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Advanced Higher Maths

Advanced Higher - Unit 2.5 Motion & Optimisation - Solutions

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Ex7 Page 20 – Motion

1. $Given \quad x = f(t) = 3 - 4t + t^2$

   $(a) \quad t = 0s\,; \;\; f(0) = 3 - 4 \times 0 + 0^2 = 3m$

   $(b) \quad t = 4s\,; \;\; f(4) = 3 - 4 \times 4 + 4^2 = 3m$

   $(c) \quad t = 3s\;; \;\; f^{'}(t) = -4 + 2t \quad ; \quad f^{'}(3) = -4 + 2 \times 3 = 2ms^{-1}$

   $(d) \quad t = 0s\;; \;\; f^{''}(t) = 2 \quad ; \quad f^{''}(0) = 2ms^{-2}$

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2. $Given \quad x = f(t) = 4t^3 - 3t^2 - 2t - 1$

   $(a) \quad t = 3s\,; \;\; f^{'}(t) = 12t^2 - 6t - 2 \;\; ; \;\; f^{'}(3) = 12(3)^2 - 6 \times 3 - 2 = 88ms^{-1}$

   $t = 4s\,; \;\; f^{'}(4) = 12(4)^2 - 6 \times 4 - 2 = 166ms^{-1}$

   $(b) \quad t = 3s\;; \;\; f^{''}(t) = 24t - 6 \quad ; \quad f^{''}(3) = 24 \times 3 - 6 = 66ms^{-2}$

   $t = 4s\;; \;\; f^{''}(3) = 24 \times 4 - 6 = 90ms^{-2}$

   $(c) \quad Average\ velocity\ during\ 4th\ second \quad v = t = \dfrac{88 + 166}{2} = 127ms^{-1}$

   $(d) \quad Average\ acceleration\ during\ 4th\ second \quad v = t = \dfrac{66 + 90}{2} = 78ms^{-1}$

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3. $Given \quad x = f(t) = \frac{1}{6}t^3 + \frac{1}{4}t^2$

$(a) \quad t = 0s\ ;\ f^{'}(t) = \dfrac{1}{2}t^2 + \dfrac{1}{2}t \ \ ;\ f^{''}(t) = t + \dfrac{1}{2} \ \ ;\ f^{'}(0) = 0 + \dfrac{1}{2} = \dfrac{1}{2}ms^{-2}$

$t = 2s\ ;\ f^{''}(2) = 2 + \dfrac{1}{2} = \dfrac{5}{2}ms^{-1}$

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4. $Given \quad x = f(t) = 9t + 3t^2 - t^3$

$(a) \quad t = 0s \ \ ;\ \ f(0) = 9 \times 0 + 3(0)^2 - t(0)^3 = 0$

$t = 0s\ ;\ f^{'}(t) = 9 + 6t - 3t^2 \ \ ;\ f^{'}(0) = 9 + 6 \times 0 - 3(0)^2 = 9ms^{-1}$

$t = 0s\ ;\ f^{''}(t) = 6 - 6t \ \ ;\ f^{''}(0) = 6 - 6 \times 0 = 6ms^{-2}$

$(b) \ \ At\ rest\ when\ f^{'}(t) = 0 \ \ ;\ \ 9 + 6t - 3t^2 = 0 \ ;\ 3 + 2t - t^2 = 0\ ;\ (3 - t)(1 + t) = 0$

$(3 - t)(1 + t) = 0\ ;\ \ t = -1s \ \ and \ \ t = 3s \ \ ;\ \ only\ feasible\ answer\ is\ 3\ seconds.$

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5. $Given \quad x = f(t) = 3t^2(3 - t) = 9t^2 - 3t^3$

$(a) \quad t = 0s\ ;\ f^{'}(t) = 18t - 9t^2 \ \ ;\ f^{'}(0) = 18 \times 0 - 9(0)^2 = 0ms^{-1}$

$t = 0s\ ;\ f^{''}(t) = 18 - 18t \ \ ;\ f^{''}(0) = 18 - 18 \times 0 = 18ms^{-2}$

$(b) \ \ At\ t = 3s \ \ ;\ \ f^{'}(3) = 18(3) - 9(3)^2 \ \ ;\ f^{'}(0) = 54 - 81 = -27ms^{-1} \quad (opposite\ direction\ !)$

$t = 3s \ \ ;\ \ f^{''}(3) = 18 - 18(3) = -36ms^{-2} \quad (\ decelerating\ !\ )$

Ex8 Stationary Points & Nature

1. $Given \quad y = x - lnx$ $\qquad \frac{dy}{dx} = 1 - \frac{1}{x} = \frac{(x-1)}{x} = 1 - x^{-1}$

$Stationary\ points\ when\ \frac{dy}{dx} = 0$

$\frac{(x-1)}{x} = 0 \quad ; \quad x - 1 = 0 \quad ; \quad x = 1 \quad ; \qquad For\ x = 1 \quad y = 1 - ln(1) = 1 \quad ; \quad SP\ \ (1,1)$

$\frac{d^2y}{dx^2} = x^{-2} = \frac{1}{x^2} \quad ; \quad x = 1 \qquad \frac{d^2y}{dx^2} = \frac{1}{x^2} = \frac{1}{1^2} > 0 \quad ; \quad Hence\ \ SP\ \ (1,1)\ is\ a\ Minimum$

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2. $Given \quad y = xlnx$ $\qquad \frac{dy}{dx} = lnx.1 + x.\frac{1}{x} = lnx + 1$

$Stationary\ points\ when\ \frac{dy}{dx} = 0$

$lnx + 1 = 0 \quad ; \quad lnx = -1 \quad ; \quad x = e^{-1} = \frac{1}{e} \quad ; \quad For\ x = \frac{1}{e} \quad y = \frac{1}{e}.(-1) \quad ; \quad SP\ \ \left(\frac{1}{e}, -\frac{1}{e}\right)$

$\frac{d^2y}{dx^2} = \frac{1}{x} \quad ; \quad x = \frac{1}{e} \qquad \frac{d^2y}{dx^2} = \frac{1}{\frac{1}{e}} = e > 0 \quad ; \quad Hence\ \ SP\ \ \left(\frac{1}{e}, -\frac{1}{e}\right)\ is\ a\ Minimum$

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3. $Given \quad y = xe^{-x}$ $\quad \frac{dy}{dx} = e^{-x}.1 - x.e^{-x} = e^{-x}(1-x)$

$Stationary\ points\ when\ \frac{dy}{dx} = 0$

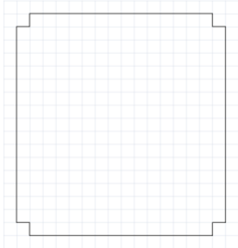
$e^{-x}(1-x) = 0 \quad ; \quad (1-x) = 0 \quad ; \quad x = 1 \quad ; \quad For\ x = 1 \quad y = 1e^{-1} = \frac{1}{e} \quad ; \quad SP\ \left(1, \frac{1}{e}\right)$

$$\frac{d^2y}{dx^2} = (1-x).-e^{-x} + e^{-x}.-1 = -e^{-x} + xe^{-x} - e^{-x} = (x-2)e^{-x} \ ;$$

$x = 1 \qquad \frac{d^2y}{dx^2} = (1-2)e^{-1} = -\frac{1}{e} < 0 \quad ; \quad Hence\ SP\ \left(1, \frac{1}{e}\right)\ is\ a\ Maximum$

Ex9 Optimisation

1.



$(a)\quad length = (16-2s)\quad ;\quad breadth = (16-2s)\ ;\quad height = s\quad ;\quad V = lbh = s(16-2s)^2$

$$V = s(256 - 64s + 4s^2)$$

$$V = 4s^3 - 64s^2 + 256s$$

$(b)\quad For\ Stationary\ Points\ \frac{dV}{ds} = 0\quad ;\quad \frac{dV}{ds} = 12s^2 - 128s + 256 = 0$

$$= 3s^2 - 32s + 64 = 0$$

$$= (3s-8)(s-8) = 0$$

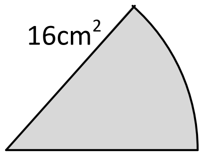
$$s = \frac{8}{3}\quad and\quad s = 8\qquad rule\ out\ 8$$

$Nature:\ \frac{d^2V}{ds^2} = 24s - 128\quad ;$

$For\ s = \frac{8}{3}\qquad \frac{d^2V}{ds^2} = 24\left(\frac{8}{3}\right) - 128 < 0\quad ;$

$Hence\ s = \frac{8}{3}\ is\ a\ maximum$

2.



$$(a)\quad Area\ of\ Sector = \frac{\theta}{360}\pi r^2 = 16 \quad ; \quad \theta = \frac{5760}{\pi r^2} \quad ; \quad Perimeter = r + r + Arc\ length$$

$$P = r + r + \frac{\theta}{360}\pi D = 2r + \frac{5760}{360\pi r^2}\pi(2r) = 2r + \frac{32}{r}$$

$$P = 2\left(r + \frac{16}{r}\right)$$

$$(b)\quad For\ Stationary\ Points\ \frac{dP}{dr} = 0 \quad ; \quad P = 2r + 32r^{-1}$$

$$\frac{dP}{dr} = 2 - 32r^{-2} = 0$$

$$\frac{dP}{dr} = 2 - \frac{32}{r^2} = 0$$

$$\frac{dP}{dr} = 2r^2 - 32 = 2(r^2 - 16) = (r-4)(r+4) = 0$$

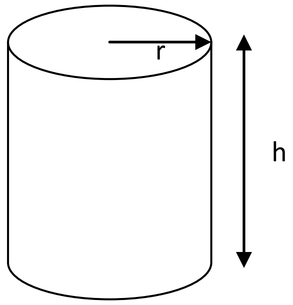
$$r = 4 \quad and \quad r = -4 \quad (\ rule\ out - 4 \quad r > 0\ )$$

$$Nature: \ \frac{d^2P}{dr^2} = \frac{64}{r^3} \quad ;$$

$$For\ s = 4 \qquad \frac{d^2P}{dr^2} = \frac{64}{4^3} > 0 \quad ;$$

$$Hence\ r = 4\ is\ a\ minimum \qquad For\ r = 4 \quad P = 2\left(r + \frac{16}{r}\right) = 2\left(4 + \frac{16}{4}\right) = 16cm$$

3.



(a) $Given\ (r+h) = 2$; $h = (2-r)$;

$$V = \pi r^2 h = \pi r^2(2-r)$$

(b) $For\ Stationary\ Points\ \frac{dV}{dr} = 0$; $V = \pi r^2(2-r) = 2\pi r^2 - \pi r^3$

$$\frac{dV}{dr} = 4\pi r - 3\pi r^2 = 0$$

$$\frac{dV}{dr} = \pi r(4-3r) = 0$$

$$\frac{dV}{dr} = r(4-3r) = 0$$

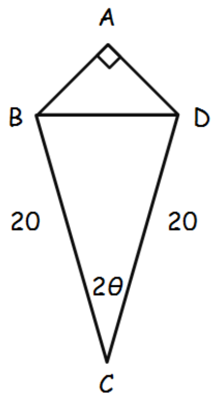
$$r = 0 \quad and \quad r = \frac{4}{3} \qquad (\,rule\ out\ 0 \quad r > 0\,)$$

$Nature:\ \frac{d^2V}{dr^2} = 4\pi - 6\pi r$;

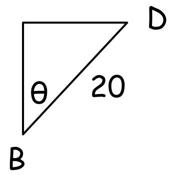
$For\ r = \frac{4}{3} \qquad \frac{d^2V}{dr^2} = 4\pi - 6\pi\left(\frac{4}{3}\right) < 0$;

$Hence\ r = \frac{4}{3}\ is\ a\ maximum \qquad For\ r = \frac{4}{3} \quad V = \pi\left(\frac{4}{3}\right)^2\left(2 - \frac{4}{3}\right) = \pi\left(\frac{16}{9}\right)\left(\frac{2}{3}\right) = \frac{32\pi}{27}m^3$

4.



(a) $Area\ of\ BCD = \frac{1}{2}absinc = \frac{1}{2}(BC)(DC)sinC = \frac{1}{2}(20)(20)sin2\theta = 200sin2\theta$



$By\ SOHCAHTOA\ ;\quad \frac{1}{2}BD = 20sin\theta\quad ;\quad BD = 40sin\theta\quad ;\quad (BD)^2 = 1600sin^2\theta$

(b) $Noting\ AB = AD\ ,\ by\ Pythagoras\ Theorem\ (AB)^2 + (AB)^2 = 1600sin^2\theta$

$$2(AB)^2 = 1600sin^2\theta$$

$$(AB)^2 = 800sin^2\theta$$

$Area\ of\ BAD = \frac{1}{2}bh = \frac{1}{2}(AB)(AD) = \frac{1}{2}(AB)^2 = \frac{1}{2}800sin^2\theta = 400sin^2\theta$

$Using\ cos2\theta = 1 - 2sin^2\theta\quad ;\quad sin^2\theta = \frac{1}{2}(1 - cos2\theta)\ ;\quad Area = 400\left(\frac{1}{2}(1 - cos2\theta)\right) = 200 - 200cos2\theta$

$Total\ Area:\ A(\theta) = 200sin2\theta + (200 - 200cos2\theta)$

$Total\ Area:\ A(\theta) = 200(1 - cos2\theta + sin2\theta)$

(c) $For\ Stationary\ Points\ \frac{dA}{d\theta} = 0$; $A(\theta) = 200(1 - cos2\theta + sin2\theta) = 200 - 200cos2\theta + 200sin2\theta$

$$\frac{dA}{d\theta} = 400sin2\theta + 400cos2\theta = 0$$

$$= sin2\theta + cos2\theta = 0$$

$$= tan2\theta = -1$$

$Higher\ work:$ $2\theta = tan^{-1}(-1) = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$; $\theta = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

$from\ diagram$ $\theta < \frac{\pi}{2}$ $\theta = \frac{3\pi}{8}$

$Nature:$ $\frac{d^2A}{d\theta^2} = 800cos2\theta - 800sin2\theta$;

$For\ \theta = \frac{3\pi}{8}$ $\frac{d^2A}{d\theta^2} = 800\left(cos\left(\frac{3\pi}{4}\right) - sin\left(\frac{3\pi}{4}\right)\right) = 800\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) < 0$;

$Hence\ \theta = \frac{3\pi}{8}\ is\ a\ maximum$ $For\ r = \frac{\pi}{8}$ $A\left(\frac{\pi}{8}\right) = 200\left(1 - cos\left(\frac{3\pi}{4}\right) + sin\left(\frac{3\pi}{4}\right)\right) = 200(1 + \sqrt{2})\ cm^2$
