ $Non\ Vertical$ $y = 1$

$x \to -\infty$ $y \to 1^+$; $x \to \infty$ $y \to 1^+$; $\mathbf{y = 1}$ $\to - \leftarrow$

3. $Stationary\ points:$ $f'(x) = 0$; $f(x) = 1 + \frac{4}{x^2+x-6}$;

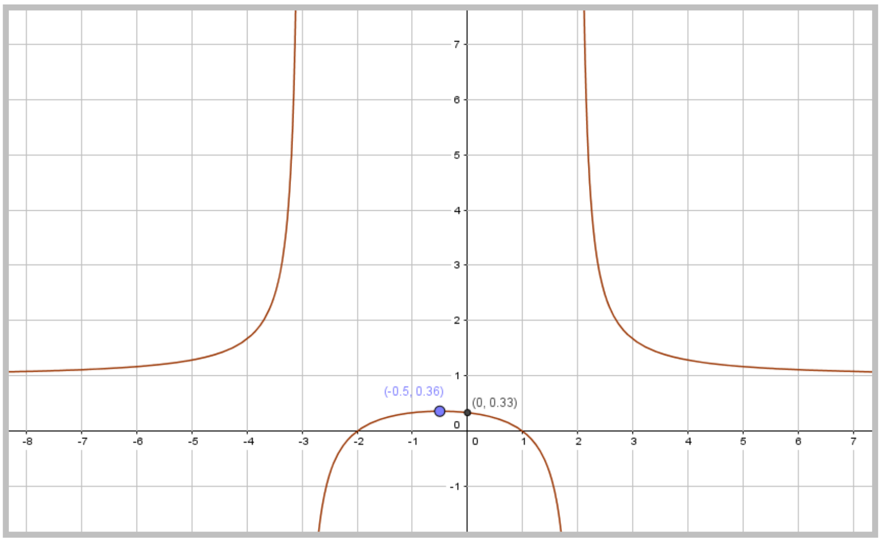
$f'(x) = \frac{(x^2+x-6).0-4.(2x+1)}{(x^2+x-6)^2} = \frac{(-8x-4)}{(x^2+x-6)^2} = 0$; $(-8x-4) = 0$; $x = -\frac{1}{2}$

$For\ x = \frac{1}{2}$ $y = 1 + \frac{4}{x^2+x-6} = 1 + \frac{4}{\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)-6} = 1 + \frac{4}{-\left(\frac{25}{4}\right)} = \frac{9}{25}$ $T.P. = \left(-\frac{1}{2}, \frac{9}{25}\right)$

Nature Table :

x	→	$-\frac{1}{2}$	→
f′(x)	+	0	-
Shape	↗	→	↘

Hence Max. T. P. at $\left(-\frac{1}{2}, \frac{9}{25}\right)$



4.

1. $f(x) = \frac{x^2+2x+5}{x+1} = (x+1) + \frac{4}{(x+1)} = (x+1) + \frac{\frac{4}{x}}{1+\frac{1}{x}}$

$x - axis\ when\ y = 0\ ;\ x^2 + x - 2 = 0\ ;\ b^2 - 4ac < 0\ ;\ no\ real\ roots$

$y - axis\ when\ x = 0 \Rightarrow y = 5\ \ (0, \mathbf{5})$

2. $Asymptotes:\quad Vertical:\ (x+1) = 0\ ;\ x = -1$

$1^- = \left(\frac{(+)}{(-)}\right)\ y \to -\infty\ ;\quad 1^+ = \left(\frac{(+)}{(+)}\right)\ y \to \infty\ ;\ x = \mathbf{-1}\quad {}_{\to}\mathbf{I}^{\leftarrow}$

$Non\ Vertical:\quad y = (x+1) + \frac{\frac{4}{x}}{1+\frac{1}{x}}\quad Non\ Vertical\ \ y = (x+1)$

$x \to -\infty\quad y \to (x+1)^-\ ;\ x \to \infty\quad y \to (x+1)^+\ ;\ \mathbf{y} = (x+1)\quad {}^{\to}/_{\leftarrow}$

3. $Stationary\ points:\ f'(x) = 0\ ;\ f(x) = (x+1) + \frac{4}{(x+1)}\ ;$

$f'(x) = 1 + \frac{(x+1).0 - 4.1}{(x+1)^2} = 1 - \frac{4}{(x+1)^2} = \frac{(x+1)^2 - 4}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2} = 0\ ;\quad x^2 + 2x - 3 = 0$

$x^2 + 2x - 3 = (x+3)(x-1) = 0\ ;\ x = 1\ \ and\ \ x = -3$

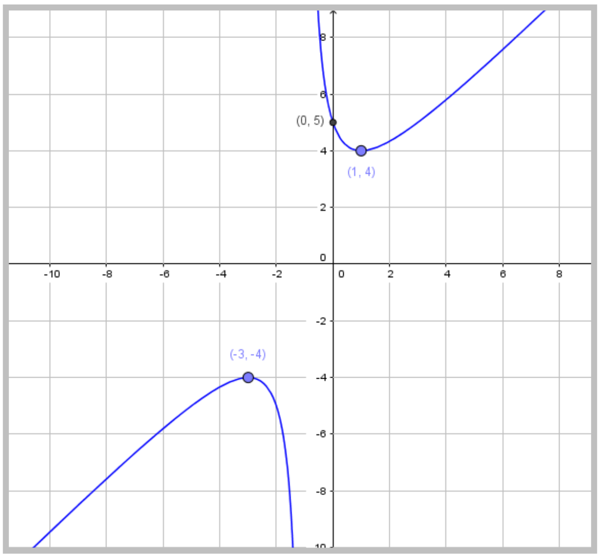
$For\ x = 1\quad y = (x+1) + \frac{4}{(x+1)} = 2 + 2 = 4\qquad T.P. = (1,4)$

$For\ x = -3\quad y = (x+1) + \frac{4}{(x+1)} = -2 - 2 = -4\qquad T.P. = (-3,-4)$

$Nature\ Table$:

x	→	-3	→	1	→
f′(x)	+	0	-	0	-
Shape	↗	→	↘	→	↗

$Hence\ Max.T.P.at\ (-3,-4)\quad and\quad Mini.TP.at\ (1,4)$



5.

1. $f(x) = \dfrac{x+1}{x^2+2x+5} = \dfrac{(x+1)}{x^2+2x+5} = \dfrac{\left(\frac{1}{x}+\frac{1}{x^2}\right)}{1+\frac{2}{x}+\frac{5}{x^2}}$

$x - axis\ when\ y = 0\ \ ;\quad x + 1 = 0\quad ;\qquad x = -1\qquad (-1{,}0)$

$y - axis\ when\ \ x = 0\ \Rightarrow y = \dfrac{1}{5}\quad (0,\dfrac{1}{5})$

2. $Asymptotes\ :\qquad Vertical:\ x^2 + 2x + 5 = 0\quad ;\quad b^2 - 4ac < 0\qquad ;\qquad no\ real\ roots$

$\boldsymbol{NO}\ Vertical\ asymptotes$

$Non\ Vertical:\qquad y = \frac{(x+1)}{x^2+2x+5} = \frac{\left(\frac{1}{x}+\frac{1}{x^2}\right)}{1+\frac{2}{x}+\frac{5}{x^2}}\qquad Non\ Vertical\quad y = 0$

$x \to -\infty\qquad y \to 0^-\quad ;\quad x \to \infty\qquad y \to\ 0^+\quad ;\quad \boldsymbol{y = 0}\qquad \underset{\rightarrow}{-}^{\leftarrow}$

3. $Stationary\ points:\ \ f'(x) = 0\qquad ;\quad \frac{x+1}{x^2+2x+5}\quad ;$

$f'(x) = \frac{(x^2+2x+5).1-2(x+1)(x+1)}{(x^2+2x+5)^2} = \frac{(x^2+2x+5)-2(x^2+2x+1)}{(x^2+2x+5)^2} = \frac{-x^2-2x+3}{(x^2+2x+5)^2} = 0\ \ ;\qquad x^2 + 2x - 3 = 0$

$x^2 + 2x - 3 = (x+3)(x-1) = 0\quad ;\quad x = 1\qquad and\quad x = -3$

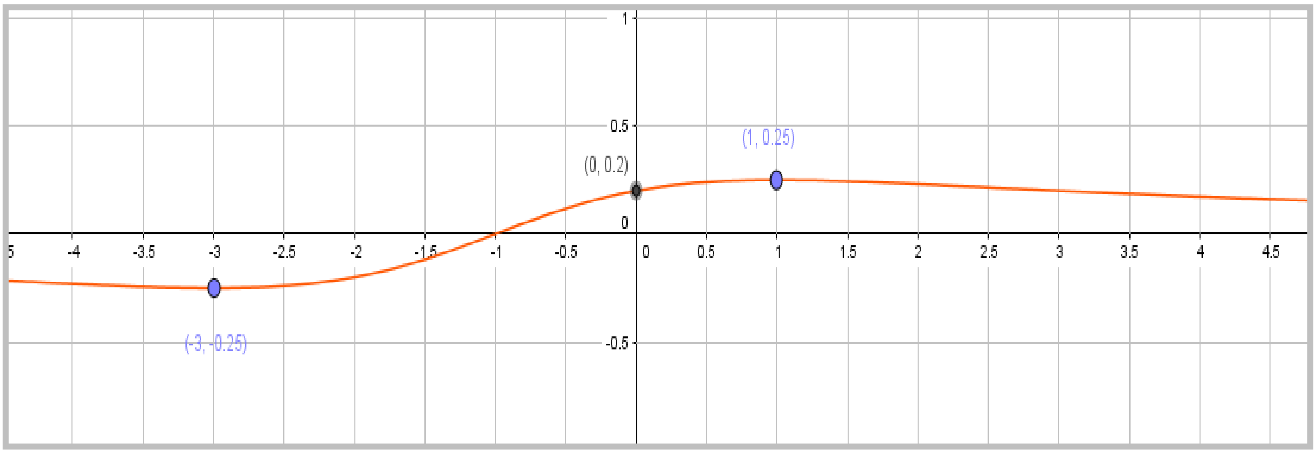
$For\ \ x = 1\qquad y = \frac{x+1}{x^2+2x+5} = \frac{2}{8} = \frac{1}{4}\qquad\qquad T.P. = \left(1,\frac{1}{4}\right)$

$For\ \ x = -3\qquad y = \dfrac{x+1}{x^2+2x+5} = \dfrac{-2}{8} = -\dfrac{1}{4}\qquad\qquad T.P. = \left(-3,-\dfrac{1}{4}\right)$

Nature Table :

x	→	-3	→	1	→
f′(x)	-	0	+	0	-
Shape	↘	→	↗	→	↘

$Hence\ Max.T.P.at\ \left(1,\frac{1}{4}\right)\ \ and\ \ Mini.TP.at\ \left(-3,-\frac{1}{4}\right)$



6.

1. $f(x) = \frac{2x^2}{x^2-1} = 2 + \frac{2}{(x^2-1)} = 2 + \frac{2}{(x+1)(x-1)} = 2 + \frac{\frac{2}{x^2}}{1-\frac{1}{x^2}}$

$x - axis\ when\ y = 0$; $\frac{2x^2}{x^2-1} = 0$; $2x^2 = 0$; $x = 0$

$y - axis\ when\ x = 0 \Rightarrow y = -1$ (0,0)

2. $Asymptotes$: $Vertical$: $(x+1)(x-1) = 0$; $x = -1$ and $x = 1$

$-1^- = \left(\frac{(+)}{(+)}\right)$ $y \to \infty$; $-1^+ = \left(\frac{(+)}{(-)}\right)$ $y \to -\infty$; $x = -1$ $\overset{\rightarrow}{\ }\mathbf{I}_{\leftarrow}$

$1^- = \left(\frac{(+)}{(-)}\right)$ $y \to -\infty$; $1^+ = \left(\frac{(+)}{(+)}\right)$ $y \to \infty$; $x = 1$ $_{\rightarrow}\mathbf{I}^{\leftarrow}$

$Non\ Vertical$: $y = 2 + \frac{\frac{2}{x^2}}{1-\frac{1}{x^2}}$ $Non\ Vertical$ $y = 2$

$x \to -\infty$ $y \to 2^+$; $x \to \infty$ $y \to 2^+$; $\mathbf{y = 2}$ $^{\rightarrow}\mathbf{-}^{\leftarrow}$

3. $Stationary\ points$: $f'(x) = 0$; $f(x) = 2 + \frac{2}{(x^2-1)}$;

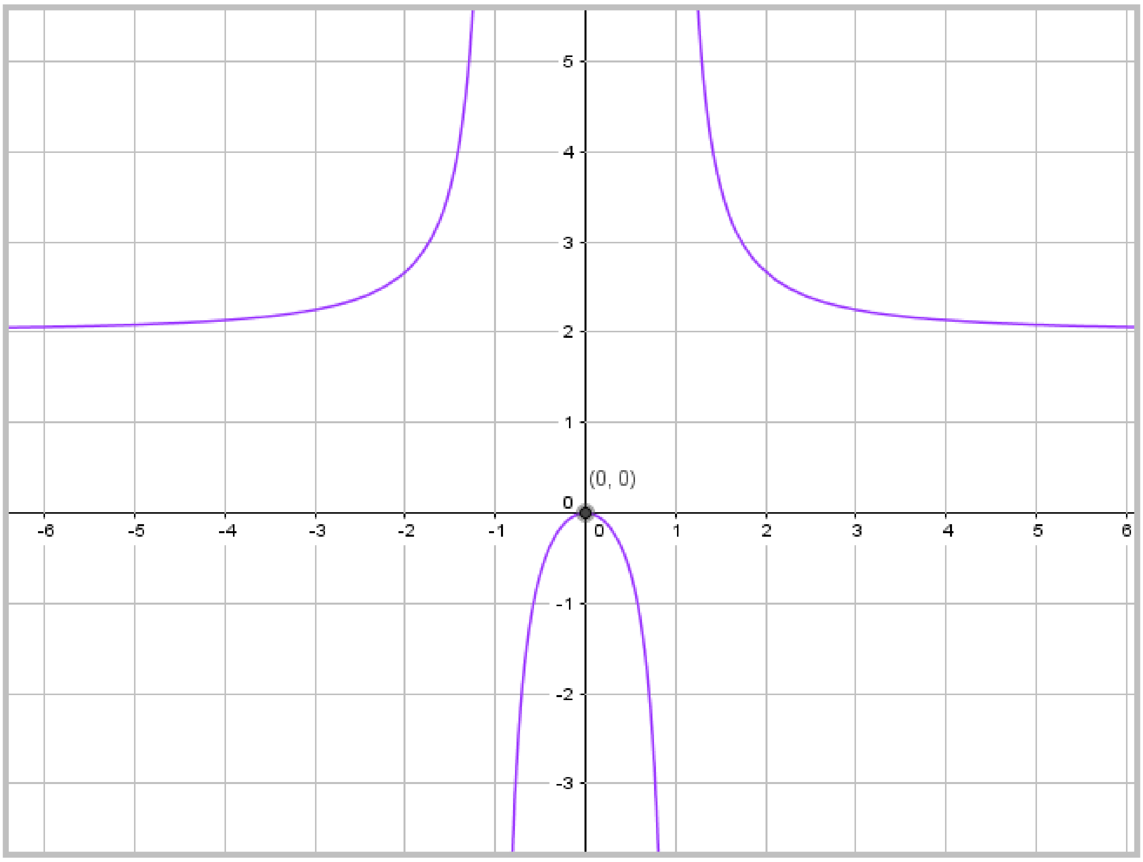
$f'(x) = \frac{(x^2-1).0-2(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2} = 0$; $x = 0$

$For\ x = 0$ $y = \frac{2x^2}{x^2-1} = 0$ $T.P. = (0,0)$

$Nature\ Table$:

x	→	0	→
f′(x)	+	0	-
Shape	↗	→	↘

$Hence\ Max.T.P.at\ (0,0)$



7.

1. $f(x)=\frac{x^2-10x+9}{x^2+10x+9}=1-\frac{20x}{x^2+10x+9}=1-\frac{20x}{(x+1)(x+9)}=1-\frac{\frac{20}{x}}{1+\frac{10}{x}+\frac{9}{x^2}}$

$x-axis\ when\ y=0$; $\frac{x^2-10x+9}{x^2+10x+9}=0$; $(x-1)(x-9)=0$; $x=1\ and\ x=9$ $(1,0),(9,0)$

$y-axis\ when\ x=0 \Rightarrow y=1$ $(0,1)$

2. $Asymptotes:$ $Vertical:$ $(x+1)(x+9)=0$; $x=-1\ and\ x=-9$

$-1^{-}=\left(\frac{(+)}{(-)}\right)$ $y\to-\infty$; $-1^{+}=\left(\frac{(+)}{(+)}\right)$ $y\to\infty$; $x=-1$ $_{\to}\mathbf{I}^{\leftarrow}$

$-9^{-}=\left(\frac{(+)}{(+)}\right)$ $y\to\infty$; $-9^{+}=\left(\frac{(+)}{(-)}\right)$ $y\to-\infty$; $x=-9$ $^{\to}\mathbf{I}_{\leftarrow}$

$Non\ Vertical:$ $y=1-\frac{20x}{(x+1)(x+9)}=1-\frac{\frac{20}{x}}{1+\frac{10}{x}+\frac{9}{x^2}}$ $Non\ Vertical$ $y=1$

$x\to-\infty$ $y\to1^{+}$; $x\to\infty$ $y\to1^{-}$; $\boldsymbol{y=1}$ $^{\to}-_{\leftarrow}$

3. $Stationary\ points:$ $f'(x)=0$; $f(x)=1-\frac{20x}{x^2+10x+9}$;

$$f'(x)=\frac{(x^2+10x+9).(-20)-(-20x)(2x+10)}{(x^2+10x+9)^2}=\frac{(-20x^2-200x-180)+(40x^2+200x)}{(x^2+10x+9)^2}$$

$$=\frac{(-20x^2-200x-180)+(40x^2+200x)}{(x^2+10x+9)^2}=\frac{20x^2-180}{(x^2+10x+9)^2}=0$$

$=\frac{20x^2-180}{(x^2+10x+9)^2}=0$; $20(x^2-9)=0$; $(x-3)(x+3)=0$; $x=3\ and\ x=-3$

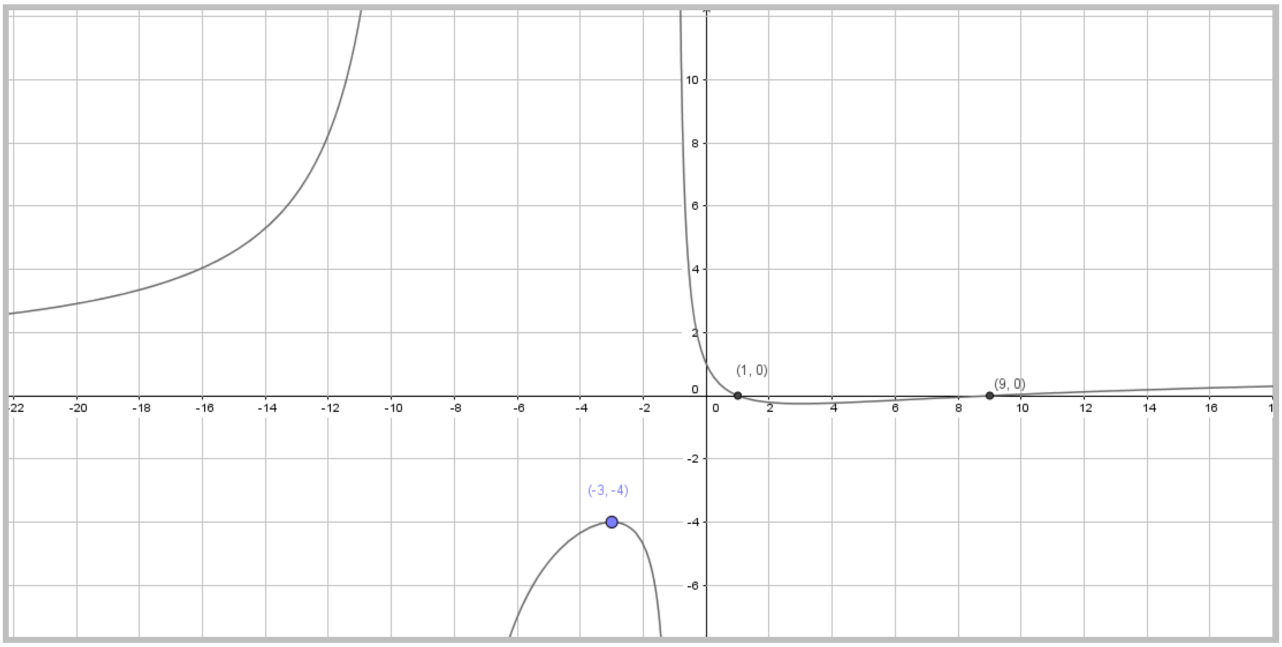
$For\ \ x = 3 \quad y = 1 - \frac{20x}{x^2+10x+9} = 1 - \frac{20(3)}{(3)^2+10(3)+9} = 1 - \frac{60}{48} = -\frac{12}{48} = -\frac{1}{4} \qquad T.P. = \left(3, -\frac{1}{4}\right)$

$For\ \ x = -3 \quad y = 1 - \frac{20x}{x^2+10x+9} = 1 - \frac{20(-3)}{(-3)^2+10(-3)+9} = 1 - \frac{-60}{-12} = -4 \qquad T.P. = (-3, -4)$

$Nature\ Table:$

x	→	-3	→	3	→
f′(x)	+	0	-	0	+
Shape	↗	→	↘	→	↗

$Hence\ Max.T.P.at\ (-3, -4)\ \ and\ Mini.T.P.at\ \left(3, -\frac{1}{4}\right)$



8.

1. $f(x) = \dfrac{2x^2 - 3x - 3}{x^2 - 3x + 2} = 2 + \dfrac{3x - 7}{x^2 - 3x + 2} = 2 + \dfrac{3x - 7}{(x-1)(x-2)} = 2 + \dfrac{\frac{3}{x} - \frac{7}{x^2}}{1 - \frac{3}{x} + \frac{2}{x^2}}$

$x - axis\ when\ y = 0\ ;\quad \dfrac{2x^2 - 3x - 3}{x^2 - 3x + 2} = 0\quad ;\quad too\ dificult\ to\ find\ roots\ so\ leave$

$y - axis\ when\ \ x = 0\ \Rightarrow y = -\dfrac{3}{2}\ \ \left(0, -\dfrac{3}{2}\right)$

2. $Asymptotes:\qquad Vertical:\ (x-1)(x-2) = 0\quad ;\quad x = 1\ \ and\ \ x = 2$

$1^- = \left(\dfrac{(-)}{(+)}\right)\ \ y \to -\infty\quad ;\qquad 1^+ = \left(\dfrac{(-)}{(-)}\right)\ \ y \to \infty\quad ;\quad x = 1\qquad {}_{\rightarrow}\mathbf{I}^{\leftarrow}$

$2^- = \left(\dfrac{(-)}{(-)}\right)\ \ y \to \infty\quad ;\qquad 2^+ = \left(\dfrac{(-)}{(+)}\right)\ \ y \to -\infty\quad ;\quad x = 2\qquad {}^{\rightarrow}\mathbf{I}_{\leftarrow}$

$Non\ Vertical:\qquad y = 2 + \frac{3x-7}{(x-1)(x-2)} = 2 + \frac{\frac{3}{x} - \frac{7}{x^2}}{1 - \frac{3}{x} + \frac{2}{x^2}}\qquad Non\ Vertical\quad y = 2$

$x \to -\infty\qquad y \to 2^-\quad ;\quad x \to \infty\qquad y \to 2^+\quad ;\quad \boldsymbol{y = 2}\qquad {}_{\rightarrow}\mathbf{-}^{\leftarrow}$

3. $Stationary\ points:\ \ f'(x) = 0\quad ;\ \ f(x) = 2 + \frac{3x-7}{x^2 - 3x + 2}\quad ;$

$$f'(x) = \frac{(x^2 - 3x + 2).(3) - (3x - 7)(2x - 3)}{(x^2 - 3x + 2)^2} = \frac{(3x^2 - 9x + 6) - (6x^2 - 23x + 21)}{(x^2 - 3x + 2)^2}$$

$$= \frac{(-3x^2 + 14x - 15)}{(x^2 + 10x + 9)^2} = 0$$

$$= 3x^2 - 14x + 15 = 0\quad ;\quad (3x - 5)(x - 3) = 0\quad ;\quad (3x - 5)(x - 3) = 0\quad ;\quad x = \frac{5}{3}\ and\ x = 3$$

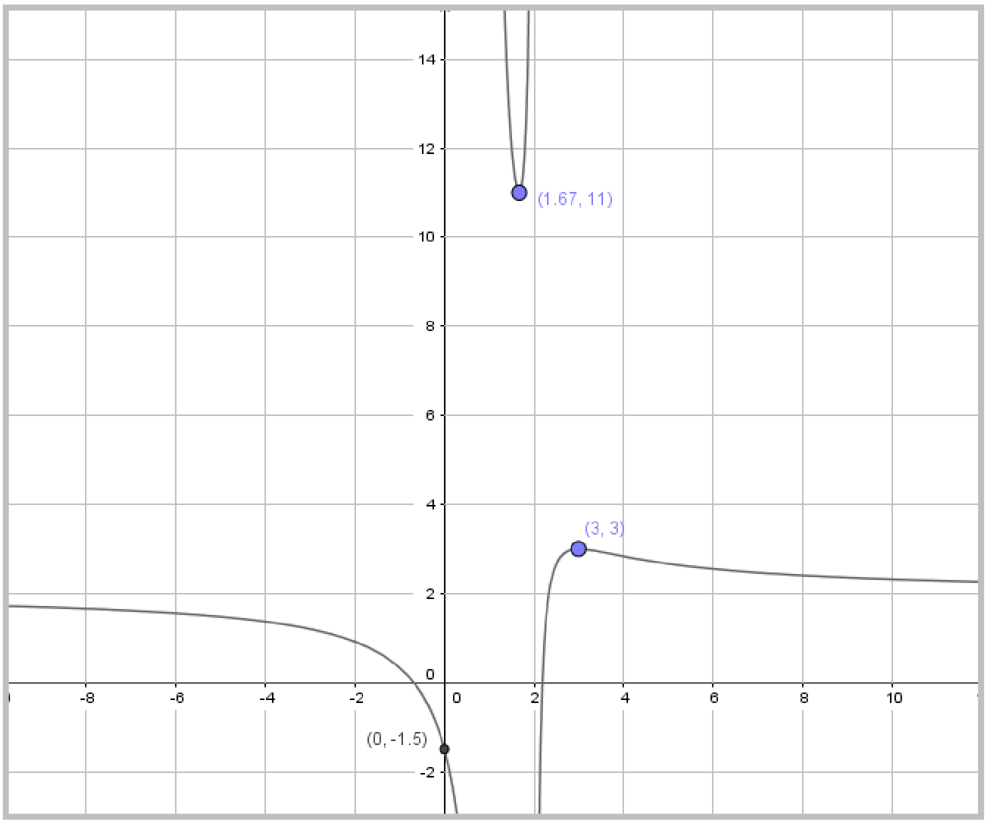
$For \ \ x = 3 \quad y = 2 + \frac{3x-7}{(x-1)(x-2)} = 2 + \frac{9-7}{(2)(1)} = 3 \qquad T.P. = (3,3)$

$For \ \ x = \frac{5}{3} \quad y = 2 + \frac{3\left(\frac{5}{3}\right) - 7}{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)} = 2 + \frac{-2}{\left(-\frac{2}{9}\right)} = 11 \qquad T.P. = \left(\frac{5}{3}, 11\right)$

$Nature\ Table$:

x	→	$\frac{5}{3}$	→	3	→
f′(x)	-	0	+	0	-
Shape	↘	→	↗	→	↘

$Hence\ Max.T.P.at\ (3,3)\ \ and\ Mini.T.P.at\ \left(\frac{5}{3}, 11\right)$



Ex 5 – Inverse, Odd and Even Functions

1. (a) $f(x) = 2x$; $y = 2x$; $x = \frac{1}{2}y$; $f^{-1}(x) = \frac{1}{2}x$

(b) $(fx) = 2 - x$; $y = 2 - x$; $x = 2 - y$; $f^{-1}(x) = 2 - x$

(c) $(fx) = \frac{2}{x}$; $y = \frac{2}{x}$; $x = \frac{2}{x}$; $f^{-1}(x) = \frac{2}{x}$

(d) $(fx) = 2^x$; $y = 2^x$; $log_2 y = log_2 2^x$; $log_2 y = x$; $log_2 y = x$; $f^{-1}(x) = log_2 x$

(e) $(fx) = 1 - 2x$; $y = 1 - 2x$; $2x = 1 - y$; $x = \frac{1-y}{2}$; $f^{-1}(x) = \frac{1-x}{2}$

(f) $(fx) = ln(x - 2)$; $y = ln(x - 2)$; $e^y + 2 = x$; $x = e^y + 2$; $f^{-1}(x) = e^x + 2$

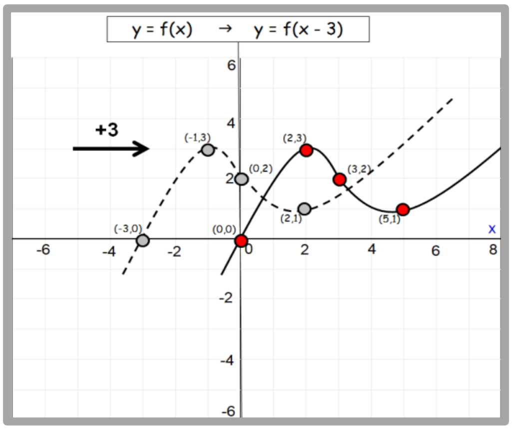
---

2. (a) $sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ (b) $tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ (c) $tan^{-1}(1) = \frac{\pi}{4}$ (d) $sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

(e) $cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ (b) $tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

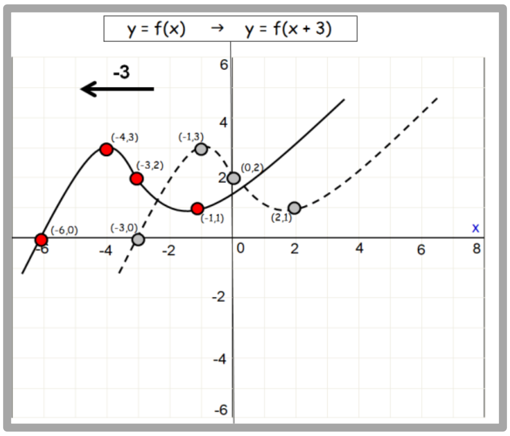
---

3. (a)



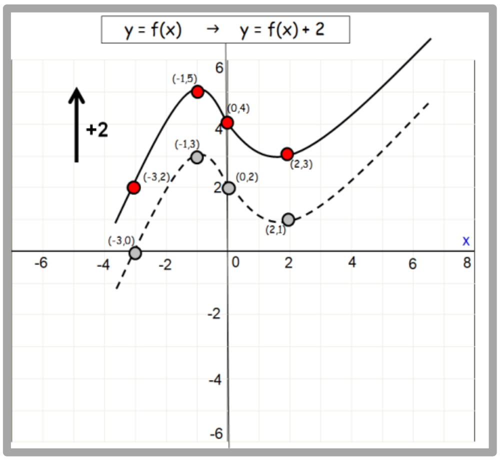
y = f(x) → y = f(x - 3)
+3
(-1,3)
(0,2)
(2,3)
(3,2)
(2,1)
(5,1)
(-3,0)
(0,0)
x

(b)



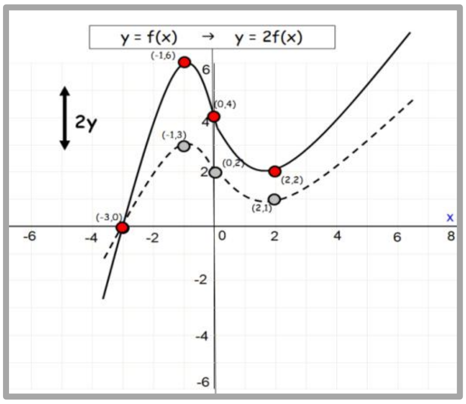
y = f(x) → y = f(x + 3)
-3
(-4,3)
(-3,2)
(-1,3)
(0,2)
(-1,1)
(2,1)
(-6,0)
(-3,0)
x

(c)

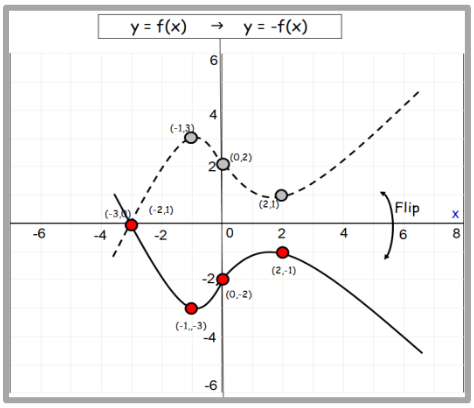


y = f(x) → y = f(x) + 2
+2
(-1,5)
(0,4)
(2,3)
(-1,3)
(-3,2)
(0,2)
(2,1)
(-3,0)
x

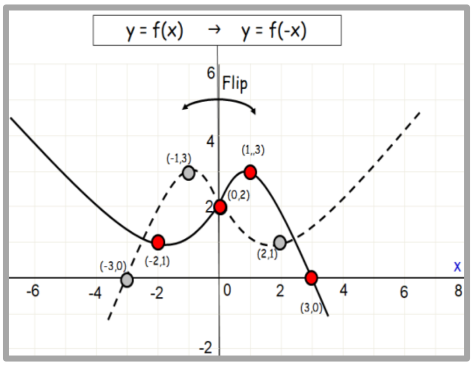
(d)


y = f(x) → y = 2f(x)
2y
(-1,6)
(0,4)
(-1,3)
(0,2)
(2,2)
(2,1)
(-3,0)

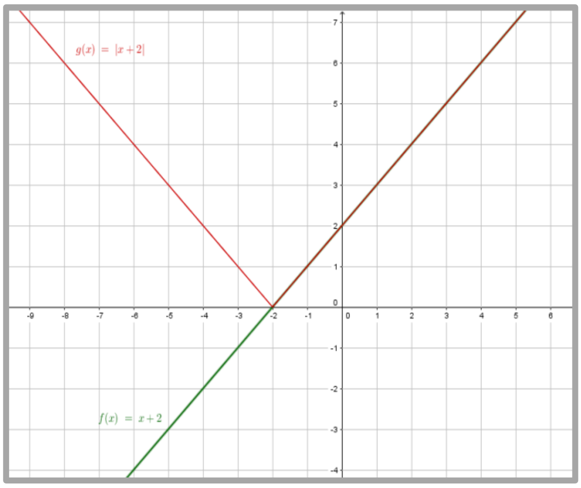
(e)


y = f(x) → y = -f(x)
(-1,3)
(0,2)
(2,1)
Flip
(-2,1)
(2,-1)
(0,-2)
(-1,-3)

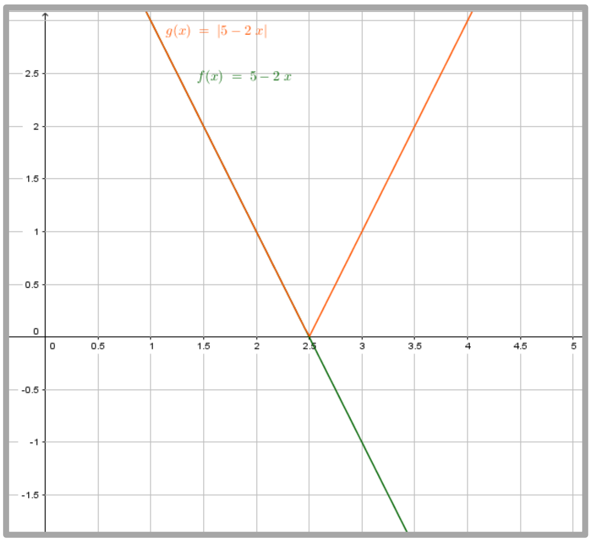
(f)


y = f(x) → y = f(-x)
Flip
(-1,3)
(1,3)
(0,2)
(-2,1)
(2,1)
(-3,0)
(3,0)

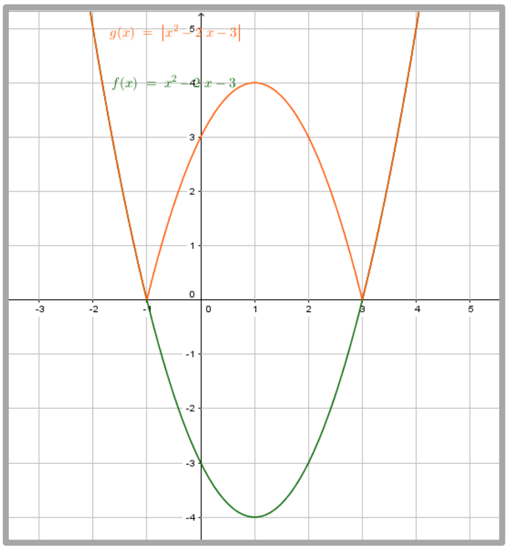
4.



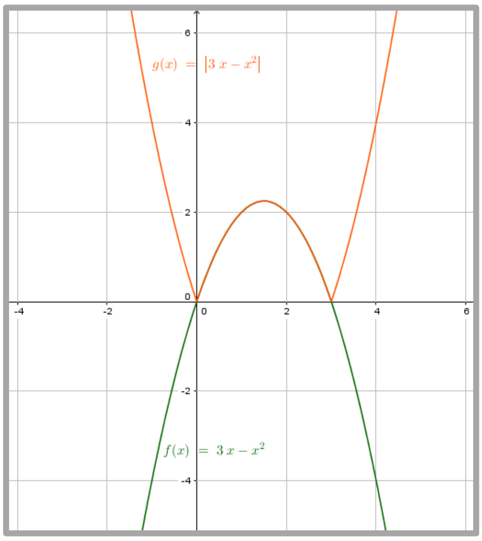
(b)



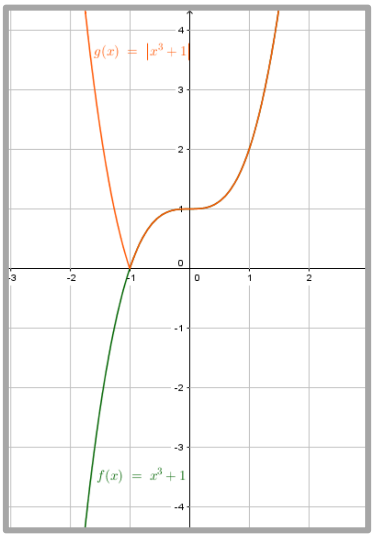
(c)



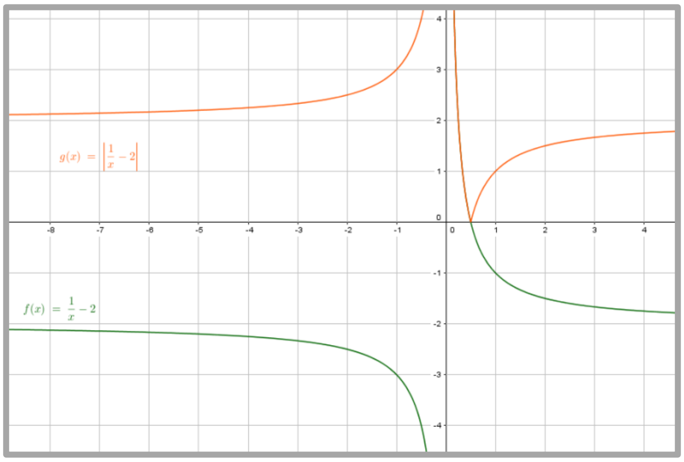
(d)



(e)



(f)



5. (a) $f(x) = (x+4)(x-2) = x^2 + 2x - 8$; $Since\ powers\ both\ odd\ and\ even : Neither$

(b) $f(x) = 3x^2 + 5$; $Even\ powers\ only$; $f(-x) = 3x^2 + 5 = f(x)$; $Even\ function$

(c) $f(x) = 2x + x^3$; $Odd\ powers\ only$; $f(-x) = -2x - x^3 = -f(x)$; $Odd\ function$

(d) $f(x) = sin2x$; $f(-x) = \sin(-2x) = -sin2x = -f(x)$; $Odd\ function$