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Advanced Higher Maths

Advanced Higher - Unit 3.2 Vector Theory - Solutions

Mr. Lafferty BSC (Hons) Open MathSci & Mrs Bissett BSC (Hons) Glasgow

Ex 1 Revision Higher of 3D Vectors

1. (a) $\underline{b}.\underline{c}+\underline{c}.\underline{a}+\underline{a}.\underline{b}$

$\underline{b}.\underline{c}=1\times0+1\times2+2\times1=4$; $\underline{c}.\underline{a}=0\times1+2\times0+1\times1=1$; $\underline{a}.\underline{b}=1\times1+0\times1+1\times2=3$

$\underline{b}.\underline{c}+\underline{c}.\underline{a}+\underline{a}.\underline{b}=4+1+3=8$; $Scalar$

(b) $(\underline{b}.\underline{c})\underline{a}+(\underline{c}.\underline{a})\underline{b}+(\underline{a}.\underline{b})\underline{c}$

$(\underline{b}.\underline{c})\underline{a}=4(\underline{i}+\underline{k})=4\underline{i}+4\underline{k}$;

$(\underline{c}.\underline{a})\underline{b}=1\left(\underline{i}+\underline{j}+2\underline{k}\right)=\underline{i}+\underline{j}+2\underline{k}$;

$(\underline{a}.\underline{b})\underline{c}=3\left(2\underline{j}+\underline{k}\right)=6\underline{j}+3\underline{k}$

$(\underline{b}.\underline{c})\underline{a}+(\underline{c}.\underline{a})\underline{b}+(\underline{a}.\underline{b})\underline{c}=(4\underline{i}+4\underline{k})+\left(\underline{i}+\underline{j}+2\underline{k}\right)+\left(6\underline{j}+3\underline{k}\right)=5\underline{i}+7\underline{j}+9\underline{k}$; $Vector$

(c) $[(\underline{b}.\underline{c})(\underline{c}.\underline{a})]\underline{a}$; $(\underline{b}.\underline{c})(\underline{c}.\underline{a})=4\times1=4$; $[(\underline{b}.\underline{c})(\underline{c}.\underline{a})]\underline{a}=4(\underline{i}+\underline{k})=4\underline{i}+4\underline{k}$; $Vector$

(d) $[(\underline{b}.\underline{c})\underline{c}+(\underline{b}.\underline{a})\underline{a}].(\underline{b}+2\underline{a})$

$(\underline{b}.\underline{c})\underline{c}=4\left(2\underline{j}+\underline{k}\right)=8\underline{j}+4\underline{k}$; $(\underline{b}.\underline{a})=1\times1+1\times0+2\times1=3$ $(\underline{b}.\underline{a})\underline{a}=3\underline{i}+3\underline{k}$

$\left[[(\underline{b}.\underline{c})\underline{c}+(\underline{b}.\underline{a})]\underline{a}\right]=\left(8\underline{j}+4\underline{k}\right)+(3\underline{i}+3\underline{k})=3\underline{i}+8\underline{j}+7\underline{k}$

$(\underline{b}+2\underline{a})=\left(\underline{i}+\underline{j}+2\underline{k}\right)+(2\underline{i}+2\underline{k})=3\underline{i}+\underline{j}+4\underline{k}$

$[(\underline{b}.\underline{c})\underline{c}+(\underline{b}.\underline{a})\underline{a}].(\underline{b}+2\underline{a})=\left(3\underline{i}+8\underline{j}+7\underline{k}\right).\left(3\underline{i}+\underline{j}+4\underline{k}\right)$

$=\left(3\underline{i}+8\underline{j}+7\underline{k}\right).\left(3\underline{i}+\underline{j}+4\underline{k}\right)=3\times3+8\times1+7\times4=45$; $Scalar$

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2. $A(1,6,-2)$, $B(2,5,4)$, $C(4,6,-3)$

$$Length\ AB = \sqrt{(2-1)^2 + (5-6)^2 + (4-(-2))^2} = \sqrt{1+1+36} = \sqrt{38}$$

$$Length\ BC = \sqrt{(4-2)^2 + (6-5)^2 + (-3-4)^2} = \sqrt{4+1+49} = \sqrt{54} = 3\sqrt{6}$$

$$Length\ AC = \sqrt{(4-1)^2 + (6-6)^2 + (-3-(-2))^2} = \sqrt{9+0+1} = \sqrt{10}$$

$$\overrightarrow{AB} = \begin{pmatrix}2\\5\\4\end{pmatrix} - \begin{pmatrix}1\\6\\-2\end{pmatrix} = \begin{pmatrix}1\\-1\\6\end{pmatrix} \quad ; \quad \overrightarrow{AC} = \begin{pmatrix}4\\6\\-3\end{pmatrix} - \begin{pmatrix}1\\6\\-2\end{pmatrix} = \begin{pmatrix}3\\0\\-1\end{pmatrix}$$

$$\overrightarrow{AB}.\overrightarrow{AC} = 1 \times 3 + (-1) \times 0 + 6 \times (-1) = 3 + 0 + (-6) = -3$$

$$cosA = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{-3}{\sqrt{38} \times \sqrt{10}} = -\frac{3}{2\sqrt{95}} \quad ; \quad \angle A = 98.85°$$

$$\overrightarrow{BA} = \begin{pmatrix}1\\6\\-2\end{pmatrix} - \begin{pmatrix}2\\5\\4\end{pmatrix} = \begin{pmatrix}-1\\1\\-6\end{pmatrix} \quad ; \quad \overrightarrow{BC} = \begin{pmatrix}4\\6\\-3\end{pmatrix} - \begin{pmatrix}2\\5\\4\end{pmatrix} = \begin{pmatrix}2\\1\\-7\end{pmatrix}$$

$$\overrightarrow{BA}.\overrightarrow{BC} = (-1) \times 2 + 1 \times 1 + (-6) \times (-7) = -2 + 1 + 42 = 41$$

$$cosB = \frac{\overrightarrow{BA}.\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{41}{\sqrt{38} \times 3\sqrt{6}} = \frac{41}{6\sqrt{57}} \quad ; \quad \angle B = 25.16°$$

$$\overrightarrow{CB} = \begin{pmatrix}2\\5\\4\end{pmatrix} - \begin{pmatrix}4\\6\\-3\end{pmatrix} = \begin{pmatrix}-2\\-1\\7\end{pmatrix} \quad ; \quad \overrightarrow{CA} = \begin{pmatrix}1\\6\\-2\end{pmatrix} - \begin{pmatrix}4\\6\\-3\end{pmatrix} = \begin{pmatrix}-3\\0\\1\end{pmatrix}$$

$$\overrightarrow{CB}.\overrightarrow{CA} = (-2) \times (-3) + (-1) \times 0 + 7 \times 1 = 6 + 0 + 7 = 13$$

$$cosC = \frac{\overrightarrow{CB}.\overrightarrow{CA}}{|\overrightarrow{CB}||\overrightarrow{CA}|} = \frac{13}{3\sqrt{6} \times \sqrt{10}} = \frac{13}{6\sqrt{15}} \quad ; \quad \angle C = 55.98°$$

$$AREA = \frac{1}{2}|CA||CB|sinC = \frac{1}{2} \times \sqrt{10} \times 3\sqrt{6} \times \frac{\sqrt{\left(6\sqrt{15}\right)^2 - 13^2}}{6\sqrt{15}} = \frac{1}{2}\sqrt{371} = 9.63$$

3. $Prove \quad |\overrightarrow{BD}|^2 + |\overrightarrow{AC}|^2 = 2|\overrightarrow{AB}|^2 + 2|\overrightarrow{AD}|^2$

$|\overrightarrow{AB}| = |\overrightarrow{CD}| = \underline{d} \quad ; \quad |\overrightarrow{AD}| = |\overrightarrow{BC}| = \underline{b} \quad ; \quad |\overrightarrow{BD}| = \underline{b} - \underline{d} \quad ; \quad \overrightarrow{AC} = \underline{b} + \underline{d}$

$LHS = |\overrightarrow{BD}|^2 + |\overrightarrow{AC}|^2 = |\underline{b} - \underline{d}|^2 + |\underline{b} + \underline{d}|^2 \qquad (Note: |\underline{x}|^2 = \underline{x}.\underline{x}$

$LHS = |\overrightarrow{BD}|^2 + |\overrightarrow{AC}|^2 = (\underline{b} - \underline{d}).(\underline{b} - \underline{d}) + (\underline{b} + \underline{d}).(\underline{b} + \underline{d})$

$= (\underline{b}.\underline{b} - 2\underline{b}\,.\,\underline{d} + \underline{d}.\underline{d}) + (\underline{b}.\underline{b} + 2\underline{b}\,.\,\underline{d} + \underline{d}.\underline{d})$

$= (\underline{b}.\underline{b} + \underline{d}.\underline{d}) + (\underline{b}.\underline{b} + \underline{d}.\underline{d})$

$= 2|\underline{b}|^2 + 2|\underline{d}|^2$

$RHS = 2|\overrightarrow{AB}|^2 + 2|\overrightarrow{AD}|^2 = 2|\underline{d}|^2 + 2|\underline{b}|^2 = 2|\underline{b}|^2 + 2|\underline{d}|^2 = LHS$

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4. $Let\ unit\ vector\ be\ \ \underline{u} = \left(x\underline{i} + y\underline{j} + z\underline{k}\right) \quad ; \quad \underline{a} = \underline{i}\ \ (45°) \quad and\ \ \underline{b} = \underline{k}\ \ (45°)$

$Using \quad \underline{a}.\underline{u} = |\underline{a}| \times |\underline{u}| \times cos\theta°$

$x = 1 \times 1 \times cos45° \qquad ; \qquad x = \frac{1}{\sqrt{2}} \quad (1)$

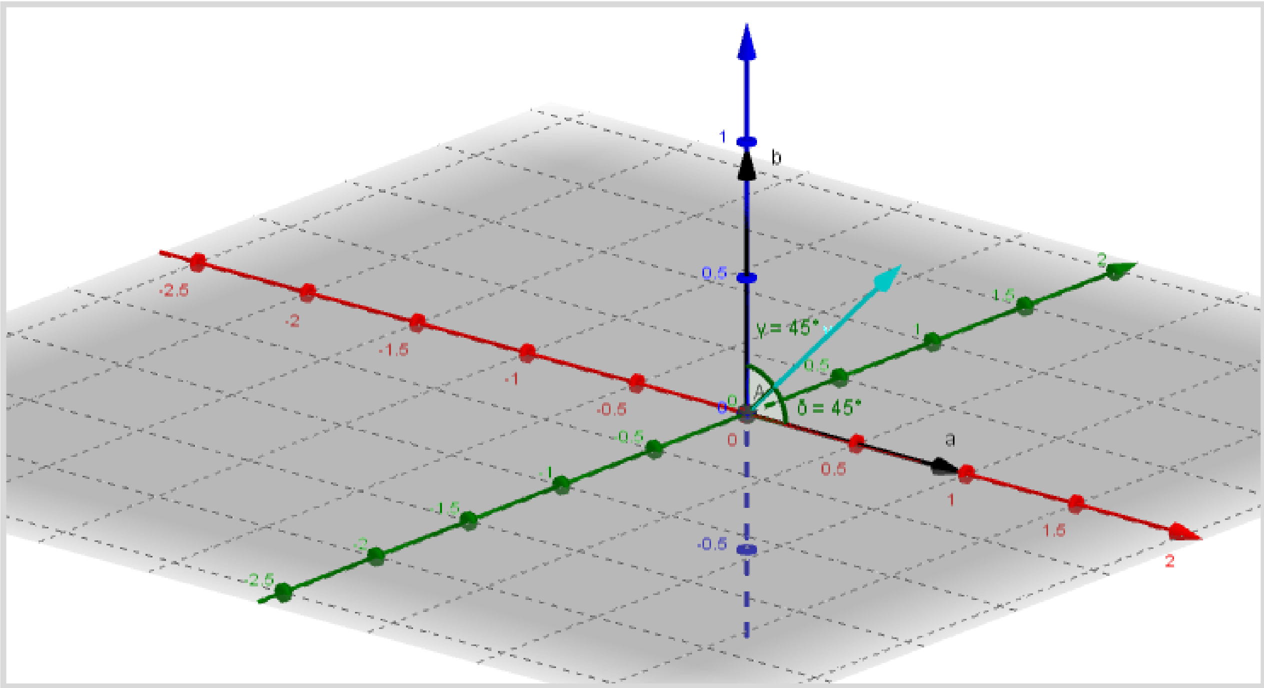
$Using \quad \underline{b}.\underline{u} = |\underline{b}| \times |\underline{u}| \times cos\theta°$

$z = 1 \times 1 \times cos45° \qquad ; \qquad z = \frac{1}{\sqrt{2}} \quad (2)$

$From\ equation\ (1)\ and\ (2) \quad x = z$

$Also \quad |\underline{u}| = x^2 + y^2 + z^2 = 1 \quad ; \quad y^2 = 0 \quad ; \quad y = 0$

$x = \frac{1}{\sqrt{2}}$; $y = 0$; $z = \frac{1}{\sqrt{2}}$ $unit\ vector\ \left(\frac{1}{\sqrt{2}}\underline{i} + \frac{1}{\sqrt{2}}\underline{k}\right)$



5. *Let unit vector be* $\underline{u} = \left(x\underline{i} + y\underline{j} + z\underline{k}\right)$; $\underline{a} = -\underline{i} + \underline{k}$ $(45°)$ *and* $\underline{b} = -2\underline{i} + 2\underline{j} + \underline{k}$ $(60°)$

Using $\underline{a}.\underline{u} = |\underline{a}| \times |\underline{u}| \times cos\theta°$

$-x + z = \sqrt{2} \times 1 \times cos45°$; $-x + z = 1$ (1)

Using $\underline{b}.\underline{u} = |\underline{b}| \times |\underline{u}| \times cos\theta°$

$-2x + 2y + z = 3 \times 1 \times cos60°$; $-2x + 2y + z = \frac{3}{2}$ (2)

From equation (1) $z = 1 + x$

From equation (2) $-2x + 2y + (1 + x) = \frac{3}{2}$; $-x + 2y = \frac{1}{2}$; $y = \frac{1}{4} + \frac{x}{2}$

Also $|\underline{u}| = x^2 + y^2 + z^2 = 1$; $x^2 + \left(\frac{1}{4} + \frac{x}{2}\right)^2 + (1 + x)^2 = 1$

$$x^2 + \left(\frac{x^2}{4} + \frac{x}{4} + \frac{1}{16}\right) + x^2 + 2x + 1 = 1$$

$\frac{9x^2}{4} + \frac{9x}{4} + \frac{1}{16} = 0$; $36x^2 + 36x + 1 = 0$

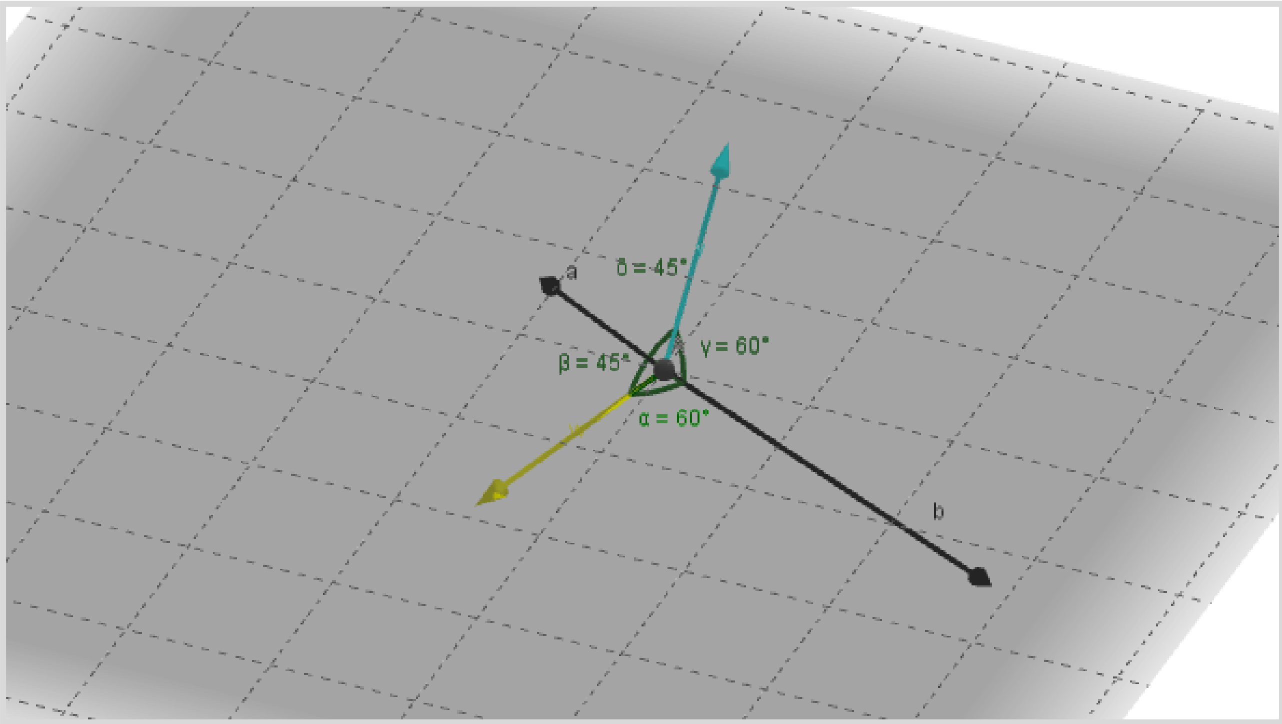
$$x = \frac{-36 \pm \sqrt{36^2 - 4 \times 36 \times 1}}{72} = \frac{-36 \pm 6\sqrt{32}}{72} = \frac{-36 \pm 24\sqrt{2}}{72} = \frac{-3 \pm 2\sqrt{2}}{6}$$

$x = \frac{-3 + 2\sqrt{2}}{6}$ *and* $x = \frac{-3 - 2\sqrt{2}}{6}$

For $x = \frac{-3 + 2\sqrt{2}}{6}$; $y = \frac{1}{4} + \frac{1}{2}\left(\frac{-3 + 2\sqrt{2}}{6}\right) = \frac{\sqrt{2}}{6}$; $z = 1 + \left(\frac{-3 + 2\sqrt{2}}{6}\right) = \frac{3 + 2\sqrt{2}}{6}$

For $x = \frac{-3 - 2\sqrt{2}}{6}$; $y = \frac{1}{4} + \frac{1}{2}\left(\frac{-3 - 2\sqrt{2}}{6}\right) = -\frac{\sqrt{2}}{6}$; $z = 1 + \left(\frac{-3 - 2\sqrt{2}}{6}\right) = \frac{3 - 2\sqrt{2}}{6}$

$$Unit\ vectors\ are\ \left(\frac{-3+2\sqrt{2}}{6}\underline{i}+\frac{\sqrt{2}}{6}\underline{j}+\frac{3+2\sqrt{2}}{6}\underline{k}\right)\ and\ \left(\frac{-3-2\sqrt{2}}{6}\underline{i}-\frac{\sqrt{2}}{6}\underline{j}+\frac{3-2\sqrt{2}}{6}\underline{k}\right)$$



$$(b)\quad Give\ vectors\ \underline{u}_1=\left(\frac{-3+2\sqrt{2}}{6}\underline{i}+\frac{\sqrt{2}}{6}\underline{j}+\frac{3+2\sqrt{2}}{6}\underline{k}\right)\ and\ \underline{u}_2=\left(\frac{-3-2\sqrt{2}}{6}\underline{i}-\frac{\sqrt{2}}{6}\underline{j}+\frac{3-2\sqrt{2}}{6}\underline{k}\right)$$

$If\ perpendicular\ then\ \underline{u}_1.\underline{u}_2=0$

$$\frac{(-3+2\sqrt{2})}{6}\times\frac{(-3-2\sqrt{2})}{6}+\frac{\sqrt{2}}{6}\times\left(-\frac{\sqrt{2}}{6}\right)+\frac{(3+2\sqrt{2})}{6}\times\frac{(3-2\sqrt{2})}{6}=0$$

$$\frac{(9-8)}{36}-\frac{2}{36}+\frac{(9-8)}{36}=\frac{2}{36}-\frac{2}{36}=0\qquad Hence\ \underline{u}_1\ and\ \underline{u}_2\ are\ perpendicular$$

Ex 2 – Vector Product in Component Form

1. $\underline{a} = \left(3\underline{i} + 2\underline{j} - \underline{k}\right) \qquad \underline{b} = \left(\underline{i} - \underline{j} - 2\underline{k}\right) \qquad \underline{c} = \left(4\underline{i} - 3\underline{j} + 4\underline{k}\right)$

(a) $\underline{b} \times \underline{c} = \left(\underline{i} - \underline{j} - 2\underline{k}\right) \times \left(4\underline{i} - 3\underline{j} + 4\underline{k}\right)$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & -2 \\ 4 & -3 & 4 \end{vmatrix} = \left(-10\underline{i} - 12\underline{j} + \underline{k}\right)$$

$\underline{a} \times \left(\underline{b} \times \underline{c}\right) = \left(3\underline{i} + 2\underline{j} - \underline{k}\right) = \left(-10\underline{i} - 12\underline{j} + \underline{k}\right)$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & -1 \\ -10 & -12 & 1 \end{vmatrix} = \left(-10\underline{i} + 7\underline{j} - 16\underline{k}\right)$$

(b) $\underline{a} \times \underline{b} = \left(3\underline{i} + 2\underline{j} - \underline{k}\right) \times \left(\underline{i} - \underline{j} - 2\underline{k}\right)$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & -1 \\ 1 & -1 & -2 \end{vmatrix} = \left(-5\underline{i} + 5\underline{j} - 5\underline{k}\right)$$

$\left(\underline{a} \times \underline{b}\right) \times \underline{c} = \left(-5\underline{i} + 5\underline{j} - 5\underline{k}\right) \times \left(4\underline{i} - 3\underline{j} + 4\underline{k}\right)$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -5 & 5 & -5 \\ 4 & -3 & 4 \end{vmatrix} = \left(5\underline{i} - 0\underline{j} - 5\underline{k}\right) = \left(5\underline{i} - 5\underline{k}\right)$$

$(c)\quad \underline{a}\times\underline{b}=\left(-5\underline{i}+5\underline{j}-5\underline{k}\right)$

$$\underline{a}\times\underline{c}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 3 & 2 & -1\\ 4 & -3 & 4\end{vmatrix}=\left(5\underline{i}-16\underline{j}-17\underline{k}\right)$$

$$\left(\underline{a}\times\underline{b}\right).\left(\underline{a}\times\underline{c}\right)=\left(-5\underline{i}+5\underline{j}-5\underline{k}\right).\left(5\underline{i}-16\underline{j}-17\underline{k}\right)=-25-80+85=-20$$

$(d)\quad \left(\underline{a}\times\underline{b}\right)=\left(-5\underline{i}+5\underline{j}-5\underline{k}\right)\qquad \left(\underline{b}\times\underline{c}\right)=\left(-10\underline{i}-12\underline{j}+\underline{k}\right)$

$$\left(\underline{a}\times\underline{b}\right).\left(\underline{a}\times\underline{c}\right)=\left(-5\underline{i}+5\underline{j}-5\underline{k}\right).\left(-10\underline{i}-12\underline{j}+\underline{k}\right)=50-60-5=-15$$

$(e)\quad \underline{a}\times\left(\underline{b}\times\underline{c}\right)=\left(-10\underline{i}+7\underline{j}-16\underline{k}\right)$

$$\left[\underline{a}\times\left(\underline{b}\times\underline{c}\right)\right].\underline{c}=\left(-10\underline{i}+7\underline{j}-16\underline{k}\right).\left(4\underline{i}-3\underline{j}+4\underline{k}\right)=-40-21-64=-125$$

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2. $\underline{a}=\left(3\underline{i}+2\underline{j}+5\underline{k}\right)\qquad \underline{b}=\left(4\underline{i}+3\underline{j}+2\underline{k}\right)\qquad \underline{c}=\left(2\underline{i}+\underline{j}+10\underline{k}\right)$

$(a)\quad \underline{a}\times\underline{b}=\left(3\underline{i}+2\underline{j}+5\underline{k}\right)\times\left(4\underline{i}+3\underline{j}+2\underline{k}\right)$

$$=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 3 & 2 & 5\\ 4 & 3 & 2\end{vmatrix}=\left(-11\underline{i}+14\underline{j}+\underline{k}\right)$$

$(b)\qquad \underline{a}\times\underline{c} = \left(3\underline{i}+2\underline{j}+5\underline{k}\right)\times\left(2\underline{i}+\underline{j}+10\underline{k}\right)$

$$\underline{a}\times\underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & 5 \\ 2 & 1 & 10 \end{vmatrix} = \left(15\underline{i}-20\underline{j}-\underline{k}\right)$$

$$\underline{b}\,.\left(\underline{a}\times\underline{c}\right) = \left(4\underline{i}+3\underline{j}+2\underline{k}\right)\times\left(15\underline{i}-20\underline{j}-\underline{k}\right) = 60-60-2 = -2$$

$$= \left(-11\underline{i}+14\underline{j}+\underline{k}\right).\left(2\underline{i}+\underline{j}+10\underline{k}\right) = -22+14+10 = 2$$

$(c)\qquad \underline{a}\times\underline{c} = \left(3\underline{i}+2\underline{j}+5\underline{k}\right)\times\left(2\underline{i}+\underline{j}+10\underline{k}\right)$

$$\underline{a}\times\underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & 5 \\ 2 & 1 & 10 \end{vmatrix} = \left(15\underline{i}-20\underline{j}-\underline{k}\right)$$

$$\underline{b}\,.\left(\underline{a}\times\underline{c}\right) = \left(4\underline{i}+3\underline{j}+2\underline{k}\right)\times\left(15\underline{i}-20\underline{j}-\underline{k}\right) = 60-60-2 = -2$$

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3. $\underline{a} = \left(3\underline{i} + \underline{j} + 2\underline{k}\right) \qquad \underline{b} = \left(2\underline{j} - \underline{k}\right) \qquad \underline{c} = \left(\underline{i} + \underline{j} + \underline{k}\right)$

$$\underline{c} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix} = \left(\underline{i} + \underline{j} - 2\underline{k}\right) \quad ; \quad \underline{a} \,.\, \underline{c} = \left(3\underline{i} + \underline{j} + 2\underline{k}\right).\left(\underline{i} + \underline{j} + \underline{k}\right) = 3 + 1 + 2 = 6$$

$$\underline{b} \times \left(\underline{c} \times \underline{a}\right) = \left(2\underline{j} - \underline{k}\right) \times \left(\underline{i} + \underline{j} - 2\underline{k}\right)$$

$$\underline{b} \times \left(\underline{c} \times \underline{a}\right) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & -2 \\ 0 & 2 & -1 \end{vmatrix} = \left(3\underline{i} + \underline{j} + 2\underline{k}\right)$$

$$\left(\underline{a} \,.\, \underline{c}\right)\underline{a} = \left(2\underline{j} - \underline{k}\right).\left(\underline{i} + \underline{j} - 2\underline{k}\right) = 6\left(3\underline{i} + \underline{j} + 2\underline{k}\right) = \left(18\underline{i} + 6\underline{j} + 12\underline{k}\right)$$

$$\underline{d} = \underline{b} \times \left(\underline{c} \times \underline{a}\right) - \left(\underline{a} \,.\, \underline{c}\right)\underline{a} = \left(3\underline{i} + \underline{j} + 2\underline{k}\right) - \left(18\underline{i} + 6\underline{j} + 12\underline{k}\right) = \left(-15\underline{i} - 5\underline{j} - 10\underline{k}\right)$$

If $\underline{b}$ *and* $\underline{d}$ *are perpendicular then* $\underline{b}.\underline{d} = 0$

$\underline{b}.\underline{d} = \left(2\underline{j} - \underline{k}\right).\left(-15\underline{i} - 5\underline{j} - 10\underline{k}\right) = 0 - 10 + 10 = 0$; *Hence* $\underline{b}$ *and* $\underline{d}$ *are perpendicular*

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4. $\underline{a} = \left(4\underline{i} - 2\underline{j} + 3\underline{k}\right) \qquad \underline{b} = \left(5\underline{i} + \underline{j} - 4\underline{k}\right)$

Vector perpendicular to $\underline{a}$ *and* $\underline{b}$ *is* $= \underline{a} \times \underline{b}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & -2 & 3 \\ 5 & 1 & -4 \end{vmatrix} = \left(5\underline{i} + 31\underline{j} + 14\underline{k}\right)$$

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5. $\underline{a} = \left(\underline{i} + \underline{j} - \underline{k}\right) \qquad \underline{b} = \left(2\underline{i} - \underline{j} + \underline{k}\right)$

$(a) \quad \underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \left(0\underline{i} - 3\underline{j} - 3\underline{k}\right) = \left(-3\underline{j} - 3\underline{k}\right)$

$(b) \quad \underline{a} + \underline{b} = \left(\underline{i} + \underline{j} - \underline{k}\right) + \left(2\underline{i} - \underline{j} + \underline{k}\right) = \left(3\underline{i}\right)$

$\underline{a} \times \left(\underline{a} + \underline{b}\right) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & -1 \\ 3 & 0 & 0 \end{vmatrix} = \left(0\underline{i} - 3\underline{j} - 3\underline{k}\right) = \left(-3\underline{j} - 3\underline{k}\right)$

$(c) \quad \underline{a}.\underline{\left(\underline{a} \times \underline{b}\right)} = \left(\underline{i} + \underline{j} - \underline{k}\right).\left(-3\underline{j} - 3\underline{k}\right) = 0 - 3 + 3 = 0$

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6. $\underline{a} = \left(4\underline{i} - \underline{k}\right) \qquad \underline{b} = \left(4\underline{i} + 3\underline{j} - 2\underline{k}\right)$

$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 0 & -1 \\ 4 & 3 & -2 \end{vmatrix} = \left(3\underline{i} + 4\underline{j} + 12\underline{k}\right) \quad ; \quad \left|\underline{a} \times \underline{b}\right| = \sqrt{3^2 + 4^2 + 12^2} = \pm 13$

$Unit\ Vectors\ are \quad \frac{1}{13}\left(3\underline{i} + 4\underline{j} + 12\underline{k}\right) \quad and \quad -\frac{1}{13}\left(3\underline{i} + 4\underline{j} + 12\underline{k}\right)$

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7. $A(4,-8,-13) \quad B(5,-2,-3) \quad C(5,4,10)$

$$\overrightarrow{AB} = \left(5\underline{i} - 2\underline{j} - 3\underline{k}\right) - \left(4\underline{i} - 8\underline{j} - 13\underline{k}\right) = \left(\underline{i} + 6\underline{j} + 10\underline{k}\right)$$

$$\overrightarrow{AC} = \left(5\underline{i} - 4\underline{j} - 10\underline{k}\right) - \left(4\underline{i} - 8\underline{j} - 13\underline{k}\right) = \left(\underline{i} + 4\underline{j} + 3\underline{k}\right)$$

$$Area\ of\ Triangle = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 6 & 10 \\ 1 & 4 & 3 \end{vmatrix} = \left(-22\underline{i} + 7\underline{j} - 2\underline{k}\right)$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-22)^2 + 7^2 + (-2)^2} = \sqrt{537}$$

$$Area\ of\ Triangle = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{537} = 11.6\ square\ units$$

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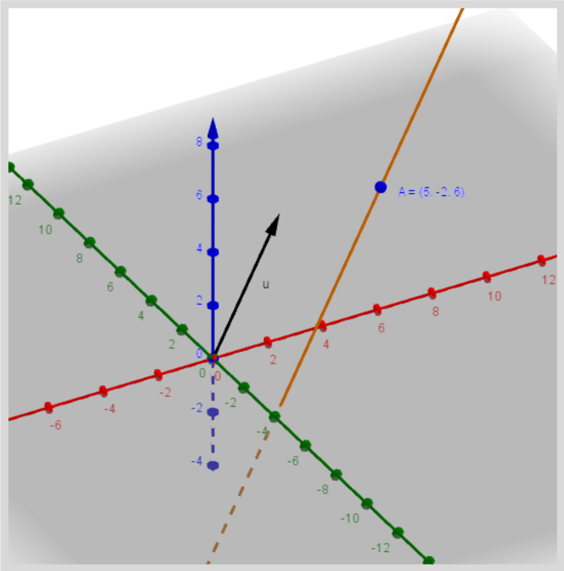
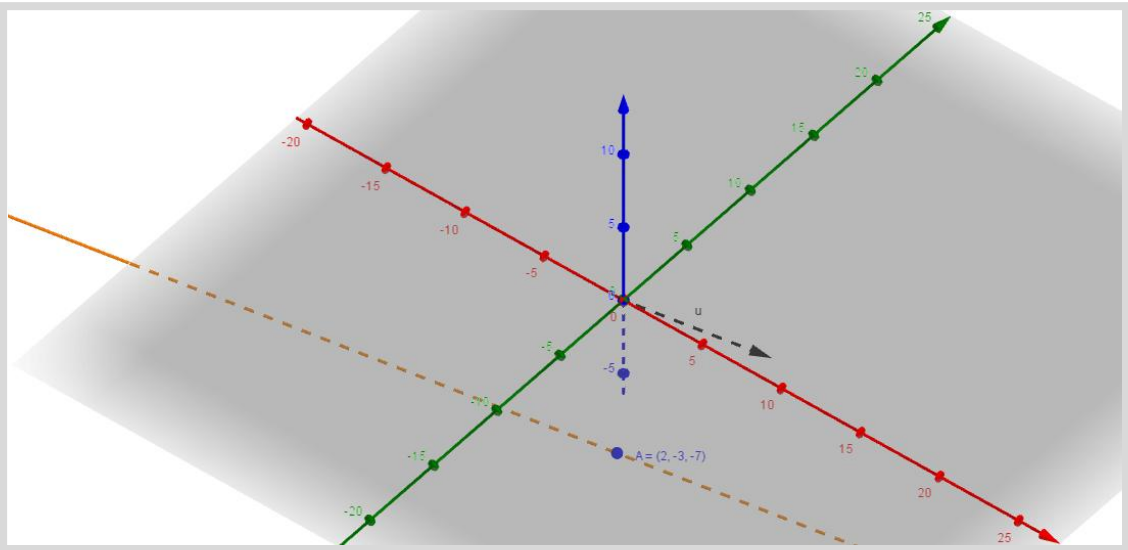
8. Leave out !!!

Ex 3 – Vector Equation of a Straight Line

1. $\underline{r} = \underline{a} + t\underline{d} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$; *Symmetrical Form* $\frac{x-5}{3} = \frac{y+2}{1} = \frac{z-6}{4}$

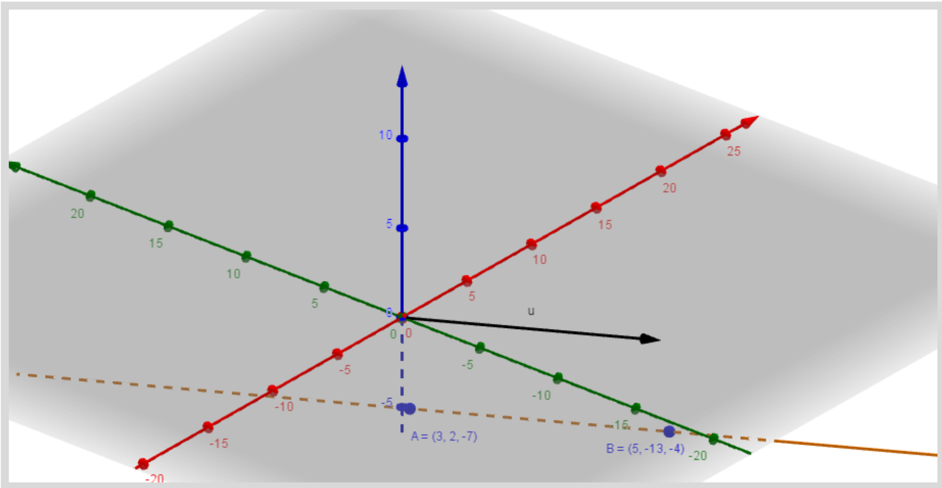


2. $\underline{r} = \underline{a} + t\underline{d} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -7 \end{pmatrix} + t\begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix}$; *Symmetrical Form* $\frac{x-2}{7} = \frac{y+3}{3} = \frac{z+7}{-2}$



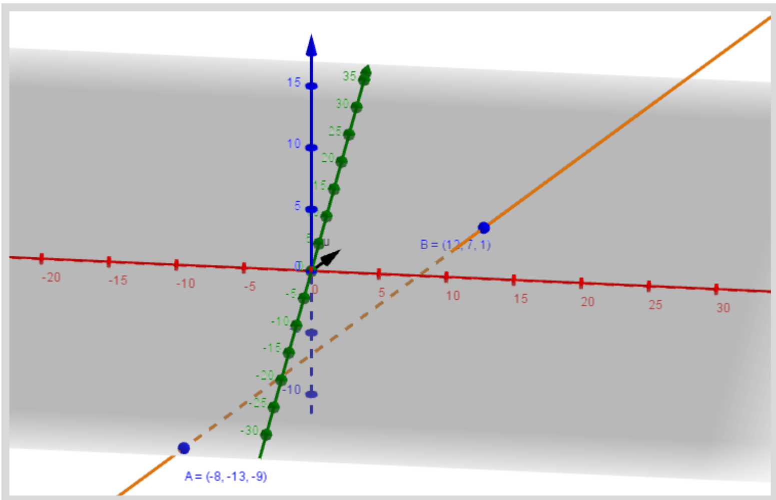
3. (a) $\underline{d} = \underline{b} - \underline{a} = \begin{pmatrix} 5 \\ -13 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ -15 \\ 3 \end{pmatrix}$

$\underline{r} = \underline{a} + t\underline{d} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -7 \end{pmatrix} + t\begin{pmatrix} 2 \\ -15 \\ 3 \end{pmatrix}$; *Symmetrical Form* $\dfrac{x-3}{2} = \dfrac{y-2}{-15} = \dfrac{z+7}{3}$



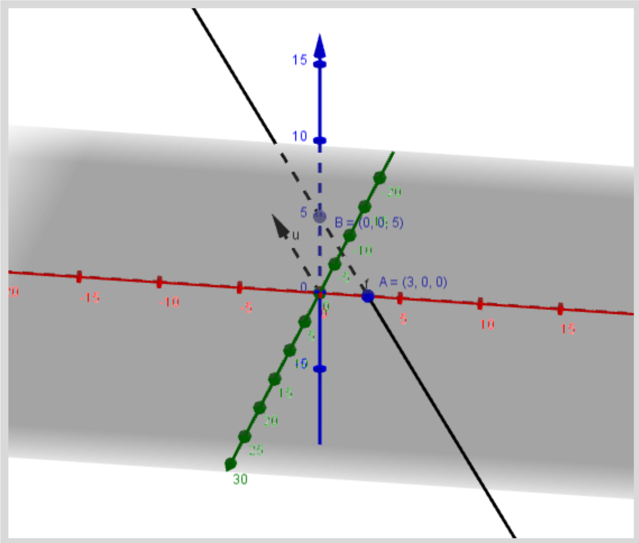
(b) $\underline{d} = \underline{b} - \underline{a} = \begin{pmatrix} 12 \\ 7 \\ 1 \end{pmatrix} - \begin{pmatrix} -8 \\ -13 \\ -9 \end{pmatrix} = \begin{pmatrix} 20 \\ 20 \\ 10 \end{pmatrix}$; *direction vector reduces to* $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$\underline{r} = \underline{a} + t\underline{d} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$; *Symmetrical Form* $\dfrac{x-12}{2} = \dfrac{y-7}{2} = \dfrac{z-1}{1}$



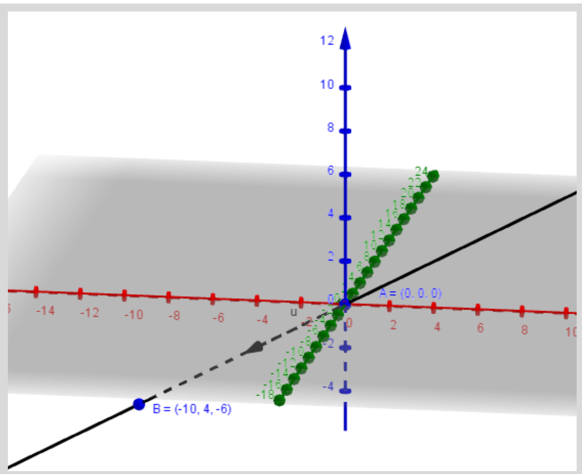
$$(c)\quad \underline{d} = \underline{b} - \underline{a} = \begin{pmatrix}0\\0\\5\end{pmatrix} - \begin{pmatrix}3\\0\\0\end{pmatrix} = \begin{pmatrix}-3\\0\\5\end{pmatrix}$$

$$\underline{r} = \underline{a} + t\underline{d} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\0\\0\end{pmatrix} + t\begin{pmatrix}-3\\0\\5\end{pmatrix} \quad ; \quad \textit{Symmetrical Form} \quad \frac{x-3}{-3} = \frac{y}{0} = \frac{z}{5}$$



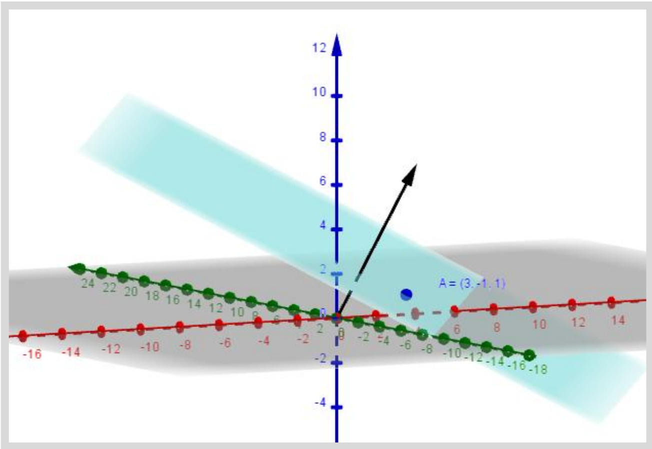
$$(d)\quad \underline{d} = \underline{b} - \underline{a} = \begin{pmatrix}-10\\4\\-6\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}-10\\4\\-6\end{pmatrix} \quad ; \quad \textit{direction vector reduces to} \begin{pmatrix}-5\\2\\-3\end{pmatrix}$$

$$\underline{r} = \underline{a} + t\underline{d} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} + t\begin{pmatrix}-5\\2\\-3\end{pmatrix} \quad ; \quad \textit{Symmetrical Form} \quad \frac{x}{-5} = \frac{y}{2} = \frac{z}{-3}$$

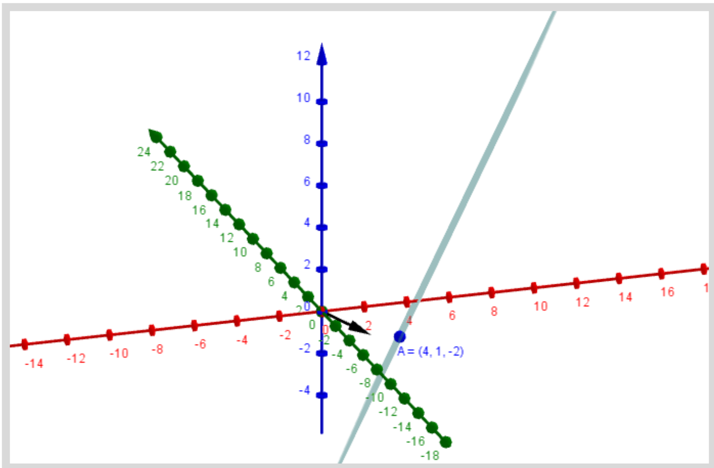


Ex 4 - Vector Equation of a Plane

1. (a) $\quad \underline{n} \cdot \underline{p} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 3 \times 3 + (-2) \times (-1) + 7 \times 1 = 18 \quad ; \; Vec.Eqn \;\; 3x - 2y + 7z = 18$



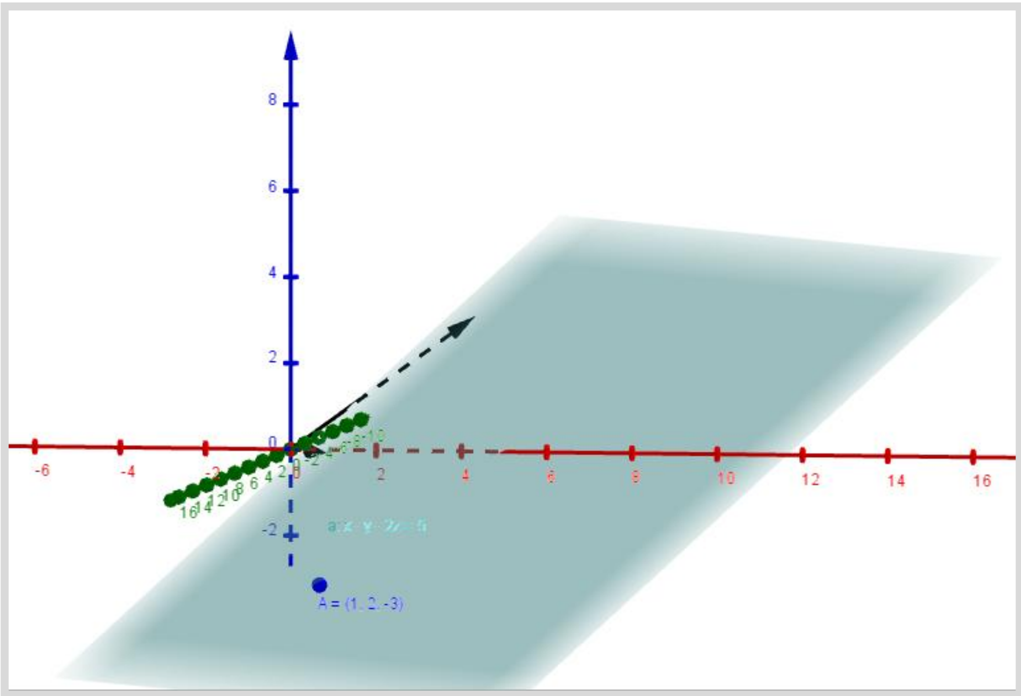
(b) $\quad \underline{n} \cdot \underline{p} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = 2 \times 4 + (-1) \times 1 + (-1) \times (-2) = 9 \quad ; \; Vec.Eqn \;\; 2x - y - z = 9$



2. (a) $\quad \underline{n} \cdot \underline{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} = 4a - 2b + 3c = 0 \quad ; \quad \underline{n} \cdot \underline{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = a + b = 0$

$$Solving\ Sim. Eqns \quad \underline{x} = \begin{pmatrix} a \\ -a \\ -2a \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$\underline{n} \cdot \underline{p} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 1 - 2 + 6 = 5 \quad ; \quad Cartesian. Eqn \quad x - y - 2z = 5$$



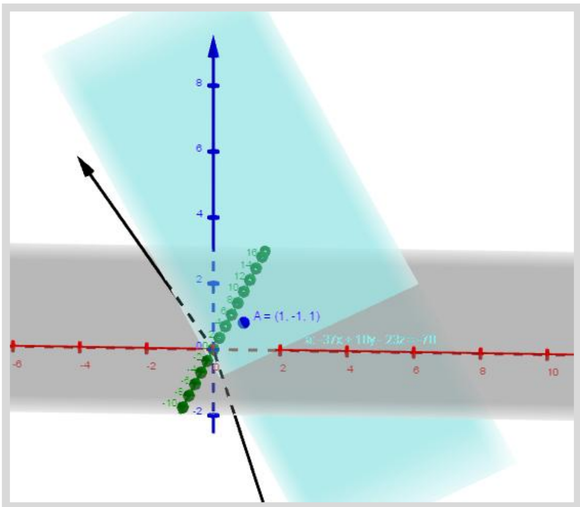
$(b)\quad \underline{n}\cdot\underline{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\cdot\begin{pmatrix} 3 \\ -5 \\ -7 \end{pmatrix} = 3a - 5b - 7c = 0 \quad ; \quad \underline{n}\cdot\underline{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\cdot\begin{pmatrix} -4 \\ -1 \\ 6 \end{pmatrix} = -4a - b + 6c = 0$

$b = -4a + 6c \quad sub\ into\ equation\ 1\ ; \quad 3a - 5(-4a + 6c) - 7c = 0 \ ; \quad 23a - 37c = 0$

$c = \frac{23}{37}a \quad ; \quad b = -4a + 6\left(\frac{23}{37}\right)a = -4a + \frac{138}{37}a \quad ; \quad b = -\frac{10}{37}a$

$Solving\ Sim.Eqns \quad \underline{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ -\frac{10}{37}a \\ \frac{23}{37}a \end{pmatrix} \qquad \begin{pmatrix} 37a \\ 37b \\ 37c \end{pmatrix} = \begin{pmatrix} 37 \\ -10 \\ 23 \end{pmatrix}$

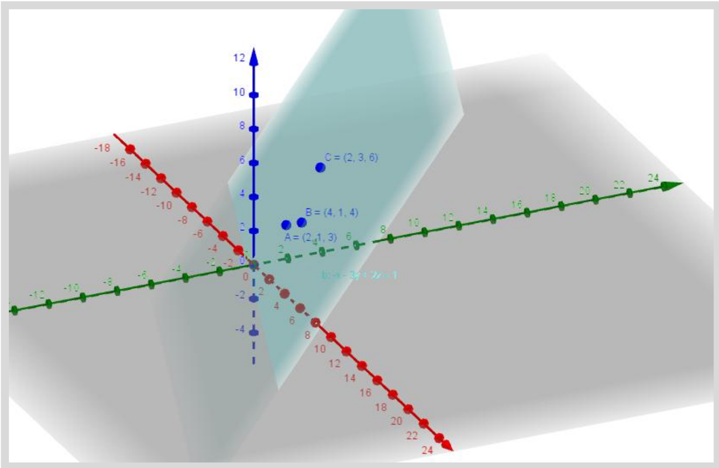
$\underline{n}\cdot\underline{p} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 37 \\ -10 \\ 23 \end{pmatrix} = 37 + 10 + 23 = 70 \quad ; \quad Cartesian.Eqn \ \ 37x - 10y + 23z = 70$



3. (a) $\quad \overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad ; \quad \overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 0 & 1 \\ 0 & 2 & 3 \end{vmatrix} = \left(-2\underline{i} - 6\underline{j} + 4\underline{k}\right) \quad ; \quad -2x - 6y + 4z = k$$

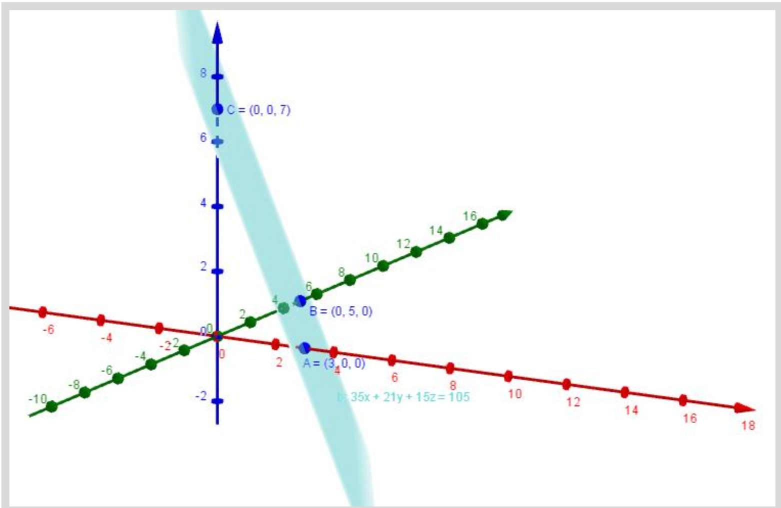
$Sub\ in\ a\ point\ (2,1,3)\ ;\quad -2(2) - 6(1) + 4(3) = 2 \quad ; \quad -2x - 6y + 4z = 2 \quad ; \quad x + 3y - 2z = -1$



(b) $\quad \overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 0 \end{pmatrix} \quad ; \quad \overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 7 \end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -3 & 5 & 0 \\ -3 & 0 & 7 \end{vmatrix} = \left(35\underline{i} + 21\underline{j} + 15\underline{k}\right) \quad ; \quad 35x + 21y + 15z = k$$

$Sub\ in\ a\ point\ (3,0,0)\ ;\quad 35(3) + 21(0) + 15(0) = 105 \quad ; \quad 35x + 21y + 15z = 105$



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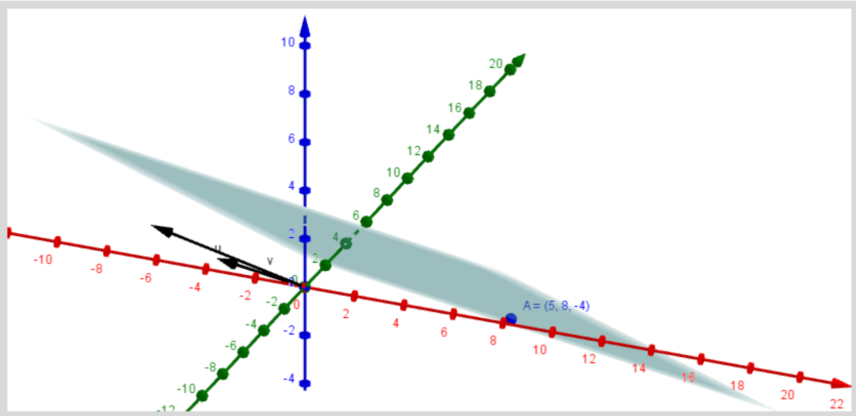
4. (a) $\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 8 \\ -4 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \\ 3 \end{pmatrix}$; $\underline{d} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$

$$\underline{n} = \overrightarrow{AB} \times \underline{d} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -5 & -3 & 3 \\ -4 & 1 & 0 \end{vmatrix} = \left(-3\underline{i} - 12\underline{j} - 17\underline{k}\right) \quad ; \quad -3x - 12y - 17z = k$$

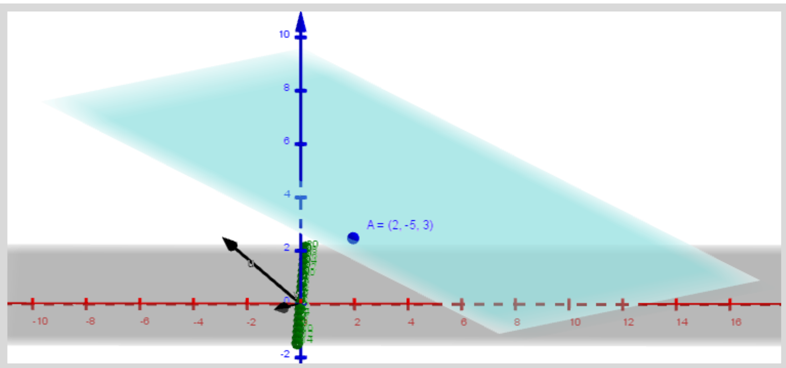
$$\underline{n} \cdot \underline{p} = \begin{pmatrix} -3 \\ -12 \\ -17 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 8 \\ -4 \end{pmatrix} = -15 - 96 + 68 = -43 \quad ; \quad Cartesian.Eqn \quad 3x + 12y + 17z = 43$$



(b) $\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 1 \\ -7 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$; $\underline{d} = \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix}$

$$\underline{n} = \overrightarrow{AB} \times \underline{d} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -3 & 5 & 2 \\ -1 & -2 & 0 \end{vmatrix} = \left(4\underline{i} - 2\underline{j} + 11\underline{k}\right) \quad ; \quad 4x - 2y + 11z = k$$

$$\underline{n} \cdot \underline{p} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ 11 \end{pmatrix} = 8 + 10 + 33 = 51 \quad ; \quad Cartesian.Eqn \quad 4x - 2y + 11z = 51$$



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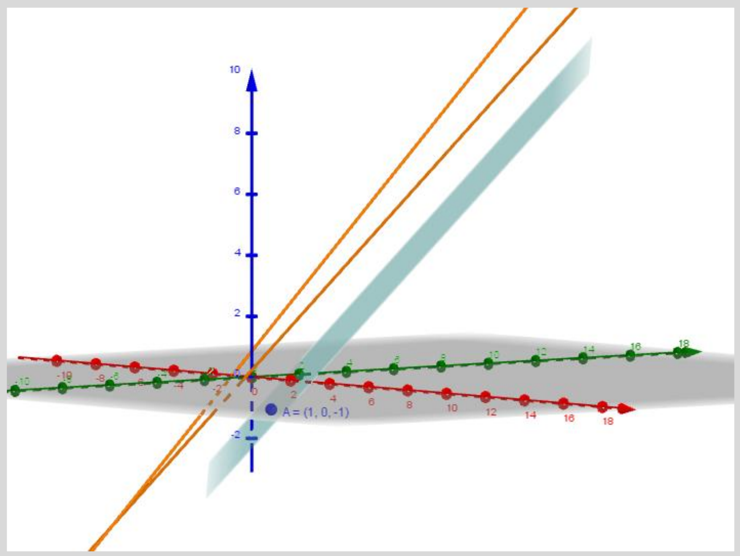
5. $\underline{d}_1 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$; $\underline{d}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

$$\underline{n} = \underline{d}_1 \times \underline{d}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 4 \\ -1 & 2 & 1 \end{vmatrix} = \left(-5\underline{i} - 6\underline{j} + 7\underline{k}\right) \quad ; \quad -5x - 6y + 7z = k$$

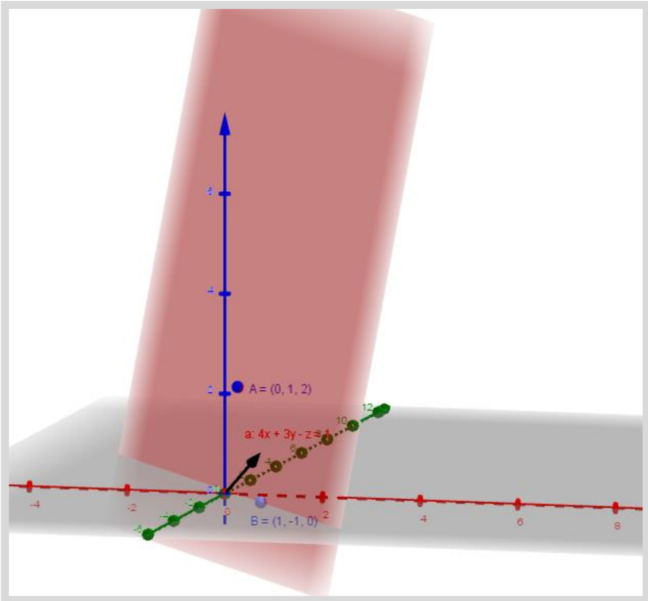
$$\underline{n} \cdot \underline{p} = \begin{pmatrix} -5 \\ -6 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = -5 - 0 - 7 = -12 \quad ; \quad Cartesian.Eqn \;\; 5x + 6y - 7z = 12$$



6. $\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$; $\underline{d} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$$\underline{n} = \overrightarrow{AB} \times \underline{d} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & -2 \\ 1 & -1 & 1 \end{vmatrix} = \left(-4\underline{i} - 3\underline{j} + \underline{k}\right) \quad ; \quad -4x - 3y + z = k$$

$$\underline{n} \cdot \underline{p} = \begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0 - 3 + 2 = -1 \quad ; \quad Cartesian.Eqn\;\; 4x + 3y - z = 1$$

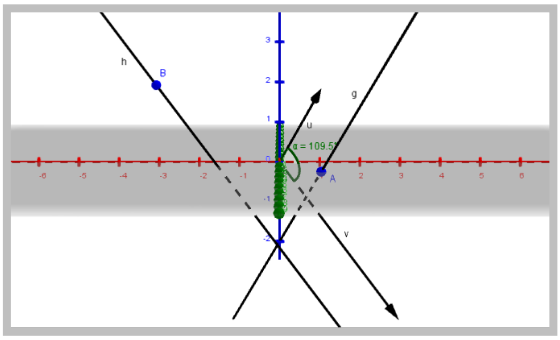


Ex 5 – Angles Intersections between Lines and Planes

1. (a) $\underline{d}_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$; $\underline{d}_2 = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$

$$\underline{d}_1 . \underline{d}_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} . \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} = 3 + 0 - 8 = -5 \quad ; \quad |\underline{d}_1| = 3 \quad |\underline{d}_2| = 5$$

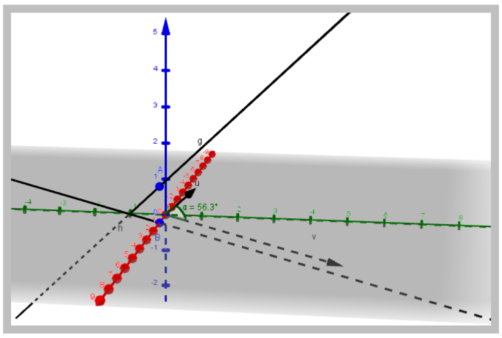
$$cos\theta = \frac{\underline{d}_1 . \underline{d}_2}{|\underline{d}_1||\underline{d}_2|} = \frac{-5}{3 \times 5} = 110° \quad ; \quad Acute\ angle\ is\ 70°$$



(b) $\underline{d}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $\underline{d}_2 = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$

$$\underline{d}_1 . \underline{d}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} = 1 + 5 - 1 = 5 \quad ; \quad |\underline{d}_1| = \sqrt{3} \quad |\underline{d}_2| = \sqrt{27}$$

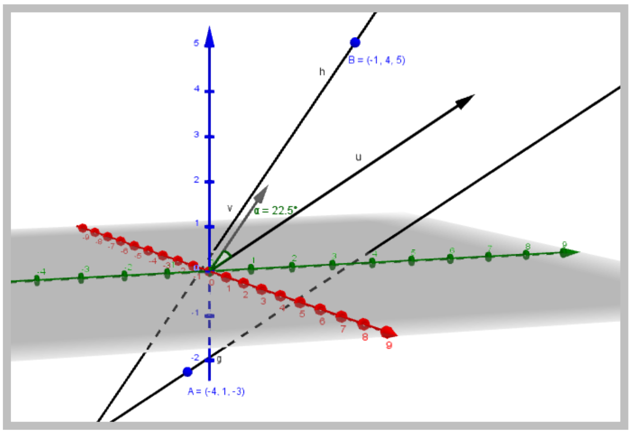
$$cos\theta = \frac{\underline{d}_1 . \underline{d}_2}{|\underline{d}_1||\underline{d}_2|} = \frac{5}{\sqrt{3} \times \sqrt{27}} = 56°$$



(c) $\underline{d}_1 = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$; $\underline{d}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$\underline{d}_1 . \underline{d}_2 = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 3 + 5 + 8 = 16$; $|\underline{d}_1| = \sqrt{50}$ $|\underline{d}_2| = \sqrt{6}$

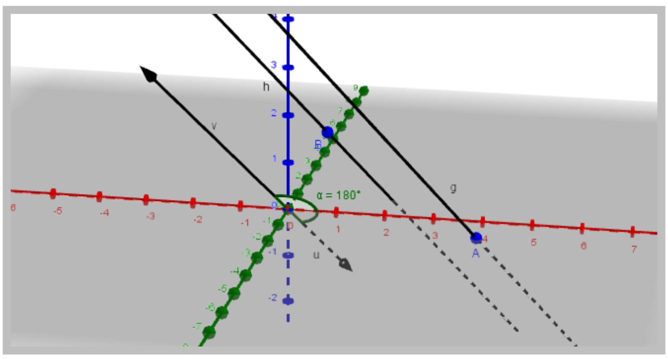
$cos\theta = \frac{\underline{d}_1 . \underline{d}_2}{|\underline{d}_1||\underline{d}_2|} = \frac{16}{\sqrt{50} \times \sqrt{6}} = 22.5°$



(d) $\underline{d}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$; $\underline{d}_2 = \begin{pmatrix} -2 \\ -4 \\ 4 \end{pmatrix}$

$\underline{d}_1 . \underline{d}_2 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} . \begin{pmatrix} -2 \\ -4 \\ 4 \end{pmatrix} = -2 - 8 - 8 = -18$; $|\underline{d}_1| = 3$ $|\underline{d}_2| = 6$

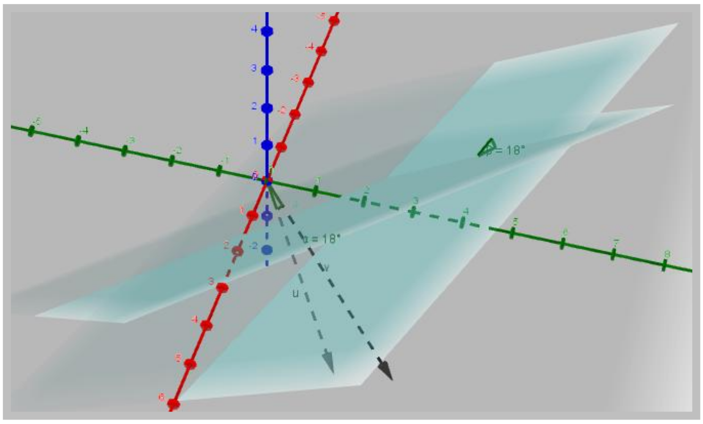
$cos\theta = \frac{\underline{d}_1 . \underline{d}_2}{|\underline{d}_1||\underline{d}_2|} = \frac{-18}{3 \times 6} = 180°$; *Acute angle is* $0°$



2. (a) $\quad \underline{d}_1 = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} \quad ; \quad \underline{d}_2 = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$

$$\underline{d}_1.\underline{d}_2 = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}.\begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} = 2 + 6 + 12 = 20 \quad ; \quad |\underline{d}_1| = \sqrt{17} \quad |\underline{d}_2| = \sqrt{26}$$

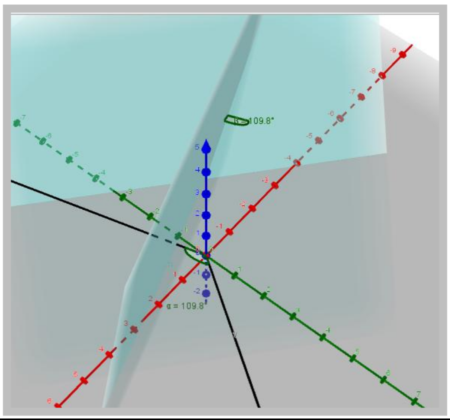
$$cos\theta = \frac{\underline{d}_1.\underline{d}_2}{|\underline{d}_1||\underline{d}_2|} = \frac{20}{\sqrt{17} \times \sqrt{26}} = 18°$$



(b) $\quad \underline{d}_1 = \begin{pmatrix} 5 \\ -14 \\ 2 \end{pmatrix} \quad ; \quad \underline{d}_2 = \begin{pmatrix} 6 \\ 7 \\ 6 \end{pmatrix}$

$$\underline{d}_1.\underline{d}_2 = \begin{pmatrix} 5 \\ -14 \\ 2 \end{pmatrix}.\begin{pmatrix} 6 \\ 7 \\ 6 \end{pmatrix} = 30 - 98 + 12 = -56 \quad ; \quad |\underline{d}_1| = 15 \quad |\underline{d}_2| = 11$$

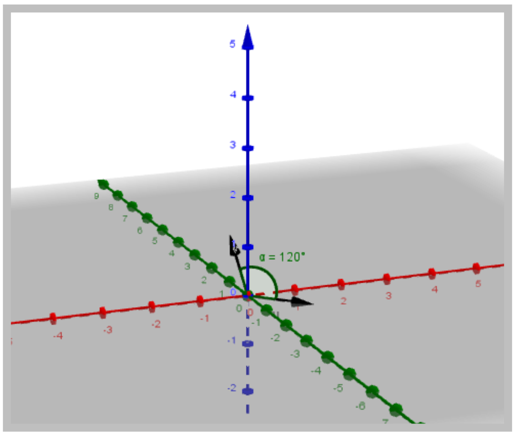
$$cos\theta = \frac{\underline{d}_1.\underline{d}_2}{|\underline{d}_1||\underline{d}_2|} = \frac{-56}{15 \times 11} = 109.8° \quad ; \quad Acute\ angle\ is\ 70.2°$$



(c) $\underline{d_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$; $\underline{d_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$\underline{d_1}.\underline{d_2} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 + 1 + 0 = -1 \quad ; \quad |\underline{d_1}| = \sqrt{2} \quad |\underline{d_2}| = \sqrt{2}$$

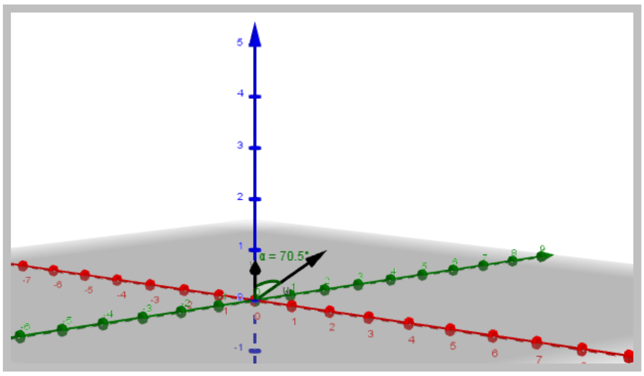
$$cos\theta = \frac{\underline{d_1}.\underline{d_2}}{|\underline{d_1}||\underline{d_2}|} = \frac{-1}{\sqrt{2} \times \sqrt{2}} = 120° \quad ; \quad Acute\ angle\ is\ 60°$$



(d) $\underline{d_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $\underline{d_2} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$$\underline{d_1}.\underline{d_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 1 - 1 + 1 = 1 \quad ; \quad |\underline{d_1}| = \sqrt{3} \quad |\underline{d_2}| = \sqrt{3}$$

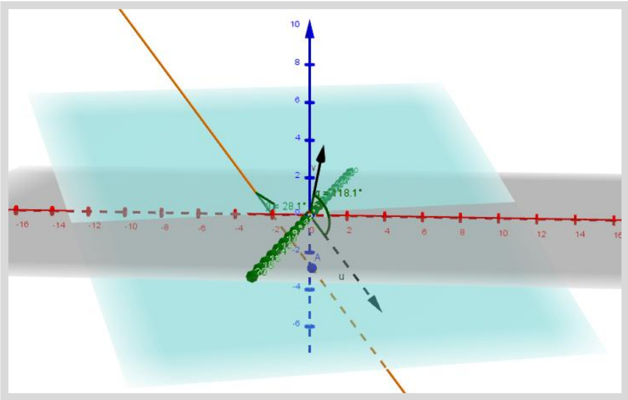
$$cos\theta = \frac{\underline{d_1}.\underline{d_2}}{|\underline{d_1}||\underline{d_2}|} = \frac{1}{\sqrt{3} \times \sqrt{3}} = 70.5°$$



3. $(a)\quad \underline{d_1} = \begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix} \quad ; \ \underline{d_2} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

$\underline{d_1}.\underline{d_2} = \begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix}.\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = 4 + 2 - 20 = -14 \quad ; \quad |\underline{d_1}| = \sqrt{42} \quad |\underline{d_2}| = \sqrt{21}$

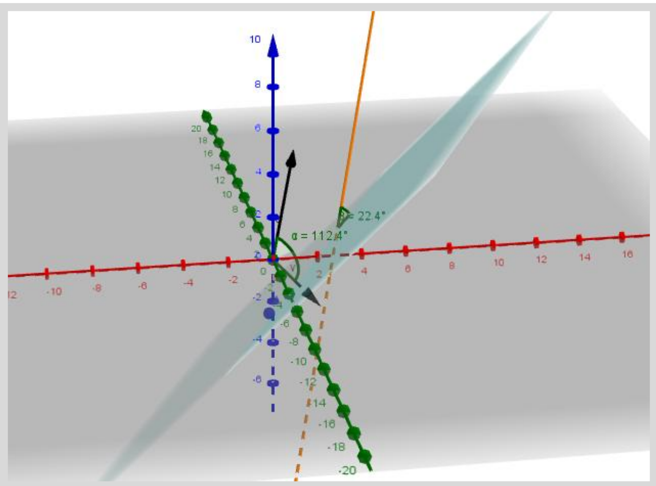
$cos\theta = \frac{\underline{d_1}.\underline{d_2}}{|\underline{d_1}||\underline{d_2}|} = \frac{-14}{\sqrt{42} \times \sqrt{21}} = 118.5° \quad ; \quad Acute\ angle\ is\ 61.5° \quad angle\ (90 - 61.5) = 28.5°$



$(b)\quad \underline{d_1} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} \quad ; \ \underline{d_2} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

$\underline{d_1}.\underline{d_2} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}.\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 4 - 6 - 6 = -8 \quad ; \quad |\underline{d_1}| = 7 \quad |\underline{d_2}| = 3$

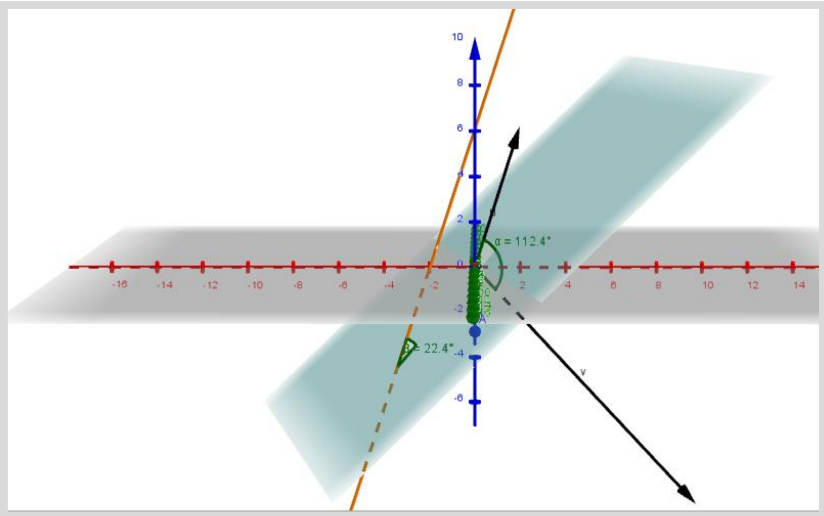
$cos\theta = \frac{\underline{d_1}.\underline{d_2}}{|\underline{d_1}||\underline{d_2}|} = \frac{-8}{7 \times 3} = 112.4° \quad ; \quad Acute\ angle\ is\ 67.6° \quad angle\ (90 - 67.6) = 22.4°$



$(c)\quad \underline{d}_1 = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \quad ; \ \underline{d}_2 = \begin{pmatrix} 10 \\ 2 \\ -11 \end{pmatrix}$

$$\underline{d}_1 . \underline{d}_2 = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} . \begin{pmatrix} 10 \\ 2 \\ -11 \end{pmatrix} = 20 + 6 - 66 = -40 \quad ; \quad |\underline{d}_1| = 7 \quad |\underline{d}_2| = 15$$

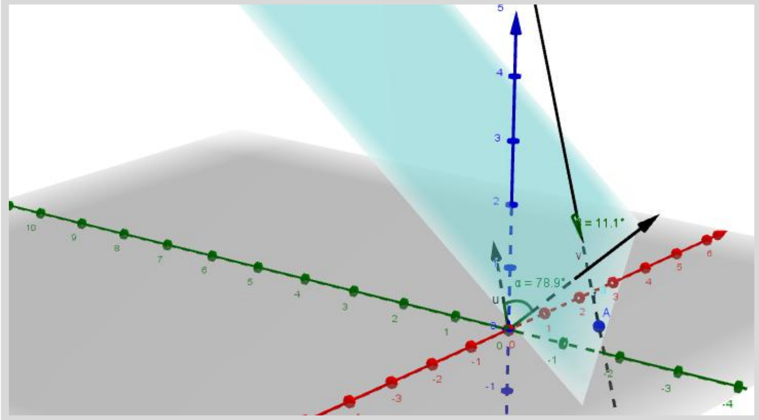
$$cos\theta = \frac{\underline{d}_1 . \underline{d}_2}{|\underline{d}_1||\underline{d}_2|} = \frac{-40}{7 \times 15} = 112.4° \quad ; \quad Acute\ angle\ is\ 67.6° \quad angle\ (90 - 67.6) = 22.4°$$



$(d)\quad \underline{d}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad ; \ \underline{d}_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$$\underline{d}_1 . \underline{d}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 1 - 2 + 2 = 1 \quad ; \quad |\underline{d}_1| = \sqrt{3} \quad |\underline{d}_2| = 3$$

$$cos\theta = \frac{\underline{d}_1 . \underline{d}_2}{|\underline{d}_1||\underline{d}_2|} = \frac{1}{\sqrt{3} \times 3} = 78.9° \quad ; \quad angle\ (90 - 78.9) = 11.1°$$



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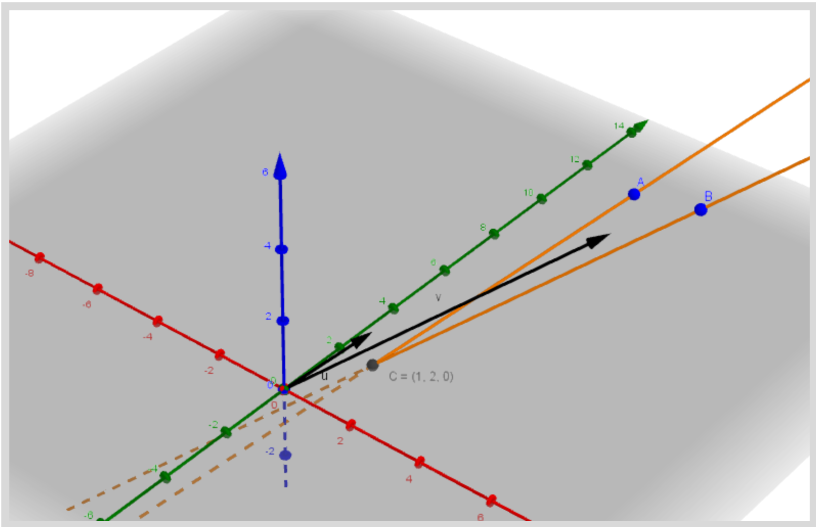
4. (a) $P(p,q,r)$; $\frac{p-4}{1}=\lambda$ $\frac{p-7}{6}=\mu$; $\lambda+4=6\mu+7$; $\lambda-6\mu=3$

; $\frac{q-8}{2}=\lambda$ $\frac{q-6}{4}=\mu$; $2\lambda+8=4\mu+6$; $\lambda-2\mu=-1$

; $\frac{r-3}{1}=\lambda$ $\frac{r-5}{5}=\mu$; $\lambda+3=5\mu+5$; $\lambda-5\mu=2$

$Solve\ sim.equations$; $\lambda=-3$ and $\mu=-1$; $Point\ is\ (1,2,0)$

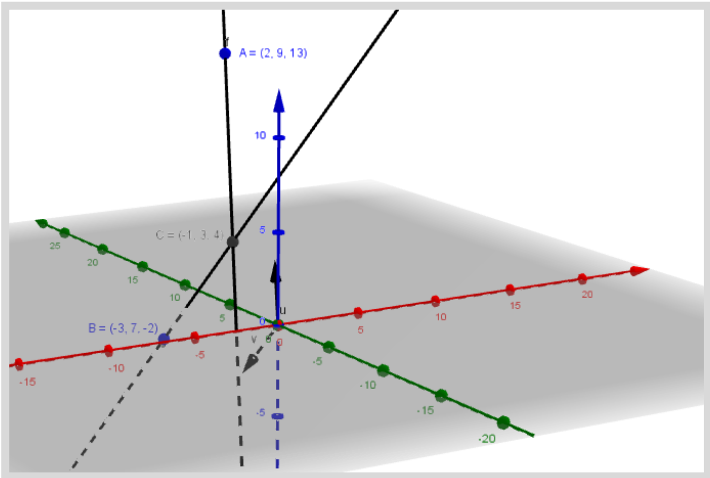


$(b)\quad P(p,q,r) \quad ; \quad \dfrac{p-2}{1} = \lambda \qquad \dfrac{p+3}{-1} = \mu \quad ; \quad \lambda + 2 = -\mu - 3 \quad ; \quad \lambda + \mu = -5$

$; \quad \dfrac{q-9}{2} = \lambda \qquad \dfrac{q-7}{2} = \mu \quad ; \quad 2\lambda + 9 = 2\mu + 7 \quad ; \quad \lambda - \mu = -1$

$; \quad \dfrac{r-13}{3} = \lambda \qquad \dfrac{r+2}{-3} = \mu \quad ; \quad 3\lambda + 13 = -3\mu - 2 \quad ; \quad \lambda + \mu = -5$

$Solve\ sim.equations\ ;\ \lambda = -3\ \ and\ \ \mu = -2\ \ ;\ \ Point\ is\ (-1,3,4)$



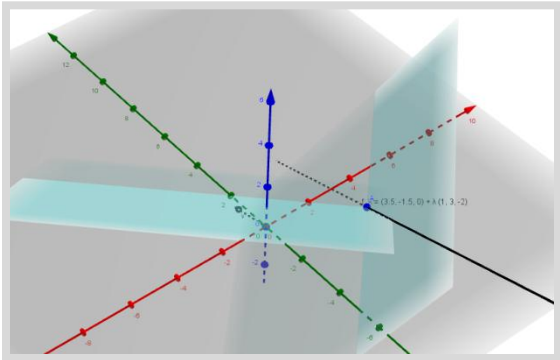
5. $(a)\quad x + y + 2z = 2 \quad and \quad x - y - z = 5$

$Eliminating\ y\ from\ the\ equations\ by\ adding\ gives\ ; \quad 2x + z = 7 \quad ; \quad z = -2x + 7$

$From\ equation\ 2\ ; \quad y = x - z - 5$

$Let\ x = t\ ; \quad z = -2t + 7 \quad ; \quad y = t - (-2t + 7) - 5 \qquad y = 3t - 12$

$Line\ \ \frac{x}{1} = \frac{y+12}{3} = \frac{z-7}{-2}\ ;\ other\ formats\ are\ possible\ !$



$(b)\quad 2x - y = 3 \quad and \quad x + y + 4z = 1$

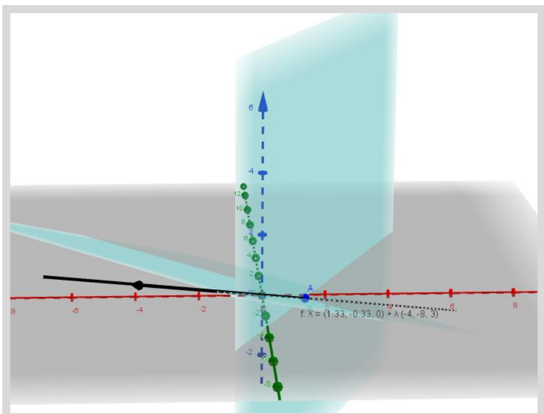
$Eliminating\ y\ from\ the\ equations\ by\ adding\ gives\ ; \quad 3x + 4z = 4 \quad ; \quad x = -\frac{4}{3}z + \frac{4}{3}$

$From\ equation\ 1\ ; \quad y = 2x - 3$

$Let\ z = t\ ; \quad x = -\frac{4}{3}t + \frac{4}{3} \quad ; \quad y = 2\left(-\frac{4}{3}t + \frac{4}{3}\right) - 3 \qquad y = -\frac{8}{3}t - \frac{1}{3}$

$\frac{x-\frac{4}{3}}{-\frac{4}{3}} = \frac{y+\frac{1}{3}}{-\frac{8}{3}} = \frac{z}{1}\quad ;\ multiply\ through\ by\ a\ \frac{1}{3}$

$Line\quad \frac{x-\frac{4}{3}}{-4} = \frac{y+\frac{1}{3}}{-8} = \frac{z}{3}\ ; \quad all\ formats\ are\ possible\ !$



$(c)\quad A:\ 2x + 3y + z = 8 \qquad B:\quad x + y + z = 10 \quad C:\ 3x + 5y + z = 6$

$The\ normal\ vector\ of\ A:\quad \underline{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

$The\ normal\ vector\ of\ B:\quad \underline{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$The\ normal\ vector\ of\ C:\quad \underline{c} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

$The\ direction\ vector\ of\ the\ line\ of\ intersection\ of\ A\ and\ B\ is\ parallel\ to\ \underline{a} \times \underline{b}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$If\ (\underline{a} \times \underline{b}).\underline{c}\ = 0,$

$the\ direction\ vector\ of\ the\ line\ of\ intersection\ is\ perpendicular\ to\ the\ normal\ vector\ of\ the\ plane\ C.$

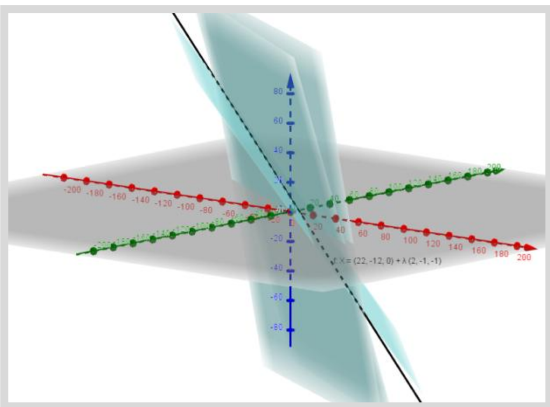
$i.e.\ the\ three\ plane\ intersect\ on\ a\ line.$

$Let\ z = 0\,, \qquad A: 2x + 3y = 8 \qquad B:\ \ x + y = 10 \qquad C: \quad 3x + 5y = 6$

$x = 22 \qquad y = -12 \qquad Hence\ point\ on\ the\ intersection\ line\ is\ (22, -12, 0)$

$The\ point\ also\ satifies\ C, since\ 3x + 5y + z = 6\ \ ;\ \ 3(22) + 5(-12) + 0 = 6$

$The\ line\ equation\ is \qquad \dfrac{x-22}{2} = \dfrac{y+12}{-1} = \dfrac{z}{-1}$



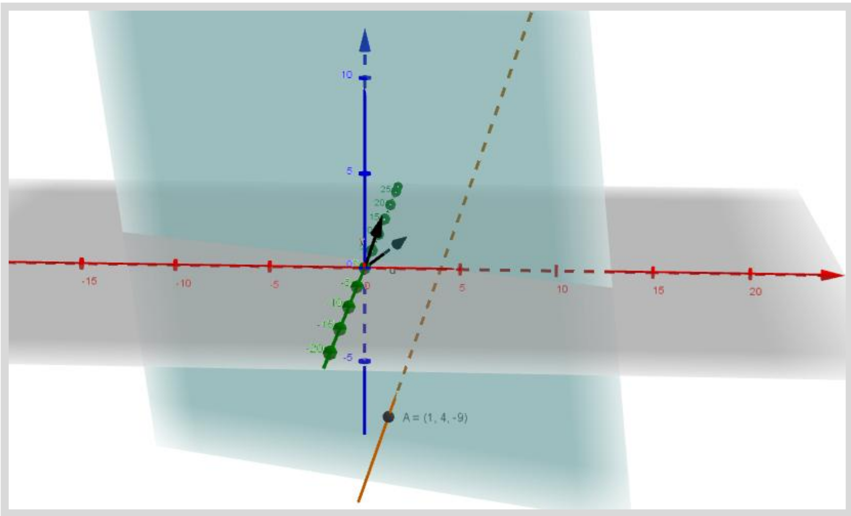
6. (a) $\quad Sub\ in\ for\ x, y\ and\ z\ in\ equation\ 2$

$$2(4+t)+4(1-t)+3t=9 \quad ; \quad 8+2t+4-4t+3t=9$$

$$t+12=9 \quad ; \quad t=-3$$

$$x=4+(-3)=1 \quad ; \quad y=1-t=1-(-3)=4 \quad ; \quad z=3t=3\times -3=-9$$

$Coordinate\ is\ (1,4,-9)$



$(b)\quad Rearrange\ equation\ 1\ into\ t =$

$x = 3t + 1 \quad ; \quad y = t + 2 \quad ; \quad z = 4t + 1$

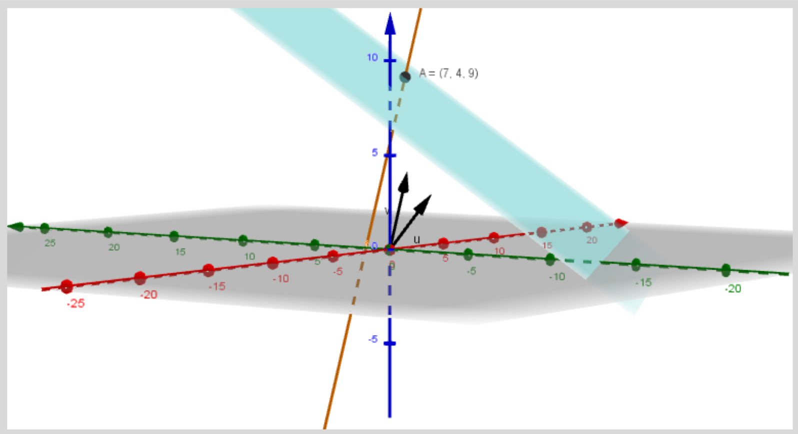
$Sub\ x, y\ and\ z\ into\ equation\ 2$

$(3t + 1) - 2(t + 2) + 3(4t + 1) = 26 \quad ; \quad 3t + 1 - 2t - 4 + 12t + 3 = 26$

$13t = 26 \quad ; \quad t = 2$

$x = 3 \times 2 + 1 = 7 \quad ; \quad y = 2 + 2 = 4 \quad ; \quad z = 4 \times 2 + 1 = 9$

$Coordinate\ is\ (7,4,9)$



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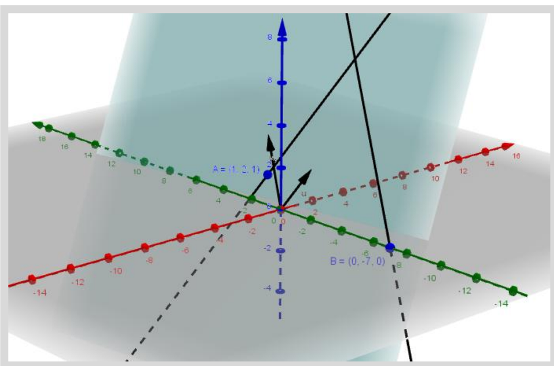
7. $P\ (1,2,1) \quad line\ direction\ \underline{d}_1 = (1,-1,2)\ \ Plane\ direction\ \underline{d}_2 = (1,2,3)$

$$\underline{n} = \underline{d}_1 \times \underline{d}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \left(-7\underline{i} - \underline{j} + 3\underline{k}\right) \quad ; \quad -7x - y + 3z = k$$

$$\underline{n}.\underline{p} = \begin{pmatrix} -7 \\ -1 \\ 3 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = -7 - 2 + 3 = -6 \quad ; \quad -7x - y + 3z = -6\ ; \quad 7x + y - 3z = 6$$



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8. (a) $\quad$ *Rearrange line equation into* $t =$

$x = 2t - 1 \quad ; \quad y = -t + 2 \quad ; \quad z = 2t - 3$

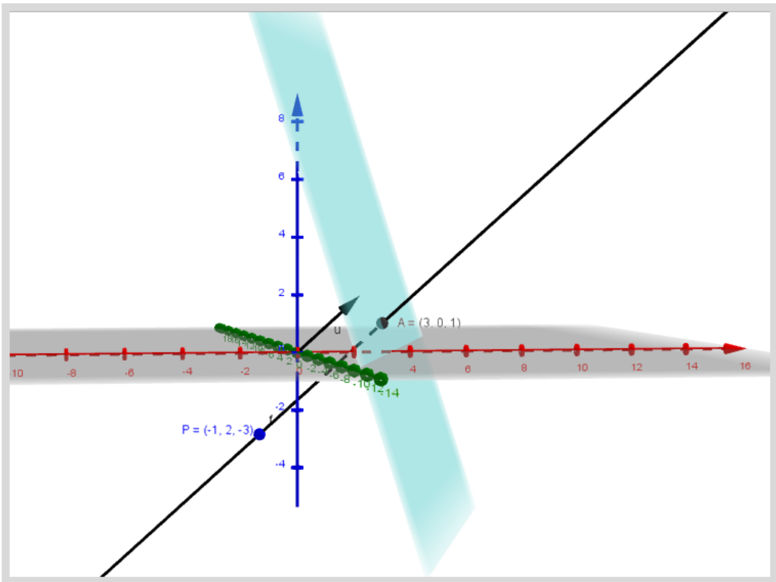
Sub x, y *and* z *into plane*

$3(2t - 1) - (-t + 2) + (2t - 3) = 10 \quad ; \quad 6t - 3 + t - 2 + 2t - 3 = 10$

$9t = 18 \quad ; \quad t = 2$

$x = 2 \times 2 - 1 = 3 \quad ; \quad y = -2 + 2 = 0 \quad ; \quad z = 2 \times 2 - 3 = 1$

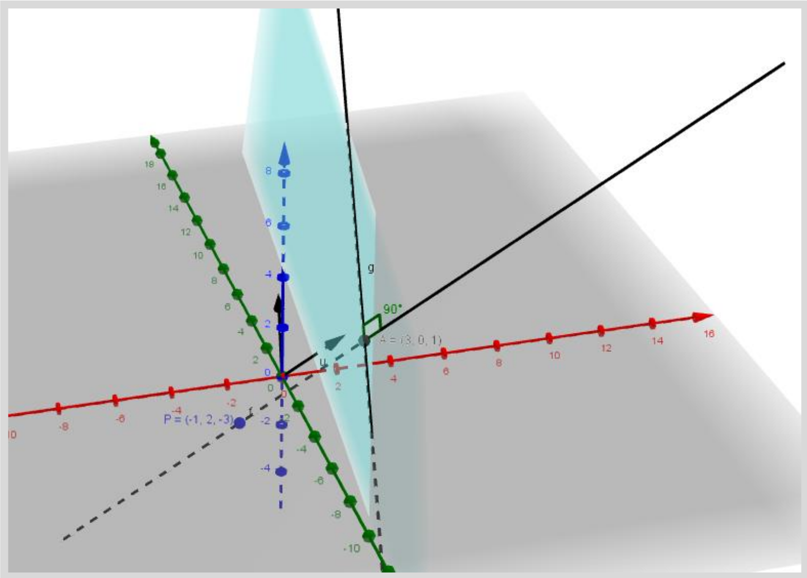
Coordinate is $(3,0,1)$



$(b)\quad \underline{d}_1 = (2,-1,2)\ \ Plane\ direction\ \underline{d}_2 = (3,-1,1)$

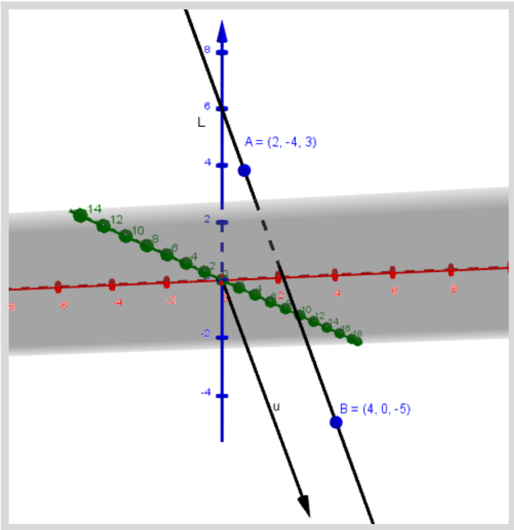
$$\underline{n} = \underline{d}_1 \times \underline{d}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 2 \\ 3 & -1 & 1 \end{vmatrix} = \left(\underline{i} + 4\underline{j} + \underline{k}\right) \quad ; \quad -7x - y + 3z = k$$

$$\underline{r} = \underline{a} + t\underline{d} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \quad ; \quad Symmetrical\ Form \quad \frac{x-3}{1} = \frac{y}{4} = \frac{z-1}{1}$$



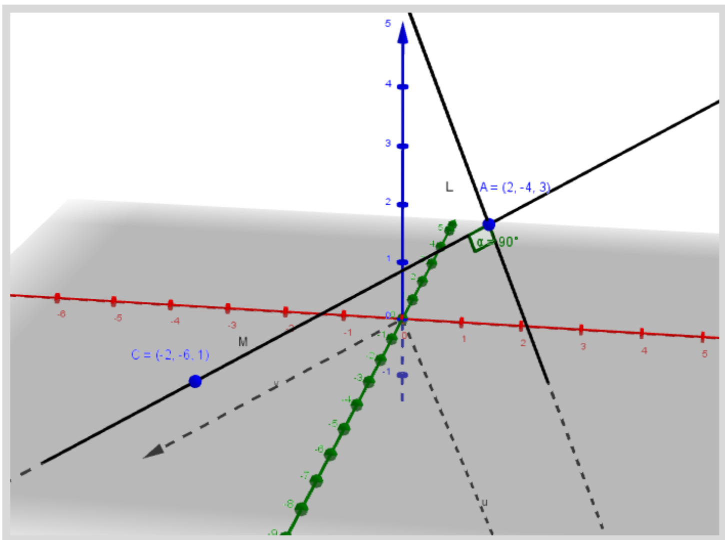
9. $(a)\quad \overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -8 \end{pmatrix} \quad ; \quad \textit{direction vector reduces to } \underline{d} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

$\underline{r} = \underline{a} + t\underline{d} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \quad ; \textit{Parametric form} \quad x = 2 + t \ ; \ y = -4 + 2t \ ; \ z = 3 - 4t$



$(b)\quad \overrightarrow{AC} = \underline{b} - \underline{a} = \begin{pmatrix} -2 \\ -6 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ -2 \end{pmatrix} \quad ; \quad \textit{direction vector reduces to } \underline{d} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$\textit{If } \perp \textit{then} \quad \underline{d_1}.\underline{d_2} = 0 \ ; \quad \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 2 + 2 - 4 = 0 \quad ; \quad \textit{Hence } L \textit{ is } \perp M$



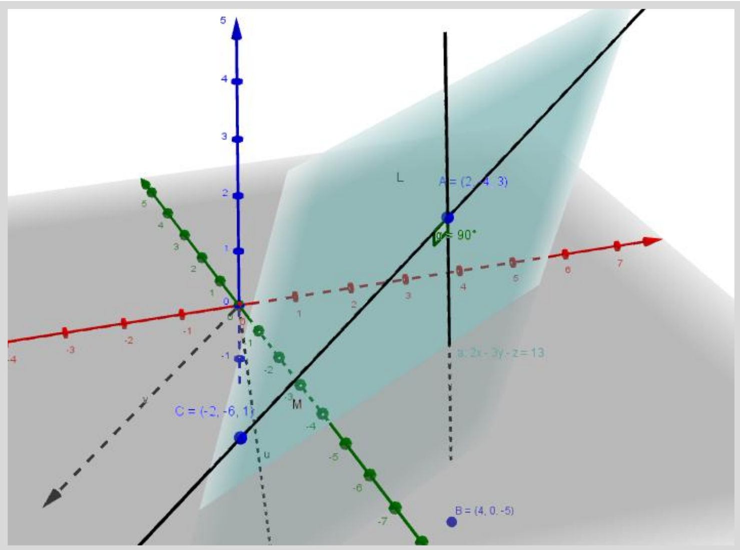
$(c)\ Plane\ containing\ L\ amd\ M$

$\underline{d}_1 = (2,1,1)\ \ Plane\ direction\ \underline{d}_2 = (1,2,-4)$

$$\underline{n} = \underline{d}_1 \times \underline{d}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 1 \\ 1 & 2 & -4 \end{vmatrix} = \left(-6\underline{i} + 9\underline{j} + 3\underline{k}\right) \quad ; \quad -6x + 9y + 3z = k$$

$$\underline{n}.\underline{p} = k \ \ ; \quad \begin{pmatrix} -6 \\ 9 \\ 3 \end{pmatrix} . \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} = -12 - 36 + 9 = -39 \quad ; \quad -6x + 9y + 3z = -39$$

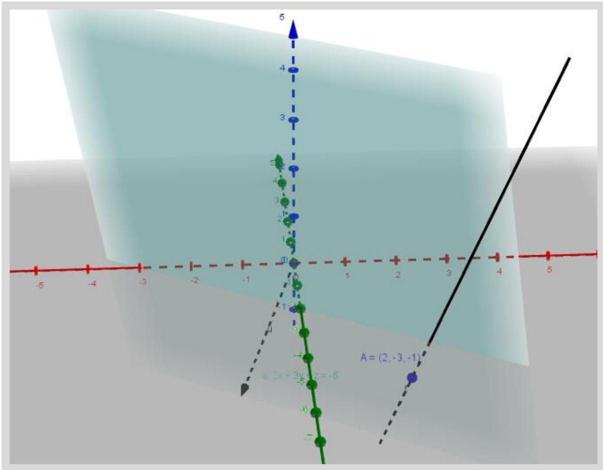
$2x - 3y - z = 13$



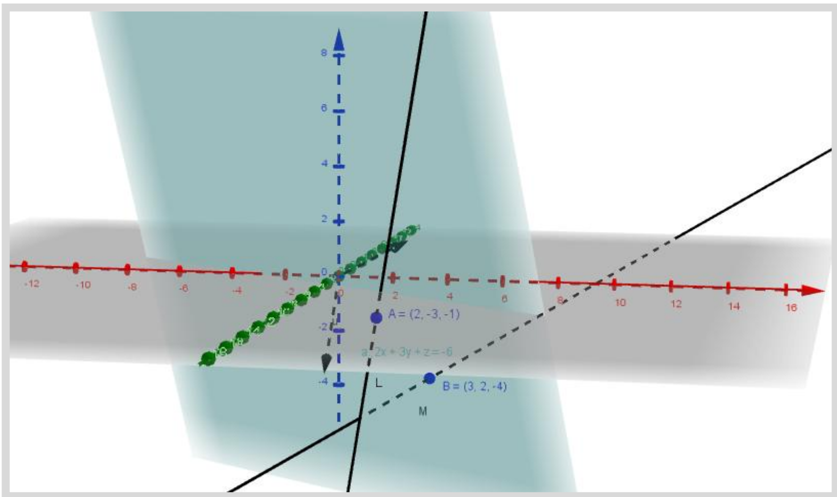
10. (a) $L\ parametric\ equations\ \ x = 2 - t \quad ; \quad y = -3 + 2t \quad ; \quad z = -1 - 4t$

$Sub\ x, y\ and\ z\ into\ plane\ \ 2x + 3y + z = -6$

$L.H.S \quad 2(2-t) + 3(-3+2t) + (-1-4t) = 4 - 2t - 9 + 6t - 1 - 4t = -6 = R.H.S$



(b) $\underline{r} = \underline{a} + t\underline{d} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad ; \quad x = 3 + 2t \quad y = 2 + 3t \quad z = -4 + t$



$(c)\quad Sub\ x, y\ and\ z\ into\ plane\ 2x + 3y + z = -6$

$2(3 + 2t) + 3(2 + 3t) + (-4 + t) = -6 \quad ; \quad 6 + 4t + 6 + 9t - 4 + t = -6$

$8 + 14t = -6 \quad ; \quad t = -1$

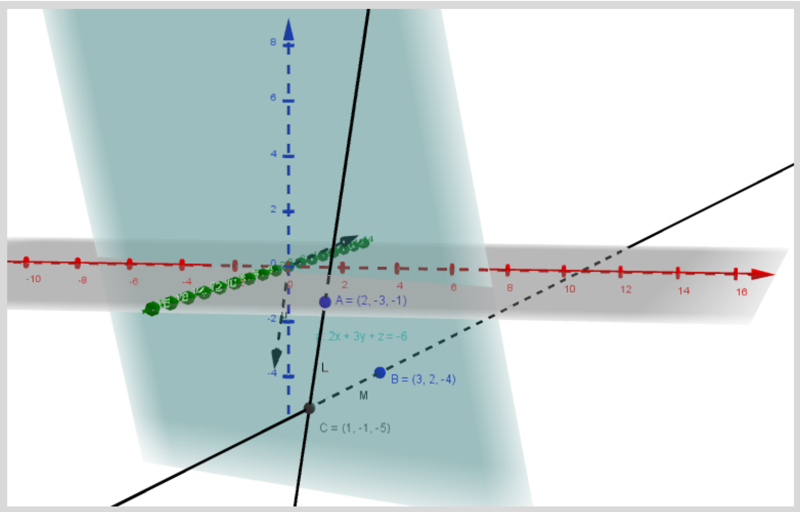
$x = 3 + 2 \times (-1) = 1 \quad ; \quad y = 2 + 3 \times (-1) = -1 \quad ; \quad z = -4 + (-1) = -5$

$Intersection\ Point\ between\ M\ and\ the\ Plane\ is\ (1, -1, -5)$

$If\ (1, -1, -5)\ lies\ on\ L\ then\ it\ will\ satisfy\ L\ equations.$

$1 = 2 - t \quad \Rightarrow t = 1 \quad ; \quad -1 = -3 + 2t \quad \Rightarrow t = 1 \quad ; \quad -5 = -1 - 4t \quad \Rightarrow t = 1$

$Hence\ (1, -1, -5)\ lies\ on\ L.$



11. (a) $\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$; $direction\ vector\ \underline{d_1} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

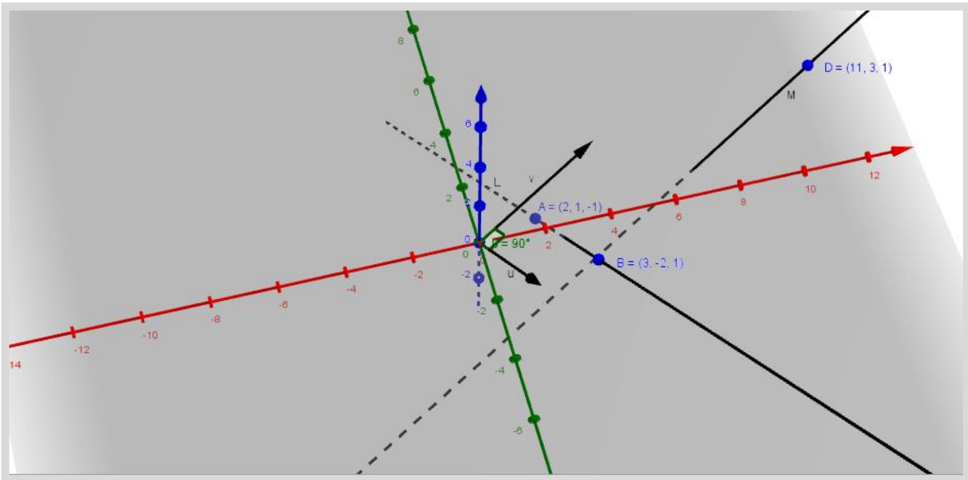
$$\underline{r} = \underline{a} + t\underline{d} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

$L\ Parametric\ form\ \ x = 2 + t\ ;\ y = -4 + 2t\ ;\ z = 3 - 4t$

$M\ Parametric\ form\ \ x = 11 + 4t\ ;\ y = 3 + 2t\ ;\ z = 1 + t$

$\frac{x - 11}{4} = \frac{y - 3}{2} = \frac{z - 1}{1}$ $direction\ vector\ \underline{d_2} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$

$If\ LM\ \perp\ then\ \ \underline{d_1}.\underline{d_2} = 0$; $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}.\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = 4 - 6 + 2 = 0$; $Hence\ L\ is\ \perp\ to\ M$



$(b)\quad P(2,1,-1)\quad \underline{d_2} = \begin{pmatrix}4\\2\\1\end{pmatrix}$ *direction of M*

Plane is $\perp$ *to M we have* $4x + 2y + z = k$

$$\underline{n}.\underline{p} = k \;\; ; \;\; \begin{pmatrix}4\\2\\1\end{pmatrix}.\begin{pmatrix}2\\1\\2\end{pmatrix} = 8 + 2 - 1 = 9 \quad ; \quad 4x + 2y + z = 9$$

Plane and line M intersect at C $\qquad M: \quad x = 11 + 4t \; ; \; y = 3 + 2t \; ; \; z = 1 + t$

Sub x, y *and* z *into plane* $4x + 2y + z = 9$

$4(11 + 4t) + 2(3 + 2t) + (1 + t) = -6 \quad ; \quad 44 + 16t + 6 + 4t + 1 + t = 9$

$51 + 21t = 9 \quad ; \quad t = -2$

$x = 11 + 4 \times (-2) = 3 \quad ; \quad y = 3 + 2 \times (-2) = -1 \quad ; \quad z = 1 + (-2) = -1$

Intersection Point C between M and the Plane is $(3, -1, -1)$

$A(2,1,-1) \qquad B(3,-2,1) \qquad C(3,-1,-1)$

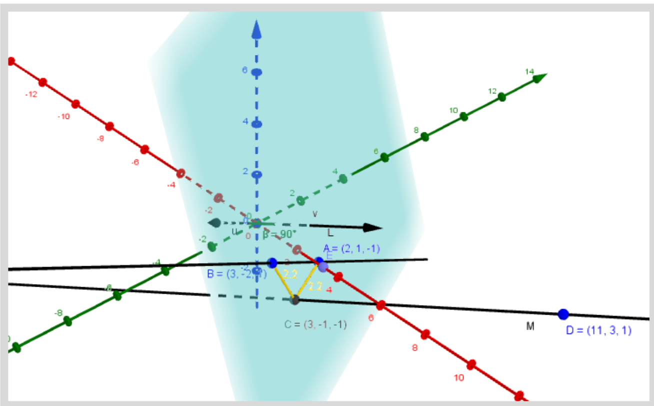
$Distance\ AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$Distance\ AC = \sqrt{(3 - 2)^2 + (-1 - 1)^2 + (-1 - (-1))^2} = \sqrt{1 + 4 + 0} = \sqrt{5}$

$Distance\ BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$Distance\ BC = \sqrt{(3 - 3)^2 + (-1 - (-2))^2 + (-1 - 1)^2} = \sqrt{0 + 1 + 4} = \sqrt{5}$

Hence C is equidistant from A and B.

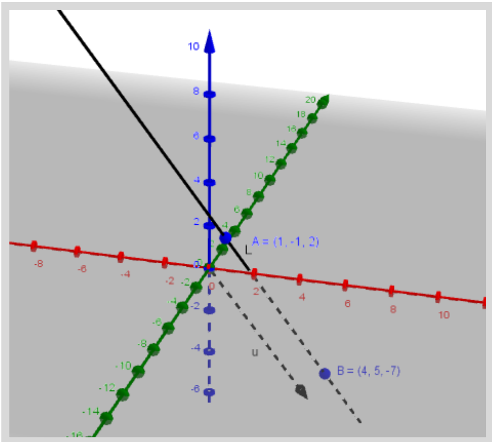


12. $(a)\quad \overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 4 \\ 5 \\ -7 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -9 \end{pmatrix} \quad ; \quad direction\ vector\ reduces\ to\ \underline{d_1} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

$$\underline{r} = \underline{a} + t\underline{d} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

$Parametric\ form\quad x = 1 + t\ ;\ y = -1 + 2t\ ;\ z = 2 - 3t$

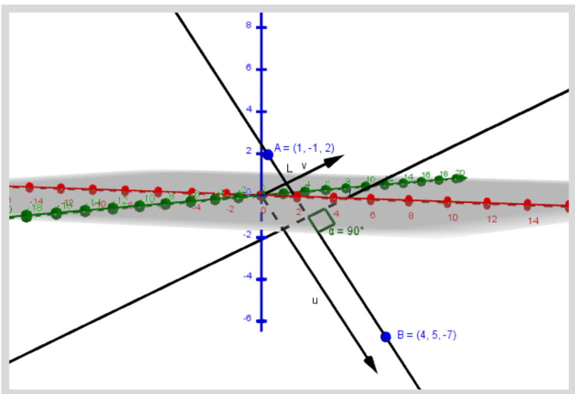
$$\frac{x-1}{1} = \frac{y+1}{2} = \frac{z-2}{-3}$$



$(b)\quad Parametric\ form\quad x = 6 + 4t\ ;\ y = 2 + t\ ;\ z = 1 + 2t$

$\frac{x-6}{4} = \frac{y-2}{1} = \frac{z-1}{2}\quad direction\ vector\ \underline{d_2} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

$If\ lines\ are\ \perp\ then\quad \underline{d_1}.\underline{d_2} = 0\ ;\quad \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}.\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = 4 + 2 - 6 = 0\quad ;\quad Hence\ Lines\ are\ \perp$



$(c) \quad P(p,q,r) \quad ; \quad \dfrac{p-1}{1}=\lambda \qquad \dfrac{p-6}{4}=\mu \quad ; \quad \lambda+1=4\mu+6 \quad ; \quad \lambda-4\mu=5$

$; \quad \dfrac{q+1}{2}=\lambda \qquad \dfrac{q-2}{1}=\mu \quad ; \quad 2\lambda-1=\mu+2 \quad ; \quad 2\lambda-\mu=3$

$; \quad \dfrac{r-2}{-3}=\lambda \qquad \dfrac{r-1}{2}=\mu \quad ; \quad -3\lambda+2=2\mu+1 \quad ; \quad 3\lambda+2\mu=1$

$Solve\ sim.equations \ ; \ \lambda=1 \ \ and \ \ \mu=-1 \quad ; \ \ Point\ is\ (2,1,-1)$