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Advanced Higher Maths

Advanced Higher - Unit 3.1 Matrices & Systems of Equations - Solutions

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Ex1 Page 78 – Matrices & System of Equations

1.
$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 3 & 3 & 1 & \vdots & 4 \\ 3 & 2 & 2 & \vdots & 7 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ 3R_1 - R_2 \\ R_3 \end{matrix}\begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 0 & 2 & \vdots & -1 \\ 3 & 2 & 2 & \vdots & 7 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ R_2 \\ 3R_1 - R_3 \end{matrix}\begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 0 & 2 & \vdots & -1 \\ 0 & 1 & 1 & \vdots & -4 \end{bmatrix}$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & -4 \\ 0 & 0 & 2 & \vdots & -1 \end{bmatrix}$$

$R_3 \ : \ 2z = -1 \ ; \ z = -\frac{1}{2}$; $R_2 : \ y + z = -4 \ ; \ y = -\frac{7}{2}$ $R_1 : \ x + y + z = 1 \ ; \ x = 5$

2.
$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{bmatrix} 1 & -2 & 1 & \vdots & 6 \\ 3 & 1 & -2 & \vdots & 4 \\ 7 & -6 & -1 & \vdots & 10 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ 3R_1 - R_2 \\ 7R_1 - R_3 \end{matrix}\begin{bmatrix} 1 & -2 & 1 & \vdots & 6 \\ 0 & -7 & 5 & \vdots & 14 \\ 0 & -8 & 8 & \vdots & 32 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ R_2 \\ 8R_2 - 7R_3 \end{matrix}\begin{bmatrix} 1 & -2 & 1 & \vdots & 6 \\ 0 & -7 & 5 & \vdots & 14 \\ 0 & 0 & -16 & \vdots & -112 \end{bmatrix}$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{bmatrix} 1 & -2 & 1 & \vdots & 6 \\ 0 & -7 & 5 & \vdots & 14 \\ 0 & 0 & -16 & \vdots & -112 \end{bmatrix}$$

$R_3 \ : -16z = -112 \ ; \ z = 7$; $R_2 : -7y + 5z = 14 \ ; \ y = 3$ $R_1 : \ x - 2y + z = 6 \ ; \ x = 5$

3.
$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{bmatrix} 5 & -1 & 2 & \vdots & 25 \\ 3 & 2 & -3 & \vdots & 16 \\ 2 & -1 & 1 & \vdots & 9 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ 3R_1 - 5R_2 \\ 2R_1 - 5R_3 \end{matrix}\begin{bmatrix} 5 & -1 & 2 & \vdots & 25 \\ 0 & -13 & 21 & \vdots & -5 \\ 0 & 3 & -1 & \vdots & 5 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ R_2 \\ 3R_2 + 13R_3 \end{matrix}\begin{bmatrix} 5 & -1 & 2 & \vdots & 25 \\ 0 & -13 & 21 & \vdots & -5 \\ 0 & 0 & 50 & \vdots & 50 \end{bmatrix}$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{bmatrix} 5 & -1 & 2 & \vdots & 25 \\ 0 & -13 & 21 & \vdots & -5 \\ 0 & 0 & 50 & \vdots & 50 \end{bmatrix}$$

R_3 : $50z = 50$; $z = 1$; R_2 : $-13y + 21z = -5$; $y = 2$ R_1 : $5x - y + 2z = 25$; $x = 5$

4.
$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{bmatrix} 1 & 1 & 1 & \vdots & 2 \\ 3 & -1 & 2 & \vdots & 4 \\ 2 & 3 & 1 & \vdots & 7 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ 3R_1 - R_2 \\ 2R_1 - R_3 \end{matrix}\begin{bmatrix} 1 & 1 & 1 & \vdots & 2 \\ 0 & 4 & 1 & \vdots & 2 \\ 0 & -1 & 1 & \vdots & -3 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ R_2 \\ R_2 + 4R_3 \end{matrix}\begin{bmatrix} 1 & 1 & 1 & \vdots & 2 \\ 0 & 4 & 1 & \vdots & 2 \\ 0 & 0 & 5 & \vdots & -10 \end{bmatrix}$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{bmatrix} 1 & 1 & 1 & \vdots & 2 \\ 0 & 4 & 1 & \vdots & 2 \\ 0 & 0 & 5 & \vdots & -10 \end{bmatrix}$$

R_3 : $5z = -10$; $z = -2$; R_2 : $4y + z = 2$; $y = 1$ R_1 : $x + y + z = 2$; $x = 3$

5.
$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{bmatrix} 5 & -3 & 6 & \vdots & 0 \\ 1 & 5 & 2 & \vdots & 0 \\ -1 & 2 & 5 & \vdots & 0 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ R_1 - 5R_2 \\ R_1 + 5R_3 \end{matrix}\begin{bmatrix} 5 & -3 & 6 & \vdots & 0 \\ 0 & -28 & -4 & \vdots & 0 \\ 0 & 7 & 31 & \vdots & 0 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ R_2 \\ R_2 + 4R_3 \end{matrix}\begin{bmatrix} 5 & -3 & 6 & \vdots & 0 \\ 0 & -28 & -4 & \vdots & 0 \\ 0 & 0 & 120 & \vdots & 0 \end{bmatrix}$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{bmatrix} 5 & -3 & 6 & \vdots & 0 \\ 0 & -28 & -4 & \vdots & 0 \\ 0 & 0 & 120 & \vdots & 0 \end{bmatrix}$$

R_3 : $120z = 0$; $z = 0$; R_2 : $4y + z = 0$; $y = 0$ R_1 : $x + y + z = 0$; $x = 0$

Ex2 Page 79 – Ill Conditioning

1. $(a)(i)$ $\begin{matrix} R_1 \\ R_2 \end{matrix}\begin{bmatrix} 1 & 1 & \vdots & 2 \\ 1 & 1.0001 & \vdots & 2.0001 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ R_1 - R_2 \end{matrix}\begin{bmatrix} 1 & 1 & \vdots & 2 \\ 0 & -0.0001 & \vdots & -0.0001 \end{bmatrix}$

$R_2: \ -0.0001y = -0.0001 \quad ; \quad y = 1 \qquad R_1: \ x + y = 2 \ ; \ x = 1$

(ii) $\begin{matrix} R_1 \\ R_2 \end{matrix}\begin{bmatrix} 1 & 1 & \vdots & 2 \\ 1 & 1.0001 & \vdots & 1.9999 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ R_1 - R_2 \end{matrix}\begin{bmatrix} 1 & 1 & \vdots & 2 \\ 0 & -0.0001 & \vdots & 0.0001 \end{bmatrix}$

$R_2: \ -0.0001y = 0.0001 \quad ; \quad y = -1 \qquad R_1: \ x + y = 2 \ ; \ x = 3 \quad ; \qquad Hence\ ill - conditioned$

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b. (i) $\begin{matrix} R_1 \\ R_2 \end{matrix}\begin{bmatrix} 1 & 0.99 & \vdots & 1.99 \\ 0.99 & 0.98 & \vdots & 1.97 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ 0.99R_1 - R_2 \end{matrix}\begin{bmatrix} 1 & 0.99 & \vdots & 1.99 \\ 0 & 0.0001 & \vdots & 0.0001 \end{bmatrix}$

$R_2: \ 0.0001y = 0.0001 \quad ; \quad y = 1 \qquad R_1: \ x + 0.99y = 1.99 \ ; \ x = 1$

(ii) $\begin{matrix} R_1 \\ R_2 \end{matrix}\begin{bmatrix} 1 & 0.99 & \vdots & 2.00 \\ 0.99 & 0.98 & \vdots & 1.97 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ 0.99R_1 - R_2 \end{matrix}\begin{bmatrix} 1 & 0.99 & \vdots & 2.00 \\ 0 & 0.0001 & \vdots & 0.01 \end{bmatrix}$

$R_2: \ 0.0001y = 0.01 \quad ; \quad y = 100 \qquad R_1: \ x + 0.99y = 2 \ ; \ x = -97 \quad ; \qquad Hence\ ill - conditioned$

Ex 1 – Matrices

1. $(a)\quad \begin{pmatrix} 3 & 1 & 4 & 2 \\ 5 & 4 & 0 & 7 \end{pmatrix}\ Order\ 2\times 4 \qquad (b)\quad \begin{pmatrix} 2 & -1 \\ 4 & 8 \\ 1 & -2 \end{pmatrix}\ Order\ 3\times 2 \qquad (c)\quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}\ Order\ 4\times 2$

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2. $(A)\ Order\ 1\times 3 \quad (B)\ Order\ 1\times 3 \quad (C)\ Order\ 1\times 3 \quad \boldsymbol{A\ and\ C\ are\ equal}$

   $(D)\ Order\ 2\times 1 \quad (E)\ Order\ 2\times 1 \quad (F)\ Order\ 2\times 1 \quad (G)\ Order\ 2\times 1 \quad \boldsymbol{F\ and\ G\ are\ equal}$

   $(H)\ Order\ 2\times 2 \quad (J)\ Order\ 2\times 2 \quad (K)\ Order\ 2\times 2 \quad (L)\ Order\ 2\times 2 \quad \boldsymbol{H\ and\ L\ are\ equal}$

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3. $(a)\quad (3x \quad -y) = (12 \quad 3) \quad ; \quad x = 4\ \ and\ y = -3$

   $(b)\quad \begin{pmatrix} x & + & 3 \\ 4 & - & y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \quad ; \quad x = 4\ \ and\ y = -1$

   $(c)\quad \begin{pmatrix} x & + & 2y \\ 2x & - & y \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix} \quad ; \quad x + 2y = 9\ \ and\ 2x - y = 8 \quad ; \quad x = 5\ \ and\ \ y = 2$

   $(d)\quad \begin{pmatrix} x^2 & y^2 \\ y^3 & x^3 \end{pmatrix} = \begin{pmatrix} 4 & 9 \\ -27 & 8 \end{pmatrix} \quad ; \quad x = 2\ \ and\ \ y = -3$

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4. $(a)\ A^T = \begin{pmatrix} 3 & 5 \\ 1 & 4 \\ 4 & 0 \\ 2 & 7 \end{pmatrix}\ Order\ 4\times 2 \quad (b)\quad A^T = \begin{pmatrix} 2 & 4 & 1 \\ -1 & 8 & -2 \end{pmatrix}\ Order\ 2\times 3$

   $(c)\ A^T = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}\ Order\ 2\times 4$

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5. $A = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 5 & 4 & 0 & 7 \end{pmatrix} \to A^T \begin{pmatrix} 3 & 5 \\ 1 & 4 \\ 4 & 0 \\ 2 & 7 \end{pmatrix} \to (A^T)^T = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 5 & 4 & 0 & 7 \end{pmatrix}$

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6. $P = \begin{pmatrix} x & 9 \\ -3 & y \end{pmatrix} \to P^T \begin{pmatrix} x & -3 \\ 9 & y \end{pmatrix} = Q \begin{pmatrix} 5 & -3 \\ 9 & -4 \end{pmatrix} \qquad x = 5\ \ and\ \ y = -4$

Ex 2 – Addition of Matrices

1. $(a)\ \begin{pmatrix}3\\1\end{pmatrix}+\begin{pmatrix}4\\5\end{pmatrix}=\begin{pmatrix}7\\6\end{pmatrix}$ $\quad(b)\ \begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}-1\\3\end{pmatrix}=\begin{pmatrix}1\\7\end{pmatrix}$ $\quad(c)\ \begin{pmatrix}2a\\b\end{pmatrix}+\begin{pmatrix}7a\\-3b\end{pmatrix}=\begin{pmatrix}9a\\-2b\end{pmatrix}$

$(d)\ \begin{pmatrix}2u\\-3v\end{pmatrix}+\begin{pmatrix}-2u\\3v\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\quad(e)\ (2\quad 5)+(1\quad 4)=(3\quad 9)$ $\quad(f)\ (2\quad -3)+(-5\quad 8)=(-3\quad 5)$

$(g)\ \begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}2&3\\4&5\end{pmatrix}=\begin{pmatrix}3&3\\4&6\end{pmatrix}$ $\quad(h)\ \begin{pmatrix}2&3&1\\5&1&0\end{pmatrix}+\begin{pmatrix}1&-2&3\\-7&1&-4\end{pmatrix}=\begin{pmatrix}3&1&4\\-2&2&-4\end{pmatrix}$

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2. $(a)\ \begin{pmatrix}3&2\\1&-1\end{pmatrix}+\begin{pmatrix}1&4\\5&2\end{pmatrix}=\begin{pmatrix}4&6\\6&1\end{pmatrix}$ $\quad(b)\ \begin{pmatrix}1&4\\5&2\end{pmatrix}+\begin{pmatrix}-2&3\\1&-4\end{pmatrix}=\begin{pmatrix}-1&7\\6&-2\end{pmatrix}$

$(c)\ \begin{pmatrix}4&6\\6&1\end{pmatrix}+\begin{pmatrix}-2&3\\1&-4\end{pmatrix}=\begin{pmatrix}2&9\\7&-3\end{pmatrix}$ $\quad(d)\ \begin{pmatrix}3&2\\1&-1\end{pmatrix}+\begin{pmatrix}-1&7\\6&-2\end{pmatrix}=\begin{pmatrix}2&9\\7&-3\end{pmatrix}$

$True : (A+B)+C = A+(B+C)$

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3. $(a)\ \begin{pmatrix}3&-4\\-5&1\end{pmatrix}+\begin{pmatrix}-3&4\\5&-1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$ $\quad(b)\ \begin{pmatrix}-3&4\\5&-1\end{pmatrix}+\begin{pmatrix}3&-4\\-5&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

$A=-B\ \ or\ B=-A$

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4. $A+B=\begin{pmatrix}3&1&4&2\\5&4&0&7\end{pmatrix}+\begin{pmatrix}2&-1&3&-2\\3&8&1&0\end{pmatrix}=\begin{pmatrix}5&0&7&0\\8&12&1&7\end{pmatrix}$ $\quad(A+B)^T=\begin{pmatrix}5&8\\0&12\\7&1\\0&7\end{pmatrix}$

$A^T+B^T=\begin{pmatrix}3&5\\1&4\\4&0\\2&7\end{pmatrix}+\begin{pmatrix}2&3\\-1&8\\3&1\\-2&0\end{pmatrix}=\begin{pmatrix}5&8\\0&12\\7&1\\0&7\end{pmatrix}$; $\quad Hence\ (A+B)^T=A^T+B^T$

Ex3 – Subtraction of Matrices

1. (a) $\begin{pmatrix}3\\1\end{pmatrix}-\begin{pmatrix}4\\5\end{pmatrix}=\begin{pmatrix}-1\\-4\end{pmatrix}$ (b) $\begin{pmatrix}2\\4\end{pmatrix}-\begin{pmatrix}-1\\3\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$ (c) $\begin{pmatrix}2a\\b\end{pmatrix}-\begin{pmatrix}7a\\-3b\end{pmatrix}=\begin{pmatrix}-5a\\4b\end{pmatrix}$

(d) $\begin{pmatrix}2u\\-3v\end{pmatrix}-\begin{pmatrix}-2u\\3v\end{pmatrix}=\begin{pmatrix}4u\\-6v\end{pmatrix}$ (e) $\begin{pmatrix}2 & 5\end{pmatrix}-\begin{pmatrix}1 & 4\end{pmatrix}=\begin{pmatrix}1 & 1\end{pmatrix}$ (f) $\begin{pmatrix}2 & -3\end{pmatrix}-\begin{pmatrix}-5 & 8\end{pmatrix}=\begin{pmatrix}7 & -11\end{pmatrix}$

(g) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}-\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}=\begin{pmatrix}-1 & -3\\-4 & -4\end{pmatrix}$ (h) $\begin{pmatrix}4 & 7\\2 & 3\end{pmatrix}-\begin{pmatrix}8 & 0\\3 & 9\end{pmatrix}=\begin{pmatrix}-4 & 7\\-1 & -6\end{pmatrix}$

(i) $\begin{pmatrix}2 & 3 & 1\\5 & 1 & 0\end{pmatrix}-\begin{pmatrix}1 & -2 & 3\\-7 & 1 & -4\end{pmatrix}=\begin{pmatrix}1 & 5 & -2\\12 & 0 & 4\end{pmatrix}$

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2. (a) $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}+\begin{pmatrix}-2 & 3\\0 & 1\end{pmatrix}=\begin{pmatrix}-1 & 5\\3 & 5\end{pmatrix}$ (b) $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}+\begin{pmatrix}5 & 2\\-1 & 0\end{pmatrix}=\begin{pmatrix}6 & 4\\2 & 4\end{pmatrix}$

(c) $\begin{pmatrix}-1 & 5\\3 & 5\end{pmatrix}+\begin{pmatrix}5 & 2\\-1 & 0\end{pmatrix}=\begin{pmatrix}4 & 7\\2 & 5\end{pmatrix}$ (d) $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}-\begin{pmatrix}-2 & 3\\0 & 1\end{pmatrix}=\begin{pmatrix}3 & -1\\3 & 3\end{pmatrix}$

(e) $\begin{pmatrix}5 & 2\\-1 & 0\end{pmatrix}-\begin{pmatrix}-2 & 3\\0 & 1\end{pmatrix}=\begin{pmatrix}7 & -1\\-1 & -1\end{pmatrix}$ (f) $\begin{pmatrix}5 & 2\\-1 & 0\end{pmatrix}-\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}=\begin{pmatrix}4 & 0\\-4 & -4\end{pmatrix}$

(g) $\begin{pmatrix}6 & 4\\2 & 4\end{pmatrix}+\begin{pmatrix}-1 & 5\\3 & 5\end{pmatrix}=\begin{pmatrix}5 & 9\\5 & 9\end{pmatrix}$ (h) $\begin{pmatrix}6 & 4\\2 & 4\end{pmatrix}-\begin{pmatrix}-1 & 5\\3 & 5\end{pmatrix}=\begin{pmatrix}7 & -1\\-1 & -1\end{pmatrix}$

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3. (a) $X+\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}0 & 2\\2 & 1\end{pmatrix}$; $X=\begin{pmatrix}0 & 2\\2 & 1\end{pmatrix}-\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}-1 & 2\\2 & 0\end{pmatrix}$

(b) $X+\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}=\begin{pmatrix}4 & -1\\3 & 2\end{pmatrix}$; $X=\begin{pmatrix}4 & -1\\3 & 2\end{pmatrix}-\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}=\begin{pmatrix}2 & -4\\-1 & -3\end{pmatrix}$

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4. (a) $A + B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & -2 \\ 1 & 5 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 4 & 9 \\ 9 & 12 \end{pmatrix}$

(b) $B - C = \begin{pmatrix} 3 & -2 \\ 1 & 5 \\ 4 & 6 \end{pmatrix} - \begin{pmatrix} 4 & 2 \\ 1 & 0 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 0 & 5 \\ 7 & 1 \end{pmatrix}$

(c) $(A + B) - C = \begin{pmatrix} 4 & 0 \\ 4 & 9 \\ 9 & 12 \end{pmatrix} - \begin{pmatrix} 4 & 2 \\ 1 & 0 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 3 & 9 \\ 12 & 7 \end{pmatrix}$

(d) $A + (B - C) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} -1 & -4 \\ 0 & 5 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 3 & 9 \\ 12 & 7 \end{pmatrix}$

$(c) and\ (d)\ tells\ us \quad (A + B) - C = A + (B - C)$

Ex 4 – Multiplication of a Matrix

1. $(a)\quad 2A = 2\begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ -2 & -4 \end{pmatrix}$ $\qquad (b)\quad 3A = 3\begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 9 & 12 \\ -3 & -6 \end{pmatrix}$

   $(c)\quad -5A = -5\begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -15 & -20 \\ 5 & 10 \end{pmatrix}$ $\qquad (d)\quad -A = -\begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 1 & 2 \end{pmatrix}$

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2. $(a)\quad A - B = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 3 \\ 4 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 5 & -2 \\ -2 & -5 & 3 \end{pmatrix}$

   $(b)\quad 2(A+B) = 2\left[\begin{pmatrix} 3 & 4 & 1 \\ 2 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 3 \\ 4 & 5 & 0 \end{pmatrix}\right] = 2\begin{pmatrix} 5 & 3 & 4 \\ 6 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 6 & 8 \\ 12 & 10 & 6 \end{pmatrix}$

   $(c)\quad 2A = 2\begin{pmatrix} 3 & 4 & 1 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 8 & 2 \\ 4 & 0 & 6 \end{pmatrix}$

   $(d)\quad 2B = 2\begin{pmatrix} 2 & -1 & 3 \\ 4 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 6 \\ 8 & 10 & 0 \end{pmatrix}$

   $(e)\quad 2A + 2B = \begin{pmatrix} 6 & 8 & 2 \\ 4 & 0 & 6 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 6 \\ 8 & 10 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 6 & 8 \\ 12 & 10 & 6 \end{pmatrix}$

   $(f)\quad 6A = 6\begin{pmatrix} 3 & 4 & 1 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 18 & 24 & 6 \\ 12 & 0 & 18 \end{pmatrix}$

   $(g)\quad 3(2A) = 3\begin{pmatrix} 6 & 8 & 2 \\ 4 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 18 & 24 & 6 \\ 12 & 0 & 18 \end{pmatrix}$

   $(h)\quad 8B = 8\begin{pmatrix} 2 & -1 & 3 \\ 4 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 16 & -8 & 24 \\ 32 & 40 & 0 \end{pmatrix}$

   $(i)\quad 4(2B) = 4\begin{pmatrix} 4 & -2 & 6 \\ 8 & 10 & 0 \end{pmatrix} = \begin{pmatrix} 16 & -8 & 24 \\ 32 & 40 & 0 \end{pmatrix}$

   $Rules:\quad k(A \pm B) = kA \pm kB \quad and \quad k(tA) = (kt)A$

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3. (a) $3A + 2B = 3\begin{pmatrix}2 & -3\\4 & 1\end{pmatrix} + 2\begin{pmatrix}-4 & 1\\3 & -2\end{pmatrix} = \begin{pmatrix}6 & -9\\12 & 3\end{pmatrix} + \begin{pmatrix}-8 & 2\\6 & -4\end{pmatrix} = \begin{pmatrix}-2 & -7\\18 & -1\end{pmatrix}$

(b) $4A - 3B = 4\begin{pmatrix}2 & -3\\4 & 1\end{pmatrix} - 3\begin{pmatrix}-4 & 1\\3 & -2\end{pmatrix} = \begin{pmatrix}8 & -12\\16 & 4\end{pmatrix} - \begin{pmatrix}-12 & 3\\9 & -6\end{pmatrix} = \begin{pmatrix}20 & -15\\7 & 10\end{pmatrix}$

(c) $5A - 4B = 5\begin{pmatrix}2 & -3\\4 & 1\end{pmatrix} - 4\begin{pmatrix}-4 & 1\\3 & -2\end{pmatrix} = \begin{pmatrix}10 & -15\\20 & 5\end{pmatrix} - \begin{pmatrix}-16 & 4\\12 & -8\end{pmatrix} = \begin{pmatrix}26 & -19\\8 & 13\end{pmatrix}$

(d) $2(A - 5B) = 2\left[\begin{pmatrix}2 & -3\\4 & 1\end{pmatrix} - 5\begin{pmatrix}-4 & 1\\3 & -2\end{pmatrix}\right] = 2\left[\begin{pmatrix}2 & -3\\4 & 1\end{pmatrix} - \begin{pmatrix}-20 & 5\\15 & -10\end{pmatrix}\right] = 2\begin{pmatrix}22 & -8\\-11 & 11\end{pmatrix}$

$= \begin{pmatrix}44 & -16\\-22 & 22\end{pmatrix}$

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4. (a) $3X = \begin{pmatrix}6 & -3\\12 & 9\end{pmatrix} = \begin{pmatrix}2 & -1\\4 & 3\end{pmatrix}$

(b) $2X + \begin{pmatrix}3 & 1\\4 & 2\end{pmatrix} = \begin{pmatrix}9 & 5\\2 & 8\end{pmatrix}$; $2X = \begin{pmatrix}9 & 5\\2 & 8\end{pmatrix} - \begin{pmatrix}3 & 1\\4 & 2\end{pmatrix} = \begin{pmatrix}6 & 4\\-2 & 6\end{pmatrix}$; $X = \begin{pmatrix}3 & 2\\-1 & 3\end{pmatrix}$

(c) $4X - \begin{pmatrix}3 & 1\\4 & 7\end{pmatrix} = \begin{pmatrix}5 & 3\\0 & 13\end{pmatrix}$; $4X = \begin{pmatrix}5 & 3\\0 & 13\end{pmatrix} + \begin{pmatrix}3 & 1\\4 & 7\end{pmatrix} = \begin{pmatrix}8 & 4\\4 & 20\end{pmatrix}$; $X = \begin{pmatrix}2 & 1\\1 & 5\end{pmatrix}$

(d) $\begin{pmatrix}7 & 1\\-4 & 3\end{pmatrix} - 3X = \begin{pmatrix}-5 & 10\\8 & 9\end{pmatrix}$; $-3X = \begin{pmatrix}-5 & 10\\8 & 9\end{pmatrix} - \begin{pmatrix}7 & 1\\-4 & 3\end{pmatrix} = \begin{pmatrix}-12 & 9\\12 & 6\end{pmatrix}$ $X = \begin{pmatrix}3 & -3\\-4 & -2\end{pmatrix}$

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5. (a) $2\begin{pmatrix}1 & -1 & 3\\2 & -7 & 5\end{pmatrix} + X = 3\begin{pmatrix}1 & 2 & -4\\3 & -5 & 1\end{pmatrix}$; $\begin{pmatrix}2 & -2 & 6\\4 & -14 & 10\end{pmatrix} + X = \begin{pmatrix}3 & 6 & -12\\9 & -15 & 3\end{pmatrix}$

$X = \begin{pmatrix}3 & 6 & -12\\9 & -15 & 3\end{pmatrix} - \begin{pmatrix}2 & -2 & 6\\4 & -14 & 10\end{pmatrix} = \begin{pmatrix}1 & 8 & -18\\5 & -1 & -7\end{pmatrix}$

(b) $5\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} - 3X = 4\begin{pmatrix}-4 & 7\\3 & 8\end{pmatrix}$; $\begin{pmatrix}5 & 10\\15 & 20\end{pmatrix} - 3X = \begin{pmatrix}-16 & 28\\12 & 32\end{pmatrix}$

$-3X = \begin{pmatrix}-16 & 28\\12 & 32\end{pmatrix} - \begin{pmatrix}5 & 10\\15 & 20\end{pmatrix} = \begin{pmatrix}-21 & 18\\-3 & 12\end{pmatrix}$; $X = \begin{pmatrix}7 & -6\\1 & 4\end{pmatrix}$

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6. $2\begin{pmatrix} p & q \\ r & s \end{pmatrix} + \begin{pmatrix} 7 & -2 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix}$; $2\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 7 & -2 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 8 \\ 6 & -4 \end{pmatrix}$

$2\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} -2 & 8 \\ 6 & -4 \end{pmatrix}$; $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 3 & -2 \end{pmatrix}$

Ex5 – Matrix Products

1. $(a)\ \ (1\ \ 2)\begin{pmatrix}3\\4\end{pmatrix}=(11)$ $\quad (b)\ \ (3\ \ 4)\begin{pmatrix}2\\5\end{pmatrix}=(26)$ $\quad (c)\ \ (5\ \ -2)\begin{pmatrix}1\\-2\end{pmatrix}=(9)$

$(d)\ \ (3\ \ 1\ \ 2)\begin{pmatrix}2\\1\\3\end{pmatrix}=(13)$ $\quad (e)\ \ (2\ \ -3\ \ 4)\begin{pmatrix}5\\1\\2\end{pmatrix}=(15)$

$(f)\ \ (8\ \ -5\ \ -1)\begin{pmatrix}x\\y\\z\end{pmatrix}=(8x-5y-z)$ $\quad (g)\ \ \begin{pmatrix}3&1\\1&2\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}9\\8\end{pmatrix}$

$(h)\ \ \begin{pmatrix}2&1\\0&1\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}10\\4\end{pmatrix}$ $\quad (i)\ \ \begin{pmatrix}2&-1\\1&2\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}1\\8\end{pmatrix}$ $\quad (j)\ \ \begin{pmatrix}1&-1\\1&0\end{pmatrix}\begin{pmatrix}5\\-2\end{pmatrix}=\begin{pmatrix}7\\5\end{pmatrix}$

$(k)\ \ \begin{pmatrix}2&3\\1&-2\end{pmatrix}\begin{pmatrix}1\\-3\end{pmatrix}=\begin{pmatrix}-7\\7\end{pmatrix}$ $\quad (l)\ \ \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}6\\-4\end{pmatrix}=\begin{pmatrix}4\\6\end{pmatrix}$ $\quad (m)\ \ \begin{pmatrix}1&-1\\2&3\end{pmatrix}\begin{pmatrix}5\\4\end{pmatrix}=\begin{pmatrix}1\\22\end{pmatrix}$

$(n)\ \ \begin{pmatrix}1\\-3\end{pmatrix}\begin{pmatrix}2&3\\1&-2\end{pmatrix}=N/A$ $\quad (o)\ \ (3\ \ 4)\begin{pmatrix}2\\5\end{pmatrix}=(26)$ $\quad (p)\ \ \begin{pmatrix}2&1&3\\3&0&1\\1&2&3\end{pmatrix}\begin{pmatrix}4\\0\\1\end{pmatrix}=\begin{pmatrix}11\\13\\7\end{pmatrix}$

$(q)\ \ \begin{pmatrix}2&1\\3&0\\1&2\end{pmatrix}\begin{pmatrix}4\\0\\1\end{pmatrix}=N/A$ $\quad (r)\ \ \begin{pmatrix}1&-2&3\\-1&4&2\\3&1&0\end{pmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}9\\8\\7\end{pmatrix}$

$(s)\ \ (cosa\ \ sina)\begin{pmatrix}cosa&-sina\\sina&cosa\end{pmatrix}=\begin{pmatrix}cos^2a+sin^2a\\-sinacosa+sinacosa\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$

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2. $(a)\ \ \begin{pmatrix}3&0\\0&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}12\\8\end{pmatrix}$; $3x=12$, $-2y=8$; $x=4\ \ y=-4$

$(b)\ \ \begin{pmatrix}x&0\\1&y\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}6\\-1\end{pmatrix}$; $2x=6$, $2+3y=-1$; $x=3\ \ y=-1$

$(c)\ \ \begin{pmatrix}2&-1\\1&-3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}11\\-7\end{pmatrix}$; $2x-y=11$, $x-3y=-7$; $x=8\ \ y=5$

$(d)\ \ \begin{pmatrix}x&y\\y&x\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}5\\-1\end{pmatrix}$; $3x+y=5$, $3y+x=-1$; $x=2\ \ y=-1$

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3. (a) $AB = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 14 & 15 \end{pmatrix}$; $BA = \begin{pmatrix} 4 & 5 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 19 & 13 \\ 2 & 4 \end{pmatrix}$; $AB \neq BA$

(b) $A(BC) = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\left[\begin{pmatrix} 4 & 5 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix}\right] = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 8 \end{pmatrix} = \begin{pmatrix} 17 \\ 11 \end{pmatrix}$

$(AB)C = \left[\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 2 & 0 \end{pmatrix}\right]\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 14 & 15 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 17 \\ 11 \end{pmatrix}$; $A(BC) = (AB)C$

(c) $(AB)^T = \left[\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 2 & 0 \end{pmatrix}\right]^T = \begin{pmatrix} 8 & 5 \\ 14 & 15 \end{pmatrix}^T = \begin{pmatrix} 8 & 14 \\ 5 & 15 \end{pmatrix}$

$(B)^T = \begin{pmatrix} 4 & 2 \\ 5 & 0 \end{pmatrix}$; $(A)^T = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$; $B^T A^T = \begin{pmatrix} 4 & 2 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 14 \\ 5 & 15 \end{pmatrix}$; $(AB)^T = B^T A^T$

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4. $A^2 = AA = \begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 25 & -1 \end{pmatrix}$; $A^3 = AAA = \begin{pmatrix} 4 & -5 \\ 25 & -1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -13 & -14 \\ 70 & -27 \end{pmatrix}$

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5. $PQ = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $QP = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} -4 & 8 \\ -2 & 4 \end{pmatrix}$

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6. (a) $\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (f) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

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7. $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $3a + c = 1$, $3b + d = 0$, $2a + c = 0$, $2b + d = 1$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$

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8. $\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $2p + 4q = 1$, $p + 3q = 0$, $2r + 4s = 0$, $r + 3s = 1$

$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}$

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9. $A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$

$pA + 4I = p\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + q\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & 2p \\ 3p & 4p \end{pmatrix} + \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} p+q & 2p \\ 3p & 4p+q \end{pmatrix}$

$\begin{pmatrix} p+q & 2p \\ 3p & 4p+q \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$ $p + q = 7$; $2p = 10$; $3p = 15$; $4p + q = 22$: $p = 5$ $q = 2$

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10. $A^2 = \begin{pmatrix} 3 & -1 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ -4 & 23 \end{pmatrix}$

$pA + 4I = p\begin{pmatrix} 3 & -1 \\ 2 & -5 \end{pmatrix} + q\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3p & -p \\ 2p & -5p \end{pmatrix} + \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} 3p+q & -p \\ 2p & -5p+q \end{pmatrix}$

$\begin{pmatrix} 3p+q & -p \\ 2p & -5p+q \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ -4 & 23 \end{pmatrix}$; $3p + q = 7$; $-p = 2$; $2p = -4$; $-5p + q = 23$

$p = -2$ $q = 13$

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11. $AB = \begin{pmatrix} 1 & 1 \\ 3 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 5 \\ 12 & 1 & 13 \\ 8 & -1 & 7 \end{pmatrix}$; $BA = \begin{pmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ 2 & 5 \end{pmatrix}$

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12. $AB = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 5 \\ 3 & -1 & 5 \\ 7 & 2 & 8 \end{pmatrix}$

$BA = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 4 & 12 \\ -1 & -2 & -2 \\ 4 & 0 & 6 \end{pmatrix}$

---

13. $M^2 = \begin{pmatrix} cos60 & sin60 \\ -sin60 & cos60 \end{pmatrix}\begin{pmatrix} cos60 & sin60 \\ -sin60 & cos60 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

$M^3 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

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14. $B^2 = 6B - 9I \quad ; \quad B^3 = 6BB - 9IB = 6B^2 - 9IB \qquad (Note : IB = B)$

$= 6(6B - 9I) - 9B = 36B - 54I - 9B = 27B - 54I \qquad ; \quad p = 27 \quad q = -54$

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15. $A^2 = 3A - 4I \quad ; \quad A^3 = 3AA - 4IA = 3A^2 - 4IA \qquad (Note : IA = A)$

$= 3(3A - 4I) - 4A = 9A - 12I - 4A = 5A - 12I \qquad ; \quad p = 5 \quad q = -12$

Ex 6 Determinant

1. (a) $\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} = 3 \times 5 - 2 \times 1 = 13$ (b) $\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} = (-1) \times (-1) - 2 \times 1 = -5$

   (c) $\begin{vmatrix} 2 & -1 \\ -4 & -1 \end{vmatrix} = 2 \times (-1) - (-1) \times (-4) = -6$ (d) $\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} = (2) \times (-2) - (-1) \times 4 = 0$

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2. (i) $|A| = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 1 \times 1 - 3 \times (-2) = 7$; $|B| = \begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix} = 2 \times 5 - 4 \times (-1) = 14$

   $|A||B| = 7 \times 14 = 98$

   $AB = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 19 \\ -5 & -3 \end{pmatrix}$; $|AB| = (-1) \times (-3) - 19 \times (-5) = 98$

   $Hence\ |AB| = |A||B|$

   (ii) $|A| = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 1 \times 1 - 3 \times (-2) = 7$; $|B| = \begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix} = 2 \times 5 - 4 \times (-1) = 14$

   $|B||A| = 14 \times 7 = 98$

   $BA = \begin{pmatrix} 2 & 4 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -6 & 10 \\ -11 & 2 \end{pmatrix}$; $|AB| = (-6) \times (2) - 10 \times (-11) = 98$

   $Hence\ |BA| = |B||A|$

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3. (a) $\begin{vmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{vmatrix} = cos^2\theta + sin^2\theta = 1$

   (b) $\begin{vmatrix} cos2\theta & sin2\theta \\ sin2\theta & cos2\theta \end{vmatrix} = cos^2 2\theta - sin^2 2\theta = cos4\theta$ $(Using\ Higher\ \ cos2\theta = cos^2\theta - sin^2\theta)$

   (c) $\begin{vmatrix} ln2 & ln4 \\ ln5 & ln6 \end{vmatrix} = ln2 \times ln6 - ln4 \times ln5 = ln2 \times ln6 - ln2^2 \times ln5 = ln2 \times ln6 - 2ln2 \times ln5$

   $= ln2(ln6 - 2ln5) = ln2(ln6 - ln25) = ln2 \times ln\left(\frac{6}{25}\right)$

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4. $(a)\ Det = \begin{pmatrix} -2 & 1 & 4 \\ 3 & -2 & 5 \\ 0 & 1 & 3 \end{pmatrix} = -2\begin{vmatrix} -2 & 5 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 5 \\ 0 & 3 \end{vmatrix} + 4\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = -2(-11) - 1(9) + 4(3) = 25$

$(b)\ Det = \begin{pmatrix} 0 & 1 & 3 \\ 0 & -1 & 4 \\ 2 & 6 & -2 \end{pmatrix} = 0\begin{vmatrix} -1 & 4 \\ 6 & -2 \end{vmatrix} - 1\begin{vmatrix} 0 & 4 \\ 2 & -2 \end{vmatrix} + 3\begin{vmatrix} 0 & -1 \\ 2 & 6 \end{vmatrix} = 0(-8) - 1(-8) + 3(2) = 14$

$(c)\ Det = \begin{pmatrix} -2 & 0 & 1 \\ 3 & -4 & 5 \\ -7 & -3 & 2 \end{pmatrix} = -2\begin{vmatrix} -4 & 5 \\ -3 & 2 \end{vmatrix} - 0\begin{vmatrix} 3 & 5 \\ -7 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & -4 \\ -7 & -3 \end{vmatrix} = -2(7) - 0(41) + 1(19) = -51$

$(d)\ \ Det = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 1(-3) - 2(-6) + 3(-3) = 0$

$(e)\ \begin{pmatrix} -7 & 14 & 7 \\ 2 & -8 & 6 \\ 9 & -3 & 12 \end{pmatrix} = -7\begin{vmatrix} -8 & 6 \\ -3 & 12 \end{vmatrix} - 14\begin{vmatrix} 2 & 6 \\ 9 & 12 \end{vmatrix} + 7\begin{vmatrix} 2 & -8 \\ 9 & -3 \end{vmatrix} = -7(-78) - 14(-30) + 7(66) = 1428$

$(e)\ \begin{pmatrix} 1 & 8 & -10 \\ 2 & 4 & 15 \\ 1 & 12 & 5 \end{pmatrix} = 1\begin{vmatrix} 4 & 15 \\ 12 & 5 \end{vmatrix} - 8\begin{vmatrix} 2 & 15 \\ 1 & 5 \end{vmatrix} - 10\begin{vmatrix} 2 & 4 \\ 1 & 12 \end{vmatrix} = 1(-160) - 8(-5) - 10(20) = -320$

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5. $(i)\ |A| = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \end{pmatrix} = 1\begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix} - 2\begin{vmatrix} 0 & 3 \\ 2 & 5 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} = 1(8) - 2(-6) + 3(-2) = 14$

$|B| = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{pmatrix} = 1\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 1(-5) - 0(-10) + 1(0) = -5$

$|A||B| = 14 \times (-5) = -70$

$|AB| = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 17 & 8 & 10 \\ 14 & 7 & 6 \\ 20 & 9 & 4 \end{pmatrix}$

$= 17\begin{vmatrix} 7 & 6 \\ 9 & 4 \end{vmatrix} - 8\begin{vmatrix} 14 & 6 \\ 20 & 4 \end{vmatrix} + 10\begin{vmatrix} 14 & 7 \\ 20 & 9 \end{vmatrix} = 17(-26) - 8(-64) + 10(-14) = -70$

$Hence\ |AB| = \ |A||B|$

5. $(ii)\quad |B||A| = (-5) \times 14 = -70$

$$|BA| = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 8 \\ 8 & 2 & 24 \\ 6 & 9 & 23 \end{pmatrix}$$

$$= 3\begin{vmatrix} 2 & 24 \\ 9 & 23 \end{vmatrix} - 1\begin{vmatrix} 8 & 24 \\ 6 & 23 \end{vmatrix} + 8\begin{vmatrix} 8 & 2 \\ 6 & 9 \end{vmatrix} = 3(-170) - 1(40) + 8(60) = -70$$

$Hence\ |BA| = \ |B||A|$

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6. $Show \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)$

$$LHS = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = 1\begin{vmatrix} y & z \\ zx & xy \end{vmatrix} - 1\begin{vmatrix} x & z \\ yz & xy \end{vmatrix} + 1\begin{vmatrix} x & y \\ yz & zx \end{vmatrix}$$

$$= 1(xy^2 - xz^2) - 1(x^2y - yz^2) + 1(x^2z - y^2z)$$

$$= xy^2 - xz^2 - x^2y + yz^2 + x^2z - y^2z$$

$$RHS = (x-y)(y-z)(z-x) = (x-y)(yz + xz - z^2 - xy)$$

$$RHS = (xyz + zx^2 - xz^2 - x^2y) - (y^2z + yzx - yz^2 - xy^2)$$

$$RHS = (x^2z - xz^2 - x^2y) - (y^2z - yz^2 - xy^2) = xy^2 - xz^2 - x^2y + yz^2 + x^2z - y^2z = LHS$$

Ex 7 – Inverse Matrix

1. $(a)\quad AB = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \qquad ; \quad Hence\ A\ is\ inverse\ of\ B\ and\ vice\ versa$

$(b)\quad AB = \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \qquad ; \quad Hence\ A\ is\ inverse\ of\ B\ and\ vice\ versa$

$(c)\quad AB = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}\begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \qquad ; \quad Hence\ A\ is\ inverse\ of\ B\ and\ vice\ versa$

$(d)\quad AB = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \qquad ; \quad Hence\ A\ is\ inverse\ of\ B\ and\ vice\ versa$

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2. $Pattern \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ; \quad M^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$(a)\quad M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad ; \quad M^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

$(b)\quad M = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \quad ; \quad M^{-1} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$

$(c)\quad M = \begin{pmatrix} 4 & 3 \\ 9 & 7 \end{pmatrix} \quad ; \quad M^{-1} = \begin{pmatrix} 7 & -3 \\ -9 & 4 \end{pmatrix}$

$(d)\quad M = \begin{pmatrix} 9 & -5 \\ -7 & 4 \end{pmatrix} \quad ; \quad M^{-1} = \begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix}$

$(e)\quad M = \begin{pmatrix} 5 & -7 \\ 3 & -4 \end{pmatrix} \quad ; \quad M^{-1} = \begin{pmatrix} -4 & 7 \\ -3 & 5 \end{pmatrix}$

$(f)\quad M = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \quad ; \quad M^{-1} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$

Ex 8 – Find Inverse Matrix if it exists

1. (a) $A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$; $|A| = ad - bc = 6 - 4 = 2 \quad A^{-1} exists$; $A^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}$

(b) $B = \begin{pmatrix} 7 & 4 \\ 16 & 9 \end{pmatrix}$; $|B| = ad - bc = 63 - 64 = -1 \ B^{-1} exists$;

$$B^{-1} = \frac{1}{-1}\begin{pmatrix} 9 & -4 \\ -16 & 7 \end{pmatrix} = \begin{pmatrix} -9 & 4 \\ 16 & -7 \end{pmatrix}$$

(c) $C = \begin{pmatrix} 4 & 2 \\ 10 & 5 \end{pmatrix}$; $|C| = ad - bc = 20 - 20 = 2$; $C^{-1}\ DOES\ NOT\ exists$

(d) $D = \begin{pmatrix} 5 & 7 \\ 6 & 9 \end{pmatrix}$; $|D| = ad - bc = 45 - 42 = 3 \quad D^{-1} exists$

$$D^{-1} = \frac{1}{3}\begin{pmatrix} 9 & -7 \\ -6 & 5 \end{pmatrix} = \begin{pmatrix} 3 & -\frac{7}{3} \\ -2 & \frac{5}{3} \end{pmatrix}$$

(e) $E = \begin{pmatrix} -2 & 4 \\ 1 & -1 \end{pmatrix}$; $|E| = ad - bc = 2 - 4 = -2 \quad E^{-1} exists$

$$E^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -4 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 2 \\ \frac{1}{2} & 1 \end{pmatrix}$$

(f) $F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$; $|F| = ad - bc = 0 - 1 = -1 \quad F^{-1} exists$

$$F^{-1} = \frac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

2. $(a)\ P = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$; $|P| = ad - bc = 2 - 0 = 2 \quad P^{-1} exists$

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 \end{pmatrix}$$

$(b)\ Q = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$; $|Q| = ad - bc = 6 - 5 = 1 \quad Q^{-1} exists$

$$Q^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$(c)\quad PQ = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 19 \\ 1 & 3 \end{pmatrix}$; $|PQ| = ad - bc = 21 - 19 = 2$; $Q^{-1} exists$

$$|PQ|^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -19 \\ -1 & 7 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{19}{2} \\ -\frac{1}{2} & \frac{7}{2} \end{pmatrix}$$

$(d)\quad P^{-1}Q^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & -11 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{11}{2} \\ -1 & 2 \end{pmatrix}$

$(e)\quad Q^{-1}P^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & -19 \\ -1 & 7 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{19}{2} \\ -\frac{1}{2} & \frac{7}{2} \end{pmatrix}$

$(f)\quad QP = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ 2 & 6 \end{pmatrix}$; $|QP| = ad - bc = 24 - 22 = 2$; $(QP)^{-1} exists$

$$|QP|^{-1} = \frac{1}{2}\begin{pmatrix} 6 & -11 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & -\frac{11}{2} \\ -1 & 2 \end{pmatrix}$$

Ex 9 – Inverse 3x3 Matrix

1. $$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 3 & 4 & 5 & 1 & 0 & 0 \\ 4 & 3 & 11 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \rightarrow \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 4 & 3 & 11 & 0 & 1 & 0 \\ 3 & 4 & 5 & 1 & 0 & 0 \end{array}\right)$$

$$\begin{matrix} \\ \\ 3R_1 - R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 4 & 3 & 11 & 0 & 1 & 0 \\ 0 & -4 & 4 & -1 & 0 & 3 \end{array}\right) \rightarrow \begin{matrix} \\ 4R_1 - R_2 \\ \end{matrix}\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & -3 & 1 & 0 & -1 & 4 \\ 0 & -4 & 4 & -1 & 0 & 3 \end{array}\right)$$

$$4R_2 - 3R_3\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & -3 & 1 & 0 & -1 & 4 \\ 0 & 0 & -8 & 3 & -4 & 7 \end{array}\right) \rightarrow 8R_1 + 3R_3\left(\begin{array}{ccc|ccc} 8 & 0 & 0 & 9 & -12 & 29 \\ 0 & -3 & 1 & 0 & -1 & 4 \\ 0 & 0 & -8 & 3 & -4 & 7 \end{array}\right)$$

$$8R_2 + R_3\left(\begin{array}{ccc|ccc} 8 & 0 & 0 & 9 & -12 & 29 \\ 0 & -24 & 0 & 3 & -12 & 39 \\ 0 & 0 & -8 & 3 & -4 & 7 \end{array}\right) \rightarrow \begin{matrix} \div 8 \\ \div -24 \\ -8 \end{matrix}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/9 & -3/2 & 29/8 \\ 0 & 1 & 0 & -1/8 & 1/2 & -13/8 \\ 0 & 0 & -8 & -3/8 & -1/2 & -7/8 \end{array}\right)$$

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2. $$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 4 & 8 & 3 & 1 & 0 & 0 \\ 3 & 5 & 1 & 0 & 1 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \rightarrow \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 0 & 0 & 1 \\ 3 & 5 & 1 & 0 & 1 & 0 \\ 4 & 8 & 3 & 1 & 0 & 0 \end{array}\right)$$

$$\begin{matrix} \\ \\ 4R_1 - R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 0 & 0 & 1 \\ 3 & 5 & 1 & 0 & 1 & 0 \\ 0 & 8 & 9 & -1 & 0 & 4 \end{array}\right) \rightarrow \begin{matrix} \\ 3R_1 - R_2 \\ \end{matrix}\left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 0 & 0 & 1 \\ 0 & 7 & 8 & 0 & -1 & 3 \\ 0 & 8 & 9 & -1 & 0 & 4 \end{array}\right)$$

$$8R_2 - 7R_3\left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 7 & -8 & -4 \\ 0 & 8 & 9 & -1 & 0 & 4 \end{array}\right) \rightarrow \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 0 & 0 & 1 \\ 0 & 8 & 9 & -1 & 0 & 4 \\ 0 & 0 & 1 & 7 & -8 & -4 \end{array}\right)$$

$$R_1 - 3R_3\left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -21 & 24 & 13 \\ 0 & 8 & 9 & -1 & 0 & 4 \\ 0 & 0 & 1 & 7 & -8 & -4 \end{array}\right) \rightarrow R_2 - 9R_3\left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -21 & 24 & 13 \\ 0 & 8 & 0 & -64 & 72 & 40 \\ 0 & 0 & 1 & 7 & -8 & -4 \end{array}\right)$$

$$2R_1 - R_2\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 22 & -24 & -14 \\ 0 & 8 & 0 & -64 & 72 & 40 \\ 0 & 0 & 1 & 7 & -8 & -4 \end{array}\right) \rightarrow \begin{matrix} \div 2 \\ \div 8 \\ \end{matrix}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & -12 & -7 \\ 0 & 1 & 0 & -8 & 9 & 5 \\ 0 & 0 & 1 & 7 & -8 & -4 \end{array}\right)$$

3.

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 0 & 2 & 3 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \rightarrow \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 3 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{matrix} \\ \\ 2R_1 - R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & -1 & 2 \end{array}\right) \rightarrow \begin{matrix} \\ \\ R_2 + R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & -1 & 2 \end{array}\right)$$

$$\begin{matrix} \\ R_2 - R_3 \\ \\ \end{matrix}\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & 1 & -1 & 2 \end{array}\right) \rightarrow \begin{matrix} 2R_1 + R_2 \\ \\ \\ \end{matrix}\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & 1 & -1 & 2 \end{array}\right)$$

$$\begin{matrix} \div 2 \\ \div 2 \\ \div 3 \end{matrix}\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & 1 & -1 & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1 \\ 0 & 0 & 1 & 1/3 & -1/3 & 2/3 \end{array}\right)$$

4.

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 4 & 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 3 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \rightarrow \begin{matrix} \\ \\ R_2 - R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 4 & 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 0 & -4 & 1 & 0 & 1 & -1 \end{array}\right)$$

$$\begin{matrix} \\ \\ 3R_1 - 4R_2 \end{matrix}\left(\begin{array}{ccc|ccc} 4 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -5 & 3 & -4 & 0 \\ 0 & -4 & 1 & 0 & 1 & -1 \end{array}\right) \rightarrow \begin{matrix} \\ \\ 2R_2 + R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 4 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -5 & 3 & -4 & 0 \\ 0 & 0 & -9 & 6 & -7 & -1 \end{array}\right)$$

$$\begin{matrix} 5R_1 + R_2 \\ \\ \\ \end{matrix}\left(\begin{array}{ccc|ccc} 20 & 12 & 0 & 8 & -4 & 0 \\ 0 & 2 & -5 & 3 & -4 & 0 \\ 0 & 0 & -9 & 6 & -7 & -1 \end{array}\right) \rightarrow \begin{matrix} \\ 9R_2 - 5R_3 \\ \\ \end{matrix}\left(\begin{array}{ccc|ccc} 20 & 12 & 0 & 8 & -4 & 0 \\ 0 & 18 & 0 & -3 & -1 & 5 \\ 0 & 0 & -9 & 6 & -7 & -1 \end{array}\right)$$

$$\begin{matrix} \div 2 \\ \\ \\ \end{matrix}\left(\begin{array}{ccc|ccc} 10 & 6 & 0 & 4 & -2 & 0 \\ 0 & 18 & 0 & -3 & -1 & 5 \\ 0 & 0 & -9 & 6 & -7 & -1 \end{array}\right) \rightarrow \begin{matrix} 3R_1 - R_2 \\ \\ \\ \end{matrix}\left(\begin{array}{ccc|ccc} 30 & 0 & 0 & 15 & -5 & -5 \\ 0 & 18 & 0 & -3 & -1 & 5 \\ 0 & 0 & -9 & 6 & -7 & -1 \end{array}\right)$$

$$\begin{matrix} \div 30 \\ \div 18 \\ \div -9 \end{matrix}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/6 & -1/6 \\ 0 & 1 & 0 & -1/6 & -1/18 & 5/18 \\ 0 & 0 & 1 & -2/3 & 7/9 & 1/9 \end{array}\right)$$

5.
$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\2&3&1&0&1&0\\3&2&1&0&0&1\end{array}\right)\rightarrow\begin{matrix}\\ \\3R_1-R_3\end{matrix}\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\2&3&1&0&1&0\\0&4&8&3&0&-1\end{array}\right)$$

$$\begin{matrix}\\2R_1-R_2\\ \\\end{matrix}\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&5&2&-1&0\\0&4&8&3&0&-1\end{array}\right)\rightarrow\begin{matrix}\\ \\4R_2-R_3\end{matrix}\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&5&2&-1&0\\0&0&12&5&-4&1\end{array}\right)$$

$$\begin{matrix}4R_1-R_3\\ \\ \\\end{matrix}\left(\begin{array}{ccc|ccc}4&8&0&-1&4&-1\\0&1&5&2&-1&0\\0&0&12&5&-4&1\end{array}\right)\rightarrow\begin{matrix}\\12R_2-5R_3\\ \\\end{matrix}\left(\begin{array}{ccc|ccc}4&8&0&-1&4&-1\\0&12&0&-1&8&-5\\0&0&12&5&-4&1\end{array}\right)$$

$$\begin{matrix}12R_1-8R_2\\ \\ \\\end{matrix}\left(\begin{array}{ccc|ccc}48&0&0&-4&-16&28\\0&12&0&-1&8&-5\\0&0&12&5&-4&1\end{array}\right)\rightarrow\begin{matrix}\div 4\\ \\ \\\end{matrix}\left(\begin{array}{ccc|ccc}12&0&0&-1&-4&7\\0&12&0&-1&8&-5\\0&0&12&5&-4&1\end{array}\right)$$

$$\begin{matrix}\div 12\\ \div 12\\ \div 12\end{matrix}\left(\begin{array}{ccc|ccc}1&0&0&-1/12&-1/3&7/12\\0&1&0&-1/12&2/3&-5/12\\0&0&1&5/12&-1/3&1/12\end{array}\right)$$

6.
$$\begin{matrix}R_1\\R_2\\R_3\end{matrix}\left(\begin{array}{ccc|ccc}1&8&5&1&0&0\\2&10&7&0&1&0\\9&7&3&0&0&1\end{array}\right)\rightarrow\begin{matrix}\\ \\9R_1-R_3\end{matrix}\left(\begin{array}{ccc|ccc}1&8&5&1&0&0\\2&10&7&0&1&0\\0&65&42&9&0&-1\end{array}\right)$$

$$\begin{matrix}\\2R_1-R_2\\ \\\end{matrix}\left(\begin{array}{ccc|ccc}1&8&5&1&0&0\\0&6&3&2&-1&0\\0&65&42&9&0&-1\end{array}\right)\rightarrow\begin{matrix}\\ \\65R_2-6R_3\end{matrix}\left(\begin{array}{ccc|ccc}1&8&5&1&0&0\\0&6&3&2&-1&0\\0&0&-57&76&-65&6\end{array}\right)$$

$$\begin{matrix}3R_1-5R_2\\ \\ \\\end{matrix}\left(\begin{array}{ccc|ccc}3&-6&0&-7&5&0\\0&6&3&2&-1&0\\0&0&-57&76&-65&6\end{array}\right)\rightarrow\begin{matrix}\\19R_2+R_3\\ \\\end{matrix}\left(\begin{array}{ccc|ccc}3&-6&0&-7&5&0\\0&114&0&114&-84&6\\0&0&-57&76&-65&6\end{array}\right)$$

$$\begin{matrix}19R_1+R_2\\ \\ \\\end{matrix}\left(\begin{array}{ccc|ccc}57&0&0&-19&11&6\\0&114&0&114&-84&6\\0&0&-57&76&-65&6\end{array}\right)\rightarrow\begin{matrix}\div 57\\ \div 114\\ \div -57\end{matrix}\left(\begin{array}{ccc|ccc}1&0&0&-1/3&11/57&2/19\\0&1&0&1&-14/19&1/19\\0&0&1&-4/3&65/57&-2/19\end{array}\right)$$

Ex 10 – Solving Sim. Equations using 2x2 Matrix

1. $\begin{matrix} x - y = 5 \\ x + y = 11 \end{matrix}$; $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$

$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 \times 1 - (-1) \times 1 = 2$; $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 11 \end{pmatrix}$; $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 11 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$; $x = 8 \quad y = 3$

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2. $\begin{matrix} 3x + y = 7 \\ 3x + 2y = 5 \end{matrix}$; $\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

$\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = 3 \times 2 - 1 \times 3 = 3$; $A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}$

$\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 5 \end{pmatrix}$; $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$; $x = 3 \quad y = -2$

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3. $\begin{matrix} 2x + y = 5 \\ 2x + 3y = -1 \end{matrix}$; $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

$\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 1 \times 2 = 4$; $A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$

$\frac{1}{4}\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \end{pmatrix}$; $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$; $x = 4 \quad y = -3$

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4. $\begin{array}{c} 3x - 4y = 18 \\ 5x + y = 7 \end{array}$; $\begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 7 \end{pmatrix}$

$\begin{vmatrix} 3 & -4 \\ 5 & 1 \end{vmatrix} = 3 \times 1 - (-4) \times 5 = 23$; $A^{-1} = \frac{1}{23}\begin{pmatrix} 1 & 4 \\ -5 & 3 \end{pmatrix}$

$\frac{1}{23}\begin{pmatrix} 1 & 4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{23}\begin{pmatrix} 1 & 4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 18 \\ 7 \end{pmatrix}$; $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{23}\begin{pmatrix} 1 & 4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 18 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$; $x = 2 \quad y = -3$

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5. $\begin{array}{c} 2x + 3y = 5 \\ 4x - 5y = 21 \end{array}$; $\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 21 \end{pmatrix}$

$\begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = 2 \times (-5) - 3 \times 4 = -22$; $A^{-1} = -\frac{1}{22}\begin{pmatrix} -5 & -3 \\ -4 & 2 \end{pmatrix}$

$-\frac{1}{22}\begin{pmatrix} -5 & -3 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{22}\begin{pmatrix} -5 & -3 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 21 \end{pmatrix}$;

$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{22}\begin{pmatrix} -5 & -3 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 21 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$; $x = 4 \quad y = -1$

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6. $\begin{array}{c} 5x - 3y = 9 \\ 7x - 6y = 9 \end{array}$; $\begin{pmatrix} 5 & -3 \\ 7 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}$

$\begin{vmatrix} 5 & -3 \\ 7 & -6 \end{vmatrix} = 5 \times (-6) - (-3) \times 7 = -9$; $A^{-1} = -\frac{1}{9}\begin{pmatrix} -6 & 3 \\ -7 & 5 \end{pmatrix}$

$-\frac{1}{9}\begin{pmatrix} -6 & 3 \\ -7 & 5 \end{pmatrix}\begin{pmatrix} 5 & -3 \\ 7 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} -6 & 3 \\ -7 & 5 \end{pmatrix}\begin{pmatrix} 9 \\ 9 \end{pmatrix}$;

$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} -6 & 3 \\ -7 & 5 \end{pmatrix}\begin{pmatrix} 9 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$; $x = 3 \quad y = 2$

Ex 11 – Solving Sim. Equations using 3x3 Matrix

1. $\begin{matrix} x-2y+z=6 \\ 3x+y-2z=4 \\ 7x-6y-z=10 \end{matrix}$; $\begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 7 & -6 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 10 \end{pmatrix}$

$$\begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 7 & -6 & -1 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ -6 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & -2 \\ 7 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 7 & -6 \end{vmatrix} = 1\times(-13) + 2\times 11 + 1\times(-25) = -16$$

Inverse exists !!!! we will assume this from now on.

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{matrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 7 & -6 & -1 \end{matrix}\,\vdots\,\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right) \rightarrow \begin{matrix} \\ \\ 7R_1 - R_3 \end{matrix}\left(\begin{matrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & -8 & 8 \end{matrix}\,\vdots\,\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & -1 \end{matrix}\right)$$

$$\begin{matrix} \\ 3R_1 - R_2 \\ \\ \end{matrix}\left(\begin{matrix} 1 & -2 & 1 \\ 0 & -7 & 5 \\ 0 & -8 & 8 \end{matrix}\,\vdots\,\begin{matrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 7 & 0 & -1 \end{matrix}\right) \rightarrow \begin{matrix} \\ \\ 8R_2 - 7R_3 \end{matrix}\left(\begin{matrix} 1 & -2 & 1 \\ 0 & -7 & 5 \\ 0 & 0 & -16 \end{matrix}\,\vdots\,\begin{matrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ -25 & -8 & 7 \end{matrix}\right)$$

$$\begin{matrix} 5R_1 - R_2 \\ \\ \\ \end{matrix}\left(\begin{matrix} 5 & -3 & 0 \\ 0 & -7 & 5 \\ 0 & 0 & -16 \end{matrix}\,\vdots\,\begin{matrix} 2 & 1 & 0 \\ 3 & -1 & 0 \\ -25 & -8 & 7 \end{matrix}\right) \rightarrow 16R_2 + 5R_3 \left(\begin{matrix} 5 & -3 & 0 \\ 0 & -112 & 0 \\ 0 & 0 & -16 \end{matrix}\,\vdots\,\begin{matrix} 2 & 1 & 0 \\ -77 & -56 & 35 \\ -25 & -8 & 7 \end{matrix}\right)$$

$$\begin{matrix} 112R_1 - 3R_2 \\ \\ \\ \end{matrix}\left(\begin{matrix} 560 & 0 & 0 \\ 0 & -112 & 0 \\ 0 & 0 & -16 \end{matrix}\,\vdots\,\begin{matrix} 455 & 280 & -105 \\ -77 & -56 & 35 \\ -25 & -8 & 7 \end{matrix}\right) \rightarrow \left(\begin{matrix} 560 & 0 & 0 \\ 0 & -112 & 0 \\ 0 & 0 & -16 \end{matrix}\,\vdots\,\begin{matrix} 455 & 280 & -105 \\ -77 & -56 & 35 \\ -25 & -8 & 7 \end{matrix}\right)$$

$$\begin{matrix} \div 560 \\ \div -112 \\ \div -16 \end{matrix}\left(\begin{matrix} 560 & 0 & 0 \\ 0 & -112 & 0 \\ 0 & 0 & -16 \end{matrix}\,\vdots\,\begin{matrix} 455 & 280 & -105 \\ -77 & -56 & 35 \\ -25 & -8 & 7 \end{matrix}\right) = \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\,\vdots\,\begin{matrix} 13/16 & 1/2 & -3/16 \\ 11/16 & 1/2 & -5/16 \\ 25/16 & 1/2 & -7/16 \end{matrix}\right)$$

$$A^{-1} = \begin{pmatrix} 13/16 & 1/2 & -3/16 \\ 11/16 & 1/2 & -5/16 \\ 25/16 & 1/2 & -7/16 \end{pmatrix}$$

$$\begin{pmatrix} 13/16 & 1/2 & -3/16 \\ 11/16 & 1/2 & -5/16 \\ 25/16 & 1/2 & -7/16 \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 7 & -6 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13/16 & 1/2 & -3/16 \\ 11/16 & 1/2 & -5/16 \\ 25/16 & 1/2 & -7/16 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 13 & 8 & -3 \\ 11 & 8 & -5 \\ 25 & 8 & -7 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \\ 10 \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 78+32-30 \\ 66+32-50 \\ 150+32-70 \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 80 \\ 48 \\ 112 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix} \;;\quad x = 1 \;\; y = 3 \;\; z = 7$$

2. $\begin{matrix} 5x - y + 2z = 25 \\ 3x + 2y - 3z = 16 \\ 2x - y + z = 9 \end{matrix}$; $\begin{pmatrix} 5 & -1 & 2 \\ 3 & 2 & -3 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 25 \\ 16 \\ 9 \end{pmatrix}$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 5 & -1 & 2 & \vdots 1 & 0 & 0 \\ 3 & 2 & -3 & \vdots 0 & 1 & 0 \\ 2 & -1 & 1 & \vdots 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{matrix} \\ \\ 2R_1 - 5R_3 \end{matrix}\begin{pmatrix} 5 & -1 & 2 & \vdots 1 & 0 & 0 \\ 3 & 2 & -3 & \vdots 0 & 1 & 0 \\ 0 & 3 & -1 & \vdots 2 & 0 & -5 \end{pmatrix}$$

$$3R_1 - 5R_2 \begin{pmatrix} 5 & -1 & 2 & \vdots 1 & 0 & 0 \\ 0 & -13 & 21 & \vdots 3 & -5 & 0 \\ 0 & 3 & -1 & \vdots 2 & 0 & -5 \end{pmatrix} \rightarrow \begin{matrix} \\ \\ 3R_2 + 13R_3 \end{matrix}\begin{pmatrix} 5 & -1 & 2 & \vdots 1 & 0 & 0 \\ 0 & -13 & 21 & \vdots 3 & -5 & 0 \\ 0 & 0 & 50 & \vdots 35 & -15 & -65 \end{pmatrix}$$

$$25R_1 - R_3 \begin{pmatrix} 125 & -25 & 0 & \vdots -10 & 15 & 65 \\ 0 & -13 & 21 & \vdots 3 & -5 & 0 \\ 0 & 0 & 50 & \vdots 35 & -15 & -65 \end{pmatrix}$$

$$\begin{matrix} \\ 50R_2 - 21R_3 \\ \end{matrix}\begin{pmatrix} 125 & -25 & 0 & \vdots -10 & 15 & 65 \\ 0 & -650 & 0 & \vdots -585 & 65 & 1365 \\ 0 & 0 & 50 & \vdots 35 & -15 & -65 \end{pmatrix}$$

$$26R_1 - R_2 \begin{pmatrix} 3250 & 0 & 0 & \vdots 325 & 325 & 325 \\ 0 & -650 & 0 & \vdots -585 & 65 & 1365 \\ 0 & 0 & 50 & \vdots 35 & -15 & -65 \end{pmatrix}$$

$$\begin{matrix} \div 3250 \\ \div -650 \\ \div 50 \end{matrix}\begin{pmatrix} 3250 & 0 & 0 & \vdots 325 & 325 & 325 \\ 0 & -650 & 0 & \vdots -585 & 65 & 1365 \\ 0 & 0 & 50 & \vdots 35 & -15 & -65 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \vdots 1/10 & 1/10 & 1/10 \\ 0 & 1 & 0 & \vdots 9/10 & -1/10 & -21/10 \\ 0 & 0 & 1 & \vdots 7/10 & -3/10 & -13/10 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1/10 & 1/10 & 1/10 \\ 9/10 & -1/10 & -21/10 \\ 7/10 & -3/10 & -13/10 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 1 & 1 & 1 \\ 9 & -1 & -21 \\ 7 & -3 & -13 \end{pmatrix}$$

$$\frac{1}{10}\begin{pmatrix} 1 & 1 & 1 \\ 9 & -1 & -21 \\ 7 & -3 & -13 \end{pmatrix}\begin{pmatrix} 5 & -1 & 2 \\ 3 & 2 & -3 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 1 & 1 & 1 \\ 9 & -1 & -21 \\ 7 & -3 & -13 \end{pmatrix}\begin{pmatrix} 25 \\ 16 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 1 & 1 & 1 \\ 9 & -1 & -21 \\ 7 & -3 & -13 \end{pmatrix}\begin{pmatrix} 25 \\ 16 \\ 9 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 25 + 16 + 9 \\ 225 - 16 - 189 \\ 175 - 48 - 117 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 50 \\ 20 \\ 10 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

$x = 5 \quad y = 2 \quad z = 1$

3. $\begin{matrix} x+y+z=2 \\ 3x-y+2z=4 \\ 2x+3y+z=7 \end{matrix}$; $\begin{pmatrix} 1 & 1 & 1 \\ 3 & -1 & 2 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 1 & 1 & 1 \vdots 1 & 0 & 0 \\ 3 & -1 & 2 \vdots 0 & 1 & 0 \\ 2 & 3 & 1 \vdots 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{matrix} \\ \\ 2R_1 - R_3 \end{matrix}\begin{pmatrix} 1 & 1 & 1 \vdots 1 & 0 & 0 \\ 3 & -1 & 2 \vdots 0 & 1 & 0 \\ 0 & -1 & 1 \vdots 2 & 0 & -1 \end{pmatrix}$$

$$3R_1 - R_2 \begin{pmatrix} 1 & 1 & 1 \vdots 1 & 0 & 0 \\ 0 & 4 & 1 \vdots 3 & -1 & 0 \\ 0 & -1 & 1 \vdots 2 & 0 & -1 \end{pmatrix} \rightarrow \begin{matrix} \\ \\ R_2 + 4R_3 \end{matrix}\begin{pmatrix} 1 & 1 & 1 \vdots 1 & 0 & 0 \\ 0 & 4 & 1 \vdots 3 & -1 & 0 \\ 0 & 0 & 5 \vdots 11 & -1 & -4 \end{pmatrix}$$

$$\begin{matrix} R_1 - R_2 \\ \\ \\ \end{matrix}\begin{pmatrix} 1 & -3 & 0 \vdots -2 & 1 & 0 \\ 0 & 4 & 1 \vdots 3 & -1 & 0 \\ 0 & 0 & 5 \vdots 11 & -1 & -4 \end{pmatrix} \rightarrow 5R_2 - R_3 \begin{pmatrix} 1 & -3 & 0 \vdots -2 & 1 & 0 \\ 0 & 20 & 0 \vdots 4 & -4 & 4 \\ 0 & 0 & 5 \vdots 11 & -1 & -4 \end{pmatrix}$$

$$\begin{matrix} 20R_1 + 3R_2 \\ \\ \\ \end{matrix}\begin{pmatrix} 20 & 0 & 0 \vdots -28 & 8 & 12 \\ 0 & 20 & 0 \vdots 4 & -4 & 4 \\ 0 & 0 & 5 \vdots 11 & -1 & -4 \end{pmatrix} \rightarrow \begin{matrix} \div 4 \\ \div 4 \\ \\ \end{matrix}\begin{pmatrix} 5 & 0 & 0 \vdots -7 & 2 & 3 \\ 0 & 5 & 0 \vdots 1 & -1 & 1 \\ 0 & 0 & 5 \vdots 11 & -1 & -4 \end{pmatrix}$$

$$\begin{matrix} \div 5 \\ \div 5 \\ \div 5 \end{matrix}\begin{pmatrix} 5 & 0 & 0 \vdots -7 & 2 & 3 \\ 0 & 5 & 0 \vdots 1 & -1 & 1 \\ 0 & 0 & 5 \vdots 11 & -1 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \vdots -7/5 & 2/5 & 3/5 \\ 0 & 1 & 0 \vdots 1/5 & -1/5 & 1/5 \\ 0 & 0 & 1 \vdots 11/5 & -1/5 & -4/5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -7/5 & 2/5 & 3/5 \\ 1/5 & -1/5 & 1/5 \\ 11/5 & -1/5 & -4/5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -7 & 2 & 3 \\ 1 & -1 & 1 \\ 11 & -1 & -4 \end{pmatrix}$$

$$\frac{1}{5}\begin{pmatrix} -7 & 2 & 3 \\ 1 & -1 & 1 \\ 11 & -1 & -4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 3 & -1 & 2 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -7 & 2 & 3 \\ 1 & -1 & 1 \\ 11 & -1 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -7 & 2 & 3 \\ 1 & -1 & 1 \\ 11 & -1 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -14+8+21 \\ 2-4+7 \\ 22-4-28 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 15 \\ 5 \\ -10 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

$x = 3 \quad y = 1 \quad z = -2$

4. $\begin{aligned} 2x+4y+5z&=-3 \\ 4x-y-7z&=6 \\ 6x+3y-z&=3 \end{aligned}$; $\begin{pmatrix} 2 & 4 & 5 \\ 4 & -1 & -7 \\ 6 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 2 & 4 & 5 & 1 & 0 & 0 \\ 4 & -1 & -7 & 0 & 1 & 0 \\ 6 & 3 & -1 & 0 & 0 & 1 \end{array}\right) \rightarrow \begin{matrix} \\ \\ 3R_1 - R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 2 & 4 & 5 & 1 & 0 & 0 \\ 4 & -1 & -7 & 0 & 1 & 0 \\ 0 & 9 & 16 & 3 & 0 & -1 \end{array}\right)$$

$$2R_1 - R_2\left(\begin{array}{ccc|ccc} 2 & 4 & 5 & 1 & 0 & 0 \\ 0 & 9 & 17 & 2 & -1 & 0 \\ 0 & 9 & 16 & 3 & 0 & -1 \end{array}\right) \rightarrow \begin{matrix} \\ \\ R_2 - R_3 \end{matrix}\left(\begin{array}{ccc|ccc} 2 & 4 & 5 & 1 & 0 & 0 \\ 0 & 9 & 17 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\begin{matrix} R_1 - 5R_3 \\ \\ \\ \end{matrix}\left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 6 & 5 & -5 \\ 0 & 9 & 17 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \rightarrow R_2 - 17R_3\left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 6 & 5 & -5 \\ 0 & 9 & 0 & 19 & 16 & -17 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$9R_1 - 4R_2\left(\begin{array}{ccc|ccc} 18 & 0 & 0 & -22 & -19 & 23 \\ 0 & 9 & 0 & 19 & 16 & -17 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \rightarrow \begin{matrix} \div 18 \\ \div 9 \\ \\ \end{matrix}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -11/9 & -19/18 & 23/18 \\ 0 & 1 & 0 & 19/9 & 16/9 & -17/9 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -11/9 & -19/18 & 23/18 \\ 19/9 & 16/9 & -17/9 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -11/9 & -19/18 & 23/18 \\ 19/9 & 16/9 & -17/9 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 5 \\ 4 & -1 & -7 \\ 6 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11/9 & -19/18 & 23/18 \\ 19/9 & 16/9 & -17/9 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11/9 & -19/18 & 23/18 \\ 19/9 & 16/9 & -17/9 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{33}{9} + \frac{-114}{18} + \frac{69}{18} \\ \frac{-57}{9} + \frac{96}{9} + \frac{-51}{9} \\ 3 - 6 + 3 \end{pmatrix} = \begin{pmatrix} \frac{7}{6} \\ \frac{-4}{3} \\ 0 \end{pmatrix}$$

$$x = \frac{7}{6} \quad y = -\frac{4}{3} \quad z = 0$$

Ex 12 – 2x2 Matrices to Represent Geometrical Transformation

1. (a) $Reflection\ in\ y-axis \quad (x,y) \to (-x,y)$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -x \\ y \end{pmatrix} \quad ; \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} \quad ; \quad matrix = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) $Reflection\ in\ line\ y = x \quad (x,y) \to (y,x)$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y \\ x \end{pmatrix} \quad ; \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \quad ; \quad matrix = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(c) $Reflection\ in\ line\ y = -x \quad (x,y) \to (y,-x)$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y \\ -x \end{pmatrix} \quad ; \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix} \quad ; \quad matrix = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(d) $Rotation\ of\ \frac{\pi}{2}\ radians\ clockwise \quad (x,y) \to (y,-x)$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y \\ -x \end{pmatrix} \quad ; \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} \quad ; \quad matrix = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(e) $Rotation\ of\ \frac{\pi}{2}\ radians\ anti-clockwise \quad (x,y) \to (y,x)$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x \end{pmatrix} \quad ; \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \quad ; \quad matrix = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(f) $A\ dilatation, about\ O, where\ the\ scale\ factor\ is\ k.$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} kx \\ ky \end{pmatrix} \quad ; \quad \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix} \quad ; \quad matrix = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

2. $x = rcos\alpha \qquad y = rsin\alpha \qquad ; \qquad x' = rcos(\alpha + \theta) \qquad y' = rsin(\alpha + \theta)$

$x' = rcos(\alpha + \theta) \;\; ; \;\; rcos\alpha cos\theta - rsin\alpha sin\theta = xcos\theta - ysin\theta$
$y' = rsin(\alpha + \theta) \;\; ; \;\; rsin\alpha cos\theta + rcos\alpha sin\theta = ycos\theta + xsin\theta$

$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad ; \qquad \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad ; \qquad matrix = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$

$(i) \;\; Rotation\ of\ \pi \; : \qquad \begin{pmatrix} cos\pi & -sin\pi \\ sin\pi & cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$(ii) \;\; Rotation\ of\ \frac{\pi}{2}\ anti-clockwise \; : \qquad \begin{pmatrix} cos\frac{\pi}{2} & -sin\frac{\pi}{2} \\ sin\frac{\pi}{2} & cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$(ii) \;\; Rotation\ of\ \frac{\pi}{2}\ clockwise \; : \qquad \begin{pmatrix} cos\left(-\frac{\pi}{2}\right) & -sin\left(-\frac{\pi}{2}\right) \\ sin\left(-\frac{\pi}{2}\right) & cos\left(-\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$