Mathematics

Mathematics 1, 2 and 3
(Supplementary Pack)
Advanced Higher

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HIGHER STILL

Mathematics

Mathematics 1, 2 and 3
(Supplementary Pack)

Advanced Higher

Support Materials

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Mathematics

Mathematics 1 (AH)

MATHEMATICS 1 (AH)

Introduction

These support materials for Mathematics were developed as part of the Higher Still Development Programme in response to needs identified at needs analysis meetings and national seminars.

Advice on learning and teaching may be found in *Achievement for All* (SOEID 1996), *Effective Learning and Teaching in Mathematics* (SOEID 1993), *Improving Mathematics* (SEED 1999) and in the Mathematics Subject Guide.

These materials were originally issued to schools to support CSYS Mathematics. They have now been updated and matched to the Arrangements for Advanced Higher Mathematics to be issued as part of the support material being provided for mathematics.

It should be noted that not all of the content of Mathematics 1 (AH) is covered by this support material. The outcomes covered are Algebra, Differentiation, Integration and Properties of Functions. At the start of each outcome a list of the content covered by the material has been included.

MATHEMATICS 1 (AH)

CONTENT
Algebra 1.1.6 express a proper rational function as a sum of partial fractions (denominator of degree at most 3 and easily factorised) **include cases where an improper rational function is reduced to a polynomial and a proper rational function by division or otherwise [A/B]**

Comment

The denominator may include a repeated linear factor or an irreducible quadratic factor. This is also required for integration of rational functions and useful for graph sketching where asymptotes are present.

The content within Algebra which, refers to the binomial theorem and corresponding notation, is **not** covered within this support material, i.e. 1.1.1, 1.1.2, 1.1.3, 1.1.4 and 1.1.5.

MATHEMATICS 1 (AH): ALGEBRA
Rational functions and partial fractions

Suppose you are given $\dfrac{x}{x^2+1}+\dfrac{1}{x-1}$

We know how to express this as a single rational function

$$\frac{x}{x^2+1}+\frac{1}{x-1}=\frac{x(x-1)+1(x^2+1)}{(x^2+1)(x-1)}=\frac{x^2-x+x^2+1}{(x^2+1)(x-1)}=\frac{2x^2-x+1}{(x^2+1)(x-1)}=\frac{2x^2-x+1}{x^3-x^2+x-1}$$

We call an expression such as the above, a **rational function**, $\dfrac{x}{x^2+1}$ and $\dfrac{1}{x-1}$ the corresponding **partial fractions.**

It is useful to be able to reverse the above process i.e. starting with the rational function, to rewrite it as a sum of partial fractions.

Suppose our rational function is $\dfrac{P(x)}{D(x)}$, where $P(x)$ and $D(x)$ are polynomials.

Step 1 - We must ensure that the degree of $P(x) <$ the degree of $D(x)$ by dividing if necessary.

Step 2 - Factorise $D(x)$. Each of the factors of $D(x)$ will become the denominator of a partial fraction.

(The corresponding numerator of the partial fractions will be of a degree less than that of the denominator.)

Example 1

Express $\dfrac{x+16}{(2x-3)(x+2)}$ as partial fractions

Let $\dfrac{x+16}{(2x-3)(x+2)}=\dfrac{A}{2x-3}+\dfrac{B}{x+2}$, where A and B are constants

Then $x + 16 = A(x + 2) + B(2x - 3)$

Put $x=-2$ gives $14=-7B \Rightarrow B=-2$

Put $x=\dfrac{3}{2}$ gives $17\tfrac{1}{2}=3\tfrac{1}{2}A \Rightarrow A=5$

Hence $\dfrac{x+16}{(2x-3)(x+2)}=\dfrac{5}{2x-3}-\dfrac{2}{x+2}$

Example 2

Express $\dfrac{2x^3+7x^2-2x-2}{2x^2+x-6}$ as partial fractions.

Here the degree of the numerator is > the degree of the denominator and so we must divide out.

$$\frac{2x^3+7x^2-2x-2}{2x^2+x-6}=x+3+\frac{x+16}{2x^2+x-6}$$

$$\begin{array}{r} x+3 \\ 2x^2+x-6\overline{\smash{)}2x^3+7x^2-2x-2} \\ \underline{2x^3+x^2-6x} \\ 6x^2+4x-2 \\ \underline{6x^2+3x-18} \\ x+16 \end{array}$$

But $\dfrac{x+16}{2x^2+x-16}=\dfrac{5}{2x-3}-\dfrac{2}{x+2}$

Hence $\dfrac{2x^3+7x^2-2x-2}{2x^2+x-6}=x+3+\dfrac{5}{2x-3}-\dfrac{2}{x+2}$

Examples for class

Express as partial fractions

(1) $\dfrac{5x+1}{(x+5)(x-3)}$ (2) $\dfrac{4x-19}{(x-1)(x-2)}$ (3) $\dfrac{14x}{x^2+x-12}$ (4) $\dfrac{8-x}{1+x-6x^2}$

(5) $\dfrac{x^2-6x-7}{(x-1)(x-2)(x+3)}$ (6) $\dfrac{x^3-2x-13}{x^2-2x-3}$

Answers

(1) $\dfrac{3}{x+5}+\dfrac{2}{x-3}$ (2) $\dfrac{15}{x-1}-\dfrac{11}{x-2}$ (3) $\dfrac{8}{x+4}+\dfrac{6}{x-3}$ (4) $\dfrac{5}{3x+1}+\dfrac{3}{1-2x}$

(5) $\dfrac{3}{x-1}-\dfrac{3}{x-2}+\dfrac{1}{x+3}$ (6) $x+2+\dfrac{9}{2(x-3)}+\dfrac{1}{2(x+1)}$

Example 3

Express $\dfrac{2x^2-5x+10}{(x+2)(x^2+x+5)}$ in partial fractions.

x^2+x+5 has no real factors (irreducible quadratic), and so we shall have two fractions with denominators $(x+2)$ and x^2+x+5.

The numerator of the first will be a constant as before but the numerator of the second might be of the first degree in x. Hence,

let $\dfrac{2x^2-5x+10}{(x+2)(x^2+x+5)}=\dfrac{A}{x+2}+\dfrac{Bx+C}{x^2+x+5}$ where A, B, C are constants.

Then $2x^2-5x+10=A(x^2+x+5)+(Bx+C)(x+2)$

Put $x=-2$ gives $28=7A \Rightarrow A=4$

Put $x=0$ gives $10=5A+2C \Rightarrow 2C=10-20=-10 \Rightarrow C=-5$

Put $x=1$ gives $7=7A+3(B+C) \Rightarrow 7=28+3(B-5) \Rightarrow 3(B-5)=-21$

$\Rightarrow B-5=-7$

$\Rightarrow B=-2$

Hence $\dfrac{2x^2-5x+10}{(x+2)(x^2+x+5)}=\dfrac{4}{x+2}+\dfrac{(-2x-5)}{x^2+x+5}=\dfrac{4}{x+2}-\dfrac{2x+5}{x^2+x+5}$

N.B. The number of constants required = the degree of the denominator of the rational function.

Example 4

Express $\dfrac{7x-10}{(3x-4)(x-1)^2}$ as partial fractions.

Let $\dfrac{7x-10}{(3x-4)(x-1)^2}=\dfrac{A}{3x-4}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$, where A, B, C are constants.

Then $7x-10=A(x-1)^2+B(3x-4)(x-1)+C(3x-4)$

Put $x=1$ gives $-3=-C \Rightarrow C=3$

Put $x=\frac{4}{3}$ gives $\frac{-2}{3}=\frac{1}{9}A \Rightarrow A=-6$

Put $x=0$ gives $-10=A+4B-4C \Rightarrow 4B=-10+6+12=8 \Rightarrow B=2$

Hence $\dfrac{7x-10}{(3x4)(x-1)^2}=\dfrac{-6}{3x-4}+\dfrac{2}{x-1}+\dfrac{3}{(x-1)^2}$

Examples for class

Express as partial fractions:

(1) $\dfrac{x^2+8x+1}{(2-x)(1+x+x^2)}$ (2) $\dfrac{3x^2+x+1}{x(x+1)^2}$ (3) $\dfrac{x^2-2x+10}{(x+2)(x-1)^2}$

(4) $\dfrac{x^2+4x+7}{(x+2)(x+3)^2}$ (5) $\dfrac{3x^2-x-2}{(1+2x)(x+2)^2}$ (6) $\dfrac{3x^2+92x}{(x^2+1)(x+6)}$

Answers

(1) $\dfrac{3}{2-x}+\dfrac{2x-1}{x^2+x+1}$ (2) $\dfrac{1}{x}+\dfrac{2}{x+1}-\dfrac{3}{(x+1)^2}$ (3) $\dfrac{2}{x+2}-\dfrac{1}{x-1}+\dfrac{3}{(x-1)^2}$

(4) $\dfrac{3}{x+2}-\dfrac{2}{x+3}-\dfrac{4}{(x+3)^2}$ (5) $\dfrac{-3}{2x+1}+\dfrac{7}{x+2}-\dfrac{4}{(x+2)^2}$ (6) $\dfrac{-12}{x+6}+\dfrac{15x+2}{x^2+1}$

MATHEMATICS 1 (AH)

CONTENT
Differentiation
1.2.1 know the meaning of the terms limit, derivative, differentiable at a point, differentiable on an interval, derived function, second derivative
1.2.2 use the notation: $f'(x), f''(x), \frac{dy}{dx}, \frac{d^2y}{dx^2}$
1.2.3 recall the derivatives of x^α (α rational), $\sin x$ and $\cos x$
1.2.4 know and use the rules for differentiating linear sums, products, quotients and composition of functions: $(f(x)+g(x))' = f'(x)+g'(x)$ $(kf(x))' = kf'(x)$, where k is a constant the chain rule: $(f(g(x)))' = f'(g(x)).g'(x)$ the product rule: $(f(x).g(x))' = f'(x).g(x) + f(x).g'(x)$ the quotient rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x).g(x) - f(x).g'(x)}{(g(x))^2}$ **differentiate given functions which require more than one application of one or more of the chain rule, product rule and the quotient rule [A/B]**
1.2.5 know the derivative of $\tan x$ the definitions and derivatives of $\sec x$, $\operatorname{cosec} x$ and $\cot x$ the derivatives of e^x ($\exp x$) and $\ln x$
1.2.6 know the definition $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
1.2.7 know the definition of higher derivatives $f^n(x), \frac{d^ny}{dx^n}$

The content listed below is **not** covered in this support material:

1.2.8 Apply differentiation to:

a) rectilinear motion
b) extrema of functions: the maximum and minimum values of a continuous function f defined on a closed interval $[a, b]$ can occur at stationary points, end points **or points where f' is not defined [A/B]**
c) optimisation problems

MATHEMATICS 1 (AH) : DIFFERENTIATION

General rules of differentiation

The following derivatives have already been met with Higher work.

$$\frac{d}{dx}x^n = n\,x^{n-1}$$

$$\frac{d}{dx}(ax+b)^n = n\,a\,(ax+b)^{n-1}$$

$$\frac{d}{dx}\sin(ax) = a\cos(ax)$$

$$\frac{d}{dx}\cos(ax) = -a\sin(ax)$$

Also the following rule:-

Chain Rule

If y is a function of u and u is a function of x, then if y is now regarded as a function of x.

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Example 1

Differentiate $y = \sin 5x$

Let $u = 5x$, then $y = \sin u$

$$\frac{du}{dx} = 5, \quad \frac{dy}{du} = \cos u = \cos 5x$$

Then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 5\cos 5x$

This process should normally be done mentally by "peeling" method.

Example 2

Differentiate $\cos^4 x$

Let $f(x) = \cos^4 x = (\cos x)^4$

Then $f'(x) = 4(\cos x)^3.(-\sin x) = -4\sin x\cos^3 x$.

Example 3

Differentiate $(2x^2-3)^7$

Let $f(x) = (2x^2-3)^7$

Then $f'(x) = 7(2x^2-3)^6.4x = 28x(2x^2-3)^6$

Examples for class.

Differentiate: (1) $\sqrt{2x+3}$ (2) $(4-5x)^{\frac{3}{2}}$ (3) $\dfrac{1}{\sqrt{2-x}}$ (4) $(x-\frac{1}{x})^3$

(5) $\dfrac{1}{(3x-5)^2}$ (6) $(2x^2-x)^{-\frac{1}{2}}$ (7) $(ax^2-b)^4$ (8) $\dfrac{a}{\sqrt{ax+b}}$

(9) $2\sin x+3\cos x$ (10) $\sin 3x$ (11) $\cos 5x$ (12) $\cos(2x+1)$

(13) $\sin(4x-7)$ (14) $\sin^2 x$ (15) $\cos^3 x$ (16) $\sin^2 3x$

Note: The derivative of a function f can either be written as

(1) $f'(x)$ or (2) $\dfrac{df}{dx}$ or (3) $\dfrac{d}{dx}(f(x))$

Answers

(1) $(2x+3)^{-½}$ (2) $-\dfrac{15}{2}(4-5x)^{½}$ (3) $\dfrac{1}{2}(2-x)^{-3/2}$ (4) $3(1+\dfrac{1}{x^2})(x-\dfrac{1}{x})^2$

(5) $-6(3x-5)^{-3}$ (6) $-\dfrac{1}{2}(4x-1)(2x^2-x)^{-3/2}$ (7) $8ax(ax^2-b)^3$

(8) $-\dfrac{1}{2}a^2(ax+b)^{-3/2}$ (9) $2\cos x-3\sin x$ (10) $3\cos 3x$ (11) $-5\sin x$

(12) $-2\sin(2x+1)$ (13) $4\cos(4x-7)$ (14) $2\sin x\cos x$

(15) $-3\cos^2 x\sin x$ (16) $6\sin 3x\cos 3x$

Definitions: New trigonometric functions

In order to extend the calculus, we require to define three new functions:

$\sec\theta=\dfrac{1}{\cos\theta}$; $\text{cosec}\,\theta=\dfrac{1}{\sin\theta}$; $\cot\theta=\dfrac{1}{\tan\theta}$

We also have $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$, $\sec^2\theta=1+\tan^2\theta$

Examples

1) $\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$; 2) $\operatorname{cosec}\frac{\pi}{2} = \frac{1}{\sin\frac{\pi}{2}} = \frac{1}{1} = 1$

3) $\cot\frac{\pi}{3}.\sin\frac{\pi}{6} = \frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}}.\sin\frac{\pi}{6} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}.\frac{1}{2} = \frac{1}{2}\times\frac{2}{\sqrt{3}}\times\frac{1}{2} = \frac{1}{2\sqrt{3}}$

4) Show that $\operatorname{cosec}^2\theta + \sec^2\theta = \operatorname{cosec}^2\theta \sec^2\theta$

$$\operatorname{cosec}^2\theta + \sec^2\theta = \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta}$$
$$= \frac{1}{\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta}.\frac{1}{\cos^2\theta}$$
$$= \operatorname{cosec}^2\theta.\sec^2\theta = \text{R.H.S.}$$

Thus identity is proved.

Examples for class

1. Evaluate

(1) $\operatorname{cosec}^2\frac{\pi}{3} - \tan^2\frac{\pi}{6}$ (2) $\cos\frac{\pi}{3}.\operatorname{cosec}^2\frac{\pi}{4}.\operatorname{cosec}^2\frac{\pi}{4}$

(3) $\sin\frac{\pi}{4}.\sin\frac{\pi}{6} + \cos\frac{\pi}{4}.\sin\frac{\pi}{3}.\tan\frac{\pi}{6}$ (4) $(1+\sin\frac{\pi}{4})(1-\cos\frac{\pi}{4})$

2. Evaluate

(1) $\cos\frac{\pi}{3}.\operatorname{cosec}^2\frac{\pi}{4}$ (2) $\sec^2\frac{\pi}{3} - \frac{\operatorname{cosec}\frac{\pi}{6}}{\sin^2\frac{\pi}{4}}$

(3) $\operatorname{cosec}^2\frac{\pi}{4}.\sin\frac{\pi}{6} - \sin^2\frac{\pi}{3} - \sin^2\frac{\pi}{6}$ (4) $(\sec\frac{\pi}{6} + \cot\frac{\pi}{3})(\operatorname{cosec}\frac{\pi}{3} - \tan\frac{\pi}{6})$

3. Simplify

(1) $\cot\theta\sin\theta$ (2) $\cot\theta\sec\theta$ (3) $\frac{\sec\theta}{\operatorname{cosec}\theta}$

(4) $\sin\theta\sec\theta$ (5) $\cos\theta\operatorname{cosec}\theta$ (6) $\frac{\cos\theta}{\sec\theta} + \frac{\sin\theta}{\operatorname{cosec}\theta}$

Answers

1\. (1) $\frac{2}{3}$ (2) 2 (3) $\frac{1}{\sqrt{2}}$ (4) $\frac{1}{2}$

2\. (1) 1 (2) 0 (3) $-\frac{3}{4}$ (4) 1

3\. (1) $\cos\theta$ (2) $\frac{1}{\sin\theta}$ (3) $\tan\theta$ (4) $\tan\theta$ (5) $\cot\theta$ (6) 1

Differentiation of a product of functions.

If u and v are functions of x, then $\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$.

Proof:

Let $f(x) = u(x).v(x)$

Then $f(x+h) = u(x+h).v(x+h)$

and $f(x+h) - f(x) = u(x+h).v(x+h) - u(x).v(x)$

$$= u(x+h).\{v(x+h) - v(x)\} + v(x).\{u(x+h) - u(x)\}$$

Hence

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{u(x+h).\{v(x+h) - v(x)\} + v(x).\{u(x+h) - u(x)\}}{h}$$

$$= \lim_{h\to 0} u(x+h).\frac{\{v(x+h) - v(x)\}}{h} + \lim_{h\to 0} v(x).\frac{\{u(x+h) - u(x)\}}{h}$$

$$= u(x)\frac{dv}{dx}(x) + v(x)\frac{du}{dx}(x)$$

Example 1

Differentiate $(3x^4 - 2)(2x^3 + 3x)$

$$\frac{d}{dx}(3x^4 - 2)(2x^3 + 3x) = (3x^4 - 2)(6x^2 + 3) + 12x^3(2x^3 + 3x)$$

$$= 42x^6 + 45x^4 - 12x^2 - 6$$

Example 2

Differentiate $\sin mx.\cos nx$

$$\frac{d}{dx}(\sin mx.\cos nx) = (\sin mx)(-n\sin nx) + (\cos nx)(m\cos mx)$$

$$= -n\sin mx\sin nx + m\cos mx\cos nx$$

$$= m\cos mx\cos nx - n\sin mx\sin nx$$

Example 3
Differentiate $u.v.w$, where u, v, w are functions of x.

$$\frac{d}{dx}\left[u.v.w\right]=\frac{d}{dx}\left[(u.v).w\right]=u.v.\frac{dw}{dx}+w\frac{d}{dx}(u.v)$$
$$=u.v.\frac{dw}{dx}+w.\left[u.\frac{dv}{dx}+v.\frac{du}{dx}\right]$$
$$=u.v.\frac{dw}{dx}+u.w.\frac{dv}{dx}+v.w.\frac{du}{dx}$$

Examples for class
Find the derivatives of:

(1) $(3x+1)(2x+1)$ (2) $(x^2+1)(\frac{1}{2}x+1)$ (3) $(3x-5)(x^2+2x)$

(4) $(x^2+3)(2x^2-1)$ (5) $(x^2-x+1)(x+1)$ (6) $(x^2+4x+5)(x^2-2)$

(7) $(x^2-x+1)(x^2+x-1)$ (8) $(x-2)(x^2+2x+4)$ (9) $(2x^2-3)(3x^2+x-1)$

(10) $\sqrt{x}\ (x^2+x+1)$ (11) $x\sin x$ (12) $\sin 2x\,.\cos 2x$

(13) $x^2\ \cos 2x$ (14) $x\sqrt{\sin x}$ (15) $(x^2+x+1)(x-1)$

(16) $(x^2+4x)(3x^2-x)$ (17) $(1-\cos x)(1+\cos x)$

(18) $2x^{\frac{3}{2}}\ (\sqrt{x}+2)(\sqrt{x}-1)$

Answers

(1) $12x+5$ (2) $\frac{3}{2}x^2+2x+\frac{1}{2}$ (3) $9x^2+2x-10$ (4) $8x^3+10x$ (5) $3x^2$

(6) $4x^3+12x^2+6x-8$ (7) $4x^3+2x$ (8) $3x^2$ (9) $24x^3+6x^2-22x-3$

(10) $\frac{5}{2}x^{3/2}+\frac{3}{2}x^{1/2}+\frac{1}{2}x^{-1/2}$ (11) $\sin x+x\cos x$ (12) $-2\sin^2 2x+2\cos^2 2x$

(13) $2x\cos 2x-2x^2\sin 2x$ (14) $\sqrt{\sin x}+\frac{x\cos x}{2\sqrt{\sin x}}$ (15) $3x^2$

(16) $12x^3+33x^2-8x$ (17) $-2\sin x\cos x$ (18) $5x^{3/2}+4x-6x^{1/2}$

Differentiation of a quotient of two functions

If u and v are functions of x, then $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Proof:

Let $f(x) = \dfrac{u(x)}{v(x)}$, then $f(x+h) = \dfrac{u(x+h)}{v(x+h)}$

$$f(x+h) - f(x) = \frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)} = \frac{u(x+h)\,v(x) - u(x)\,v(x+h)}{v(x+h)\,v(x)}$$

$$\begin{aligned}
\text{Hence } f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{v(x)\{u(x+h)-u(x)\} - u(x)\{v(x+h)-v(x)\}}{h\,v(x+h)\,v(x)} \\
&= \lim_{h\to 0}\frac{v(x)\left\{\dfrac{u(x+h)-u(x)}{h}\right\} - u(x)\left\{\dfrac{v(x+h)-v(x)}{h}\right\}}{v(x+h)\,v(x)} \\
&= \frac{v(x)\dfrac{d}{dx}u(x) - u(x)\dfrac{d}{dx}v(x)}{v^2(x)}
\end{aligned}$$

Example 1

Differentiate $\dfrac{x^2-1}{x^2+1}$

$$\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{(x^2+1)\,2x - (x^2-1)\,2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

Example 2

Differentiate $\cot x$

$$\begin{aligned}
\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) &= \frac{\sin x\,(-\sin x) - \cos x.\cos x}{\sin^2 x} \\
&= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\
&= \frac{-1}{\sin^2 x} \\
&= -\operatorname{cosec}^2 x
\end{aligned}$$

Examples for class

Differentiate

(1) $\frac{3}{2x-1}$ (2) $\frac{1}{1-3x^2}$ (3) $\frac{x}{x-2}$ (4) $\frac{x^2}{x-4}$ (5) $\frac{x^2+x+1}{x^2-x+1}$

(6) $\frac{2x^2-x+1}{3x^2+x-1}$ (7) $\frac{1+x+x^2}{x}$ (8) $\tan x$ (9) $\sec x$

(10) $\operatorname{cosec} x$ (11) $\frac{\sin^2 x}{1+\sin x}$ (12) $\frac{1-\cos x}{1+\cos x}$

N.B. $\frac{d}{dx}\tan x = \sec^2 x$

Answers

(1) $\frac{-6}{(2x-1)^2}$ (2) $\frac{6x}{(1-3x^2)^2}$ (3) $\frac{-2}{(x-2)^2}$ (4) $\frac{x^2-8x}{(x-4)^2}$ (5) $\frac{-2x^2+2}{(x^2-x+1)^2}$

(6) $\frac{5x^2-7x}{(3x^2+x-1)^2}$ (7) 1 - $\frac{1}{x^2}$ (8) $\sec^2 x$ (9) $\frac{\sin x}{\cos^2 x}(\equiv \tan x \sec x)$

(10) $-\frac{\cos x}{\sin^2 x}(\equiv -\cot x \operatorname{cosec} x)$ (11) $\frac{\sin x \cos x(2+\sin x)}{(1+\sin x)^2}$ (12) $\frac{2\sin x}{(1+\cos x)^2}$

Example 3

Differentiate $\frac{2x}{\sqrt{x^2+1}}$

$$\frac{d}{dx}\frac{2x}{\sqrt{x^2+1}} = \frac{(\sqrt{x^2+1}).2-(2x).\frac{1}{2_1.\sqrt{x^2+1}}.2_1 x}{(\sqrt{x^2+1})^2} = \frac{2.\sqrt{x^2+1}-\frac{2x^2}{\sqrt{x^2+1}}}{(x^2+1)^1}.\frac{\frac{\sqrt{x^2+1}}{1}}{\frac{\sqrt{x^2+1}}{1}}$$

$$= \frac{2(x^2+1)-2x^2}{(x^2+1)^{\frac{3}{2}}} = \frac{2}{(x^2+1)^{\frac{3}{2}}}$$

Note: We could have used the product rule by taking $\frac{u}{v}$ as $u.v^{-1}$.

Examples for class

Differentiate

(1) $\dfrac{\sqrt{x}}{x+1}$ (2) $\dfrac{\sqrt{x+1}}{\sqrt{x-1}}$ (3) $\dfrac{x^{\frac{1}{2}}+2}{x^{\frac{3}{2}}}$

(4) $\dfrac{1}{\sqrt{x^2-1}}$ (5) $\dfrac{1}{\sqrt{2x^2-3x+4}}$ (6) $\dfrac{\sqrt{x}}{\sin x}$

Answers

(1) $\dfrac{x^{-\frac{1}{2}}-x^{\frac{1}{2}}}{2(x+1)^2}$ (2) $-\dfrac{1}{(x-1)^{\frac{1}{2}}(x+1)^{\frac{3}{2}}}$ (3) $-\dfrac{1}{x^2}-\dfrac{3}{x^{\frac{5}{2}}}=\dfrac{-x^{\frac{1}{2}}-3}{x^{\frac{5}{2}}}$

(4) $\dfrac{-x}{(x^2-1)^{\frac{3}{2}}}$ (5) $\dfrac{-4x+3}{2(2x^2-3x+4)^{\frac{3}{2}}}$ (6) $\dfrac{x^{\frac{1}{2}}(\frac{1}{2}x\sin x-\cos x)}{\sin^2 x}$

Higher derivatives

Let $y = 6x^2 + 3x + 4$ then $\frac{dy}{dx} = 12x + 3$

Here $\frac{dy}{dx}$ is defined by the above equation as a function of x and so we can differentiate this expression with respect to x.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 12$$

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is usually written as $\frac{d^2y}{dx^2}$ or y'' and is called the **second derivative of** y.

This process can be repeated to get 3rd, 4th, ..., nth derivatives which can be written as $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots\ldots, \frac{d^ny}{dx^n}$

Note: $\frac{d^2y}{dx^2} \neq \frac{1}{\frac{d^2x}{dy^2}}$

Example 1

$y = x^4 + 3x^3 + 1$

$$\frac{dy}{dx} = 4x^3 + 9x^2$$

$$\frac{d^2y}{dx^2} = 12x^2 + 18x$$

$$\frac{d^3y}{dx^3} = 24x + 18$$

$$\frac{d^4y}{dx^4} = 24$$

$$\frac{d^5y}{dx^5} = 0$$

$$\frac{d^ny}{dx^n} = 0$$

Example 2

$y = \log x$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}$$

$$\frac{d^3y}{dx^3} = \frac{2}{x^3}$$

$$\frac{d^4y}{dx^4} = \frac{-3.2}{x^4}$$

$$\frac{d^5y}{dx^5} = \frac{4.3.2}{x^5}$$

$$\frac{d^ny}{dx^n} = \frac{n!(-1)^{n+1}}{x^n}$$

Examples for class

(1) Write down the first and second derivatives of :

(a) $x^2(x-1)$ (b) $5x^4 - 3x^3 + 2x^2 - x + 1$ (c) $10x^4 - 4x^3 + 5x + 2$

(d) $\frac{1}{x}$ (e) $\sqrt{x}$ (f) $\sqrt{2x+1}$ (g) $\frac{1}{x^3}$ (h) $\frac{\sin x}{\sin x + \cos x}$

(2) Verify that $y = x \sin x$ is a solution of the differentiable equation

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2 + 2)y = 0$$

(3) Show that $y = \cos kx$ satisfies the differentiable equation $\frac{d^2y}{dx^2} + k^2 y = 0$

Answers

(1) (a) $\frac{dy}{dx} = 3x^2 - 2x$; $\frac{d^2y}{dx^2} = 6x - 2$

(b) $\frac{dy}{dx} = 20x^3 - 9x^2 + 4x - 1$; $\frac{d^2y}{dx^2} = 60x^2 - 18x + 4$

(c) $\frac{dy}{dx} = 40x^3 - 12x^2 + 5$; $\frac{d^2y}{dx^2} = 120x^2 - 24x$

(d) $\frac{dy}{dx} = -x^{-2}$; $\frac{d^2y}{dx^2} = 2x^{-3}$

(e) $\frac{dy}{dx} = \frac{1}{2}x^{-½}$; $\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-3/2}$

(f) $\frac{dy}{dx} = \frac{1}{2}(2x+1)^{-½}$; $\frac{d^2y}{dx^2} = -(2x+1)^{-3/2}$

(g) $\frac{dy}{dx} = -3x^{-4}$; $\frac{d^2y}{dx^2} = 12x^{-5}$

(h) $\frac{dy}{dx} = \frac{1}{\sin x + \cos x}$; $\frac{d^2y}{dx^2} = \frac{\sin x - \cos x}{(\sin x + \cos x)^2}$

(2) Proof (3) Proof

The Exponential and Logarithmic functions

The exponential function, exp x or e^x

For all finite x, we define $\exp x = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, where

$n! = n(n-1)(n-2)\ldots 3.2.1$

Exp x is called the exponential function to the base e and is quite frequently denoted by the symbol e^x.

Basic properties of e^x

1) Replace x by 1 in the series to obtain $e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = 2.7182812$

   We usually take $e = 2.72$, to 3 sig figs.

2) e is an irrational number.

3) $e^0 = 1 + \frac{0}{1!} + \frac{0^2}{2!} + \ldots = 1$

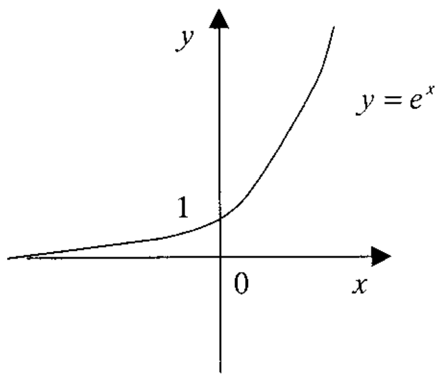
4) $$\begin{aligned} e^x.e^y &= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)\left(1 + \frac{y}{1!} + \frac{y^2}{2!} + \ldots\right) \\ &= 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{x}{1!} + \frac{xy}{1!} + \frac{x^2}{2!} + \ldots \\ &= 1 + \left(\frac{x}{1!} + \frac{y}{1!}\right) + \left(\frac{x^2}{2!} + \frac{xy}{1!} + \frac{y^2}{2!}\right) + \ldots \\ &= 1 + \frac{(x+y)}{1!} + \frac{(x^2 + 2xy + y^2)}{2!} + \ldots \\ &= 1 + \frac{(x+y)}{1!} + \frac{(x+y)^2}{2!} + \ldots \\ &= e^{x+y} \end{aligned}$$

5) $\frac{1}{e^x} = e^{-x}$

6) $e^x \to \infty$ as $x \to \infty$ and $e^x \to 0$ as $x \to -\infty$

7) e^x is always positive.

The graph of y = e^x



The derivative of e^x

$$\lim_{h\to 0}\frac{e^h-1}{h}=1$$

Proof: $$\frac{e^h-1}{h}=\frac{\left\{1+\frac{h}{1!}+\frac{h^2}{2!}+\ldots\right\}-1}{h}=\frac{\frac{h}{1!}+\frac{h^2}{2!}+\frac{h^3}{3!}+\ldots}{h}$$

$$=\frac{h\left(\frac{1}{1!}+\frac{h}{2!}+\frac{h^2}{3!}+\ldots\right)}{h}$$

$$=1+\frac{h}{2!}+\frac{h^2}{3!}+\ldots$$

as $h\to 0, \frac{e^h-1}{h}\to 1$, $\quad\therefore \lim_{h\to 0}\frac{e^h-1}{h}=1$

Theorem: $\frac{d}{dx}e^x=e^x$

Proof: Let $f(x)=e^x$, then $f(x+h)=e^{x+h}$

$$\frac{f(x+h)-f(x)}{h}=\frac{e^{x+h}-e^x}{h}=e^x\left(\frac{e^h-1}{h}\right)$$

Hence $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}e^x\left(\frac{e^h-1}{h}\right)=e^x\lim_{h\to 0}\left(\frac{e^h-1}{h}\right)$

$$=e^x$$

Examples

1) $\frac{d}{dx}e^{2x} = 2e^{2x}$

2) $\frac{d}{dx}e^{-x} = -e^{-x}$

3) $\frac{d}{dx}e^{\tan x} = \sec^2 x\, e^{\tan x}$

4) $\frac{d}{dx}xe^x = x\frac{d}{dx}e^x + e^x\frac{d}{dx}x = xe^x + e^x$

5) $\frac{d}{dx}\tan(e^x) = \left[\sec^2(e^x)\right]e^x = e^x\sec^2(e^x)$

Examples for class

Differentiate with respect to x:

1) (a) e^{5x} (b) $e^{\frac{x}{2}}$ (c) $e^{\sqrt{x}}$

2) (a) e^{-2x} (b) $e^{\frac{-5x}{2}}$ (c) $e^{(5-2x)}$

3) (a) xe^x (b) xe^{-x} (c) x^2e^{-x}

4) (a) $(x+4)e^x$ (b) $e^x\sin x$ (c) $10e^x$

5) (a) $e^{\sin x}$ (b) $e^{\cos x}$ (c) $e^{\tan x}$

6) $\frac{e^{ax}}{x^{\frac{1}{2}}}$ 7) x^2e^{4x} 8) $e^{ax}\sin^2 x$

9) xe^{x-x^2} 10) $x^2e^{\sin^2 x}$

Answers

1) (a) $5e^{5x}$ (b) $\frac{1}{2}e^{x/2}$ (c) $\frac{1}{2}x^{-1/2}e^{\sqrt{x}}$

2) (a) $-2e^{-2x}$ (b) $-\frac{5}{2}e^{-5/2x}$ (c) $-2e^{(5-2x)}$

3) (a) $(1+x)e^{x}$ (b) $(1-x)e^{-x}$ (c) $2xe^{-x}-x^2e^{-x}$

4) (a) $(5+x)e^{x}$ (b) $(\cos x+\sin x)e^{x}$ (c) $10e^{x}$

5) (a) $\cos x\, e^{\sin x}$ (b) $-\sin x\, e^{\cos x}$ (c) $\sec^2 x\, e^{\tan x}$

6) $\left(-\frac{1}{2}x^{-3/2}+ax^{-1/2}\right)e^{ax}$ 7) $(2x+4x^2)e^{4x}$ 8) $(a\sin^2 x+2\sin x\cos x)e^{ax}$

9) $(1+x-2x^2)\,e^{x-x^2}$ 10) $(1+x\sin x\cos x)2x\,e^{\sin^2 x}$

The Logarithmic Function (ln x).

The logarithmic function is defined to be the inverse function of the exponential function.

Thus $y=\ln x \Leftrightarrow x=e^{y}$.

When the base of a logarithm is e the logarithms are called **natural** or Naperian logarithms.

Basic properties.

1) $\ln ab=\ln a+\ln b$

2) $\ln\frac{1}{b}=-\ln b$

3) $\ln\frac{a}{b}=\ln a-\ln b$

4) $\ln a^{\frac{p}{q}}=\frac{p}{q}\ln a$

5) $\ln e=1$

6) $\ln 1=0$

Examples 1) $\ln e^3 = 3\ln e = 3$

2) $\ln\sqrt[3]{e} = \ln e^{\frac{1}{3}} = \frac{1}{3}\ln e = \frac{1}{3}$

3) Find x if $\ln x = 2$. $\ln x = 2 \Leftrightarrow x = e^2$

Examples for class

1) Find the values of (a) $\ln e^2$ (b) $\ln\frac{1}{e}$ (c) $\ln\sqrt{e}$

2) Find x in terms of e if

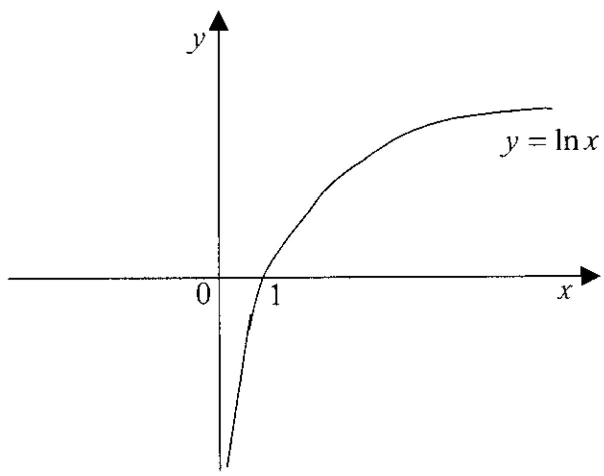
(a) $\ln x = 3$ (b) $\ln x = -2$ (c) $\ln x = 1 + \ln 2$ (d) $\ln(\ln x) = 0$

Answers

1) (a) 2 (b) –1 (c) $\frac{1}{2}$

2) (a) $x = e^3$ (b) $x = e^{-2}$ (c) $x = 2e$ (d) $x = e$

The graph of y = ln x



The derivative of ln x

Let $y = \ln x$

then $x = e^y$

and so $\dfrac{dx}{dy} = e^y = x$

Hence $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{x}$

i.e. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

Examples: 1) $\dfrac{d}{dx}\ln 2x = \dfrac{1}{2x}.2 = \dfrac{1}{x}$

2) $\dfrac{d}{dx}\ln 3x = \dfrac{1}{3x}.3 = \dfrac{1}{x}$

3) $\dfrac{d}{dx}\ln x^2 = \dfrac{1}{x^2}.2x = \dfrac{2}{x}$

4) $\dfrac{d}{dx}\ln\left(\dfrac{1+x}{1-x}\right) = \dfrac{d}{dx}[\ln(1+x) - \ln(1-x)$

$= \dfrac{1}{1+x} + \dfrac{1}{1-x} = \dfrac{2}{(1+x)(1-x)}$

Examples for class

Differentiate

1) (a) $\ln\dfrac{x}{a}$ (b) $\ln(ax^2+bx+c)$

2) (a) $\ln x^2$ (b) $\ln(x^3+3)$

3) $x\ln x$ 4) (a) $\ln\sin x$ (b) $\ln\cos x$ 5) $\ln\left(\dfrac{a+x}{a-x}\right)$

6) $\ln\left(e^x+e^{-x}\right)$ 7) $\ln\left(x+\sqrt{x^2+1}\right)$ 8) $\ln\sqrt{x^2+1}$

9) $\ln\tan\left(\dfrac{x}{2}\right)$ 10) $\ln\sqrt{\sin x}$ 11) $\ln\left(\dfrac{e^x}{1+e^x}\right)$

12) $\ln\left\{\sqrt{x-1}+\sqrt{x+1}\right\}$ 13) $\ln(\sec x+\tan)$

14) $\ln(\operatorname{cosec}x+\cot x)$ 15) $\ln(1-2\cos 2x)$

Answers

1) (a) $\dfrac{1}{x}$ (b) $\dfrac{2ax+b}{ax^2+bx+c}$ 2) (a) $\dfrac{2}{x}$ (b) $\dfrac{3x^2}{x^3+3}$

3) $1+\ln x$ 4) (a) $\dfrac{\cos x}{\sin x}\equiv\cot x$ (b) $-\dfrac{\sin x}{\cos x}\equiv-\tan x$

5) $-\dfrac{2x}{(a+x)(a-x)}$ 6) $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ 7) $\dfrac{1}{(x^2+1)^{½}}$

8) $\dfrac{x}{x^2+1}$ 9) $\dfrac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\equiv\dfrac{1}{\sin x}$ 10) $\dfrac{\cos x}{2\sin x}\equiv\dfrac{1}{2}\cot x$

11) $\dfrac{1}{1+e^x}$ 12) $\dfrac{1}{2\sqrt{(x-1)(x+1)}}$ 13) $\sec x$

14) $\dfrac{\cos x-1}{\sin x(\cos x+1)}$ 15) $\dfrac{4\sin 2x}{1-2\cos 2x}$

MATHEMATICS 1 (AH)

CONTENT

Integration

1.3.1 know the meaning of the terms integrate, integrable, integral, indefinite integral, definite integral and constant of integration

1.3.2 recall standard integrals of x^{α} ($\alpha \in \mathbf{Q}$, $\alpha \neq -1$), $\sin x$ and $\cos x$

$$\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx, a, b \in \mathbf{R}$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < b < c$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx\ , b \neq a$$

$$\int_b^a f(x)dx = F(b) - F(a)\ , \text{ where } F'(x) = f(x)$$

1.3.3 know the integrals of e^x, x^{-1}, $\sec^2 x$

1.3.4 integrate by substitution:
expressions requiring a simple substitution
expressions where the substitution will be given

the following special cases of substitution $\int f(ax+b)dx$ $\quad \int \frac{f'(x)}{f(x)}dx$

The content listed below is **not** covered within this support material:

1.3.5 use an elementary treatment of the integral as a limit using rectangles

1.3.6 apply integration to the evaluation of areas **including integration with respect to *y* [A/B]**

MATHEMATICS 1 (AH): INTEGRATION.

Introduction

Integration is defined to be the inverse process to differentiation.
Indeed it is often called **antidifferentiation.**

Just as we had 9 standard derivatives, so we have a set of standard integrals together with a set of methods.

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where C is a constant, $n \neq -1$

2) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a\,(n+1)} + C$, where C is a constant

Example 1

$$\int \left(x + \frac{1}{x^2}\right)^3 dx = \int \left(x^3 + 3 + \frac{3}{x^3} + \frac{1}{x^6}\right) dx$$

$$= \int \left(x^3 + 3 + 3x^{-3} + x^{-6}\right) dx$$

$$= \frac{1}{4}x^4 + 3x - \frac{3}{2}x^{-2} - \frac{1}{5}x^{-5} + C, \text{ where } C \text{ is a constant.}$$

$$= \frac{1}{4}x^4 + 3x - \frac{3}{2x^2} - \frac{1}{5x^5} + C$$

Example 2

$$\int \sqrt{1 + \frac{x}{2}}\, dx = \int \left(\frac{1}{2}x + 1\right)^{\frac{1}{2}} dx = \frac{\left(\frac{1}{2}x+1\right)^{\frac{3}{2}}}{\frac{3}{2}.\frac{1}{2}} + C, \text{ where } C \text{ is a constant}$$

$$= \frac{4}{3}\left(1 + \frac{1}{2}x\right)^{\frac{3}{2}} + C$$

Example 3

$$\int \frac{x^3 + 3x^2}{x} dx = \int \left(\frac{x^3}{x} + \frac{3x^2}{x}\right) dx = \int \left(x^2 + 3x\right) dx$$

$$= \frac{1}{3}x^3 + \frac{3}{2}x^2 + C, \text{ where } C \text{ is a constant.}$$

Examples for class

Integrate:

(1) $\int 3x\,dx$ (2) $\int 5x^2\,dx$ (3) $\int \frac{dx}{2}$ (4) $\int d\theta$ (5) $\int (4x^2 - 5x + 1)\,dx$

(6) $\int x\left(8x - \frac{1}{2}\right)dx$ (7) $\int (2x-3)(x+4)\,dx$ (8) $\int \sqrt[3]{x}\,dx$ (9) $\int \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx$

(10) $\int a\,dt$ (11) $\int \left(\frac{1}{x^3} - \frac{1}{x^2} - 1\right)dx$ (12) $\int \left(\frac{x^2+3x}{x}\right)dx$

(13) $\int \frac{x^3 - x^2 + 1}{x^2}\,dx$ (14) $\int \sqrt{2x+3}\,dx$ (15) $\int \frac{dx}{(3-2x)^2}$

(16) $\int \sqrt{ax+b}\,dx$ (17) $\int (4x+5)^2\,dx$

(18) $\int x(1+x)(1+x^2)\,dx$ (19) $\int (4x^3 - 6x^2 + 5)\,dx$ (20) $\int \frac{x^4+1}{x^2}\,dx$

(21) $\int \frac{dx}{(2x-1)^4}$ (22) $\int \frac{dx}{(3-2x)^3}$ (23) $\int \left(\frac{x^4+3x+1}{x^3}\right)dx$

Answers

(1) $\frac{3x^2}{2} + C$ (2) $\frac{5x^3}{3} + C$ (3) $\frac{x}{2} + C$ (4) $\theta + C$

(5) $\frac{4x^3}{3} - \frac{5x^2}{2} + x + C$ (6) $\frac{8x^3}{3} - \frac{x^2}{4} + C$ (7) $\frac{2x^3}{3} + \frac{5x^2}{2} - 12x + C$

(8) $\frac{3x^{4/3}}{4} + C$ (9) $\frac{2}{3}x^{3/2} + 2x^{1/2} + C$ (10) $at + C$ (11) $-\frac{1}{2x^2} + \frac{1}{x} - x + C$

(12) $\frac{x^2}{2} + 3x + C$ (13) $\frac{x^2}{2} - x - \frac{1}{x} + C$ (14) $\frac{1}{3}(2x+3)^{3/2} + C$

(15) $\frac{1}{2(3-2x)} + C$ (16) $\frac{2(ax+b)^{3/2}}{3a} + C$ (17) $\frac{1}{12}(4x+5)^3 + C$

(18) $\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + C$ (19) $x^4 - 2x^3 + 5x + C$ (20) $\frac{x^3}{3} - \frac{1}{x} + C$

(21) $-\frac{1}{6(2x-1)^3} + C$ (22) $\frac{1}{4(3-2x)^2} + C$ (23) $\frac{x^2}{2} - \frac{3}{x} - \frac{1}{2x^2} + C$

Example 4

If $\dfrac{d^2y}{dx^2} = 3x^2$, find y in terms of x.

$\dfrac{d^2y}{dx^2} = 3x^2 \Rightarrow \dfrac{dy}{dx} = \int 3x^2\,dx = x^3 + A$, where A is a constant.

$\Rightarrow y = \dfrac{1}{4}x^4 + Ax + B$, where B is a constant.

Example 5

If $\dfrac{dy}{dx} = \dfrac{1}{(3-2x)^{\frac{3}{2}}}$, and if $y = 3$ when $x = 1$, find y in terms of x.

$\dfrac{dy}{dx} = \dfrac{1}{(3-2x)^{\frac{3}{2}}} \Rightarrow y = \int \dfrac{dx}{(3-2x)^{\frac{3}{2}}} = \int (3-2x)^{-\frac{3}{2}}\,dx$

$= \dfrac{-2}{-2}(3-2x)^{-\frac{1}{2}} + A$, where A is a constant

$\Rightarrow y = \dfrac{1}{(3-2x)^{\frac{1}{2}}} + A$

When $x = 1, y = 3$, giving $3 = \dfrac{1}{1} + A \Rightarrow A = 2$

Hence $\Rightarrow y = \dfrac{1}{(3-2x)^{\frac{1}{2}}} + 2$

Examples for class

(1) If $\dfrac{dy}{dx} = 6x^2$, find y in terms of x, when $y = 5$ if $x = 1$.

(2) If $\dfrac{d^2y}{dx^2} = 5x$, find y in terms of x, when it is known that if $x = 2$, $\dfrac{dy}{dx} = 12$ and also when $x = 1, y = 1$.

(3) The gradient of a curve is given by $\dfrac{dy}{dx} = 9x^2 - 10x + 4$.
If the curve passes through (1,6), find y.

Answers

(1) $y = 2x^2 + 3$ (2) $y = \dfrac{5x^3}{6} + 2x - \dfrac{11}{6}$ (3) $y = 3x^3 - 5x^2 + 4x + 4$

Two more standard integrals

3) $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b) + C$

4) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$

Example 1

$\int \frac{dx}{x} = \ln x + C$, where C is a constant.

Example 2

$\int \frac{dx}{2x-3} = \frac{1}{2}\ln(2x-3) + C$, where C is a constant.

Example 3

$\int e^{4x}dx = \frac{1}{4}e^{4x} + C$, where C is a constant.

N.B. When dealing with $\int \frac{dx}{(ax+b)}$, always ensure that $(ax + b)$ is a positive quantity.

Examples for class

(1) $\int \frac{1.4}{x}dx$	(2) $\int \frac{dx}{x+3}$	(3) $\int \left(\frac{3}{x-1} - \frac{4}{x-2}\right)dx$
(4) $\int \frac{dx}{3-2x}$	(5) $\int 3e^{2x}dx$	(6) $\int e^{3x-1}dx$
(7) $\int \left(e^{x} + e^{-x}\right)^2 dx$	(8) $\int e^{3x}dx$	(9) $\int e^{\frac{x}{2}}\,dx$
(10) $\int \left[\frac{1+e^{x}}{e^{x}}\right]dx$	(11) $\int \frac{dx}{1-x}dx$	(12) $\int \left(-e^{-\frac{2}{3}x}\right)dx$
(13) $\int 3e^{-\frac{4}{3}(1-x)}\,dx$	(14) $\int \frac{dx}{3x+4}$	(15) $\int \frac{dx}{-2x-3}$

Answers

(1) $1.4\ln x + C$ (2) $\ln(x + 3) + C$ (3) $3\ln(x - 1) - 4\ln(x - 2) + C$

(4) $-\frac{1}{2}\ln(3-2x)+C$ (5) $\frac{3}{2}e^{2x}+C$ (6) $\frac{1}{3}e^{3x-1}+C$

(7) $\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}+C$ (8) $\frac{1}{3}e^{3x}+C$ (9) $2e^{x/2}+C$

(10) $-e^{-x}+x+C$ (11) $-\ln(1-x)+C$ (12) $x+\frac{3}{2}e^{-2/3x}+C$

(13) $\frac{9}{4}e^{-4/3(1-x)}+C$ (14) $\frac{1}{3}\ln(3x+4)+C$ (15) $-\frac{1}{2}\ln(-2x-3)+C$

Example 4

$$\int \frac{x^2}{x+1}\,dx$$

For this type of example, we use the following rule:

> If the degree of the numerator is greater than or equal to the degree of the denominator, divide out then integrate.

$$\int \frac{x^2}{x+1}\,dx = \int \left[x-1+\frac{1}{x+1}\right]dx$$

$$= \frac{1}{2}x^2 - x + \ln(x+1) + A$$

where A is a constant.

$$\begin{array}{r} x-1 \\ x+1\overline{)x^2} \\ \underline{x^2+x} \\ -x \\ \underline{-x-1} \\ 1 \end{array}$$

Example 5

$$\int \frac{x}{x+1}\,dx = \int \left[1-\frac{1}{x+1}\right]dx$$

$$\begin{array}{r} 1 \\ x+1\overline{)x} \\ \underline{x+1} \\ -1 \end{array}$$

$= x - \ln(x+1) + C$, where C is a constant.

Examples for class

Integrate:

(1) $\int \frac{6x^2+1}{2x-1}\,dx$ (2) $\int \frac{1+x^2}{1-x}\,dx$ (3) $\int \frac{6x^2-x+7}{2x+1}\,dx$

(4) $\int \frac{x^2}{1-2x}\,dx$ (5) $\int \frac{3-4x^2}{2-3x}\,dx$ (6) $\int \frac{x-2}{x+1}\,dx$

(7) $\int \frac{x+5}{x+2}\,dx$ (8) $\int \frac{x}{3x+1}\,dx$

Answers

(1) $\frac{3x^2}{2}+\frac{3x}{2}+\frac{5}{4}\ln(2x-1)+C$ (2) $-\frac{x^2}{2}-x-2\ln(1-x)+C$

(3) $\frac{3x^2}{2}-2x+\frac{9}{2}\ln(2x+1)+C$ (4) $-\frac{x^2}{4}-\frac{x}{4}-\frac{1}{8}\ln(1-2x)+C$

(5) $\frac{2x^2}{3}+\frac{8x}{9}-\frac{11}{27}\ln(2-3x)+C$ (6) $x-3\ln(x+1)+C$

(7) $x+3\ln(x+2)+C$ (8) $\frac{1}{3}x-\frac{1}{9}\ln(3x+1)+C$

More standard integrals

5) $\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$

6) $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$

7) $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c$

Examples for class

Integrate:

(1) $\int \cos 5x\, dx$ (2) $\int \sin 3x\, dx$ (3) $\int \cos(2x+a)\, dx$

(4) $\int \sin\left(\frac{1}{3}x\right) dx$ (5) $\int \sec^2\left(\frac{x}{3}\right) dx$ (6) $\int \left(\cos 3x - \sin\left(\frac{x}{3}\right)\right) dx$

(7) $\int \sin 2x\, dx$ (8) $\int \cos(1-3x)\, dx$ (9) $\int \sec^2(2x+1)\, dx$

Answers

(1) $\frac{1}{5}\sin 5x + C$ (2) $-\frac{1}{3}\cos 3x + C$ (3) $\frac{1}{2}\sin(2x+a) + C$

(4) $-3\cos\frac{1}{3}x + C$ (5) $3\tan\frac{x}{3} + C$ (6) $\frac{1}{3}\sin 3x + 3\cos\frac{x}{3} + C$

(7) $-\frac{1}{2}\sin 2x + C$ (8) $-\frac{1}{3}\sin(1-3x) + C$ (9) $\frac{1}{2}\tan(2x+1) + C$

Another standard integral

8) $\int \frac{f'(x)}{f(x)}\, dx = \ln f(x) + C$

Example 1

$\int \frac{\cos x}{\sin x}\, dx = \ln(\sin x) + C$, where C is a constant.

Here $f(x) = \sin x$ and $f'(x) = \cos x$

Example 2

$\int \frac{\sin x}{\cos x}\, dx = -\int \frac{(-\sin x)}{\cos x}\, dx$

$= -\ln(\cos x) + C$, where C is a constant. Here $f(x) = \cos x$ and $f'(x) = -\sin x$

$= \ln\left(\frac{1}{\cos x}\right) + C$

$= \ln \sec x + C$

Example 3

$$\int \frac{3x^2}{x^3+1}\,dx = \ln(x^3+1) + C \text{, where } C \text{ is a constant.}$$

Here $f(x) = x^3 + 1$ and $f'(x) = 3x^2$

Example 4

$$\int \frac{x}{3x^2+1}\,dx = \frac{1}{6}\int \frac{6x}{3x^2+1}\,dx \qquad \text{Here } f(x) = 3x^2+1 \text{ and } f'(x) = 6x$$

$$= \frac{1}{6}\ln(3x^2+1) + C \text{, where } C \text{ is a constant.}$$

Examples for class

Integrate:

(1) $\int \frac{6x+5}{3x^2+5x+1}\,dx$ (2) $\int \frac{x}{2x^2+3}\,dx$ (3) $\int \frac{e^x}{e^x+1}\,dx$ (4) $\int \frac{e^{3t}}{e^{3t}+6}\,dt$

(5) $\int \frac{x}{x^2-1}\,dx$ (6) $\int \frac{1+\cos 2x}{2x+\sin 2x}\,dx$ (7) $\int \tan x\,dx$ (8) $\int \cot x\,dx$

Answers

(1) $\ln(3x^2 + 5x + 1) + C$ (2) $\frac{1}{4}\ln(2x^2+3) + C$ (3) $\ln(e^x + 1) + C$

(4) $\frac{1}{3}\ln(e^{3t}+6) + C$ (5) $\frac{1}{2}\ln(x^2 - 1) + C$ (6) $\frac{1}{2}\ln(2x + \sin 2x) + C$

(7) $-\ln(\cos x) + C$ **(8) $\ln(\sin x) + C$**

Standard Integrals

These integrals can be collected in a table as follows:

Function	Integral
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n+1}, \quad n \neq -1$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln(ax+b)$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b)$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b)$
$\sec^2(ax+b)$	$\frac{1}{a}\tan(ax+b)$
e^{ax}	$\frac{1}{a}e^{ax}$
$\frac{f'(x)}{f(x)}$	$\ln(f(x))$

Integration by substitution

Indefinite integral

It sometimes happens that an integrand which is difficult to reduce directly to one of the standard forms, may be reduced more easily by the choice of a suitable new independent variable.

Example 1

$\int \frac{x^4}{(x^5+1)^3}\,dx$ Let $u = x^5 + 1$

$= \int \frac{\frac{1}{5}\,du}{u^3}$ Then $\frac{du}{dx} = 5x^4$

$= \frac{1}{5}\int u^{-3}\,du$ i.e. $du = 5x^4\,dx$

$= \frac{1}{5}.-\frac{1}{2}u^{-2} + C$, where C is a constant. i.e. $\frac{1}{5}\,du = x^4\,dx$

$= -\frac{1}{10u^2} + C$

$= -\frac{1}{10(x^5+1)^2} + C$

Example 2

$\int \sin^7 x.\cos x\,dx$ Let $u = \sin x$

$= \int u^7\,du$ Then $du = \cos x\,dx$

$= \frac{1}{8}u^8 + C$, where C is a constant.

$= \frac{1}{8}\sin^8 x + C$

Example 3

$\int xe^{x^2}\,dx$ Let $u = x^2$

$= \int \frac{e^u}{2}\,du$ Then $\frac{du}{dx} = 2x$

$= \frac{1}{2}\int e^u\,du$ i.e. $du = 2x\,dx$

$= \frac{1}{2}e^u + C$, where C is a constant i.e. $\frac{du}{2} = x\,dx$

$= \frac{1}{2}e^{x^2} + C$

Examples for class

Integrate:

(1) $\int \dfrac{\sin x}{1+\cos x}\,dx$ (2) $\int \dfrac{1-2\sin x}{x+2\cos x}\,dx$ (3) $\int \sin x.\cos^4 x\,dx$

(4) $\int \sin^3 x.\cos^2 x\,dx$ (5) $\int \dfrac{\sin 2x}{a+b\cos^2 x}\,dx$ (6) $\int \dfrac{2x^3}{1+x^4}\,dx$

(7) $\int x^2 e^{x^3}\,dx$ (8) $\int 3xe^{x^2}\,dx$ (9) $\int xe^{(x^2+1)}\,dx$

Answers

(1) $-\ln(\cos x) + C$ (2) $\ln(x + \cos x) + C$ (3) $-\dfrac{1}{5}\cos^5 x + C$

(4) $\dfrac{1}{3}\cos^3 x - \dfrac{2}{5}\cos^5 x + C$ (using the substitution $u = \cos^2 x$)

(5) $-\dfrac{1}{b}\ln(a+b\cos^2 x)+C$ (6) [illegible] (7) [illegible]

(8) [illegible] (9) $\dfrac{1}{2}e^{(x^2+1)}+C$

Definite integrals

Example 4

$\int_0^{\frac{\pi}{2}} \sin^4 x \cos x\,dx$

Let $u = \sin x$

$du = \cos x\,dx$

[illegible]

Example 5

$$\int_2^8 \frac{dx}{(5x+2)^3}$$

Let $u = 5x + 2$

$du = 5\,dx$

i.e. $dx = \frac{1}{5}du$

x	2	8
u	12	42

$$= \frac{1}{5}\int_{12}^{42} \frac{du}{u^3}$$

$$= \frac{1}{5}\left[-\frac{1}{2}u^{-2}\right]_{12}^{42}$$

$$= -\frac{1}{10}\left[\frac{1}{u^2}\right]_{12}^{42}$$

$$= -\frac{1}{10}\left\{\frac{1}{42^2} - \frac{1}{12^2}\right\}$$

$$= \frac{1}{1568}$$

Examples for class

Evaluate:

(1) $\int_0^2 \frac{2x^3}{1+x^4}\,dx$ (2) $\int_1^3 x^2 e^{x^3}\,dx$ (3) $\int_0^2 x e^{(x^2+1)} dx$

(4) $\int_0^1 \frac{x^2}{1+x^3}dx$ (5) $\int_1^e \frac{\ln x}{x^2}dx$ (6) $\int_0^1 \frac{x}{\sqrt{3+x^2}}dx$

(7) $\int_0^{\pi/4} \tan^3 x dx$

Answers

(1) $\frac{1}{2}\ln 17$ (2) $\frac{1}{3}e(e^{26}-1)$ (3) $\frac{1}{2}e(e^4-1)$

(4) $\frac{1}{3}\ln 2$ (5) $\frac{e-2}{e}$ (6) $2 - \sqrt{3}$ (7) $\frac{1}{2} - \frac{1}{2}\ln 2$

MATHEMATICS 1 (AH)

CONTENT
Properties of functions 1.4.1 know the meaning of the terms function, domain, range, inverse function, critical point, stationary point, point of inflexion, concavity, local maxima and minima, global maxima and minima, continuous, discontinuous, asymptote 1.4.2 determine the domain and the range of a function 1.4.3 use the derivative tests for locating and identifying stationary points 1.4.9 sketch graphs of real rational functions using available information, derived from calculus and/or algebraic arguments, on zeros, asymptotes (vertical and non-vertical), critical points, symmetry

The content listed below is **not** included within this support material:

1.4.4 sketch the graphs of $\sin x$, $\cos x$, $\tan x$, e^x, $\ln x$ and their inverse functions, simple polynomial functions

1.4.5 know and use the relationship between the graph of $y = f(x)$ and the graphs of $y = kf(x), y = f(x) + k, y = f(x + k), y = f(kx)$, where k is a constant

1.4.6 know and use the relationship between the graph of $y = f(x)$ and $y = |f(x)|$ and $y = f^{-1}(x)$

1.4.7 given the graph of a function f, sketch the graph of a related function

1.4.8 determine whether a function is even or odd or neither and (symmetrical) and use these properties in graph sketching

Note that symmetrical is included in the covered content.

MATHEMATICS 1 (AH) : PROPERTIES OF FUNCTIONS

Application of second derivatives to distinguish types of stationary values.

If, at $x = a$,
1) $f''(a) < 0$, then $f(x)$ has a maximum stationary value.
2) $f''(a) > 0$, then $f(x)$ has a minimum stationary value.
3) $f''(a) = 0$, no information – table of signs must be used.

N.B. At a point of inflexion, the tangent need not be horizontal i.e. $\frac{dy}{dx} \neq 0$

But $\frac{d^2y}{dx^2} = 0$. In this case, the point of inflexion is not stationary.

Example 1

Determine the stationary points of $y = -x^4$.

$y = -x^4$

$\frac{dy}{dx} = -4x^3$

$\frac{d^2y}{dx^2} = -12x^2$

$\frac{dy}{dx} = 0 \Leftrightarrow -4x^3 = 0$

$\Leftrightarrow x = 0$

At $x = 0$, $\frac{d^2y}{dx^2} = 0 \rightarrow$ Table of signs

x	$\rightarrow$	0	$\rightarrow$
$\frac{dy}{dx}$	+	0	–
	$\nearrow$	$\rightarrow$ Max	$\searrow$

y has a maximum stationary point (0, 0).

Example 2

Find the stationary values of $f(x) = 2x^5 - x^4 - 2x^3$ and determine their nature.

$f(x) = 2x^5 - x^4 - 2x^3$

$f'(x) = 10x^4 - 4x^3 - 6x^2$

$f''(x) = 40x^3 - 12x^2 - 12x$

$$f'(x) = 0 \Leftrightarrow 10x^4 - 4x^3 - 6x^2 = 0$$
$$\Leftrightarrow 2x^2(5x^2 - 2x - 3) = 0$$
$$\Leftrightarrow 2x^2(5x + 3)(x - 1) = 0$$
$$\Leftrightarrow x = 0\text{, } twice \quad or\ x = -\frac{3}{5} \quad or \quad x = 1$$

At $x = 0$, $f''(x) = 0 \rightarrow$ table of signs
At $x = 0$, $f(x) = 0$ – **Point of inflexion**

x	$\rightarrow$	0	$\rightarrow$
$f'(x)$	−	0	−
	↘	→	↘

At $x = -\frac{3}{5}$, $f''(x) = -\frac{144}{25} < 0$

Hence at $x = -\frac{3}{5}$, Maximum stationary point.

At $x = -\frac{3}{5}$, $f(x) = \frac{509}{3125}$ – **Maximum**

At $x = 1$, $f''(x) = 16 > 0$

Hence at $x = 1$, Minimum Stationary Point.

At $x = 1$, $f(x) = -1$ – **Minimum**

Examples for class

Investigate stationary values of each of the following, stating
(a) the value of x for which it occurs;
(b) the stationary value;
(c) whether it is a maximum or minimum or a point of inflexion.

(1) $(x-2)^2(x-1)$ (2) $x^3(x-4)$ (3) $x\sqrt{1-x}$ (4) $\frac{x^2}{\sqrt{x^2+1}}$

(5) $x^4 + 4x^3$ (6) $x(x^4 - 5)$ (7) $3x^4 - 8x^3 + 6x^2$ (8) $\frac{x^2}{1-x}$

(9) $\frac{x^5}{(x+2)^2}$ (10) $x - 2\cos x$ (11) $2x - \tan x$

Answers

(1) (a) $x = 2, \frac{4}{3}$ (b) $(2, 0), (\frac{4}{3}, \frac{4}{27})$ (c) $x = 2$ maximum, $x = \frac{4}{3}$ minimum

(2) (a) $x = 0, 3$ (b) $(0, 0), (3, -27)$ (c) $x = 0$ pt of inflexion, $x = 3$ minimum

(3) (a) $x = \frac{2}{3}$ (b) $(\frac{2}{3}, \frac{2}{3\sqrt{3}})$ (c) minimum

(4) (a) $x = 0$ (b) $(0, 0)$ (c) minimum

(5) (a) $x = 0, -3$ (b) $(0, 0), (-3, -27)$ (c) $x = 0$ pt of inflexion, $x = -3$ minimum

(6) (a) $x = 1, -1$ (b) $(1, -4), (-1, 4)$ (c) $x = 1$ minimum, $x = -1$ maximum

(7) (a) $x = 0, 1$ (b) $(0, 0), (1, 1)$ (c) $x = 0$ minimum, $x = 1$ pt of inflexion

(8) (a) $x = 0, 2$ (b) $(0, 0), (2, -4)$ (c) $x = 0$ minimum, $x = 2$ maximum

(9) (a) $x = 0, -\frac{10}{3}$ (b) $(0, 0), (-\frac{10}{3}, -\frac{10^5}{432})$

(c) $x = 0$ pt of inflexion, $x = -\frac{10}{3}$ maximum

(10) $x = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$ (b) $(-\frac{5\pi}{6}, -\frac{5\pi}{6} - \frac{\sqrt{3}}{2}), (-\frac{\pi}{6}, -\frac{\pi}{6} + \frac{\sqrt{3}}{2}), \ldots$

(d) $x = -\frac{5\pi}{6}$ maximum, $x = -\frac{\pi}{6}$ minimum, $x = \frac{7\pi}{6}$ maximum, $x = \frac{11\pi}{6}$ minimum …

(11) $x = \frac{\pi}{4}, -\frac{\pi}{4}$ (b) $(\frac{\pi}{4}, \frac{\pi}{2} - 1), (-\frac{\pi}{4}, -\frac{\pi}{2} + 1)$

(c) $x = \frac{\pi}{4}$ maximum, $x = -\frac{\pi}{4}$ minimum

Example 3

Find the points of inflexion of the function $y = x^4 - 6x^2 + 8x + 5$.

$$y = x^4 - 6x^2 + 8x + 5$$

$$\frac{dy}{dx} = 4x^3 - 12x + 8$$

$$\frac{d^2y}{dx^2} = 12x^2 - 12$$

At points of inflexion, $\frac{d^2y}{dx^2} = 0 \Leftrightarrow 12x^2 - 12 = 0$

$$\Leftrightarrow 12(x-1)(x+1) = 0$$

$$\Leftrightarrow x = 1 \text{ or } -1$$

At stationary points, $\frac{dy}{dx} = 0 \Leftrightarrow 4x^3 - 12x + 8 = 0$

$$\Leftrightarrow 4(x-1)(x-1)(x+2) = 0$$

$$\Leftrightarrow x = 1, \text{twice} \quad \text{or } x = -2$$

x	$\rightarrow$	1	$\rightarrow$
$\frac{dy}{dx}$	+	0	+
	$\nearrow$	$\rightarrow$	$\nearrow$

Hence points of inflexion are (-1, -8) and (1, 8).

Examples for class

Find any points of inflexion on:

(1) $y = 3x^3 - 4x + 5$ (2) $y = x^4 - 6x^2 - 8x$

(3) $y = x(x^2 - 1)$ (4) $y = x^2(x + 6)$

Answers

(1) (0, 5) (2) (-1, 3), (1, -13) (3) (0, 0) (4) (-2, 16)

Curve sketching

1. **Symmetry about x-axis.**

   A curve is symmetrical about the x-axis if both the point (h, k) and the point (h, -k) satisfy its equation. This will always be true if the equation contains only even powers of y.

   Example: $y^2 = x$

2. **Symmetry about y-axis.**

   A curve is symmetrical about the y-axis when its equation contains only even powers of x.

   Example: $y = x^2$, $y = 4x^4 + 2x^2 + 3$

3. **Symmetry about the origin.**

   This symmetry will exist if both (h, k) and $(-h, -k)$ lie on the curve. This will happen if the equation contains even powers of x and y.

Examples for class

What symmetry is possessed by the following ?

(1) $y = \dfrac{1}{x^2+1}$ (2) $x^2 = \dfrac{1}{y^2+1}$ (3) $y = x(x^2-1)$

(4) $x^5 + y^5 - 5x^2y = 0$ (5) $x^2 + y^2 = 4$ (6) $\dfrac{x^2}{4} + \dfrac{y^2}{12} = 1$

Answers

(1) symmetrical about y axis (2) symmetrical about y axis and x axis and the origin

(3) symmetrical about origin (4) no symmetry

(5) symmetrical about y axis and x axis and the origin

(6) symmetrical about x axis and y axis

Points of inflexion

We have already seen that if a point of inflexion occurs on the curve $y = f(x)$ at $x = a$, $f''(a) = 0$. If also $f'(a) = 0$, then the point of inflexion is also a stationary point i.e. at this point the tangent to the curve is horizontal.

Example 1

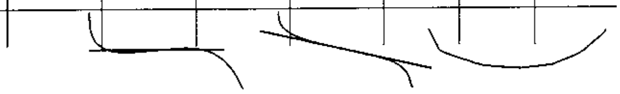
If $f'(x) = (x-1)^4(x-8)^3$, find the x co-ordinates of the points of inflexion on The graph of f and determine the shape of the graph in the neighbourhood of each point.

$$f'(x) = (x-1)^4(x-8)^3$$
$$f''(x) = 7(x-8)^2(x-1)^3(x-5)$$

At the point of inflexion, $f''(x) = 0 \Leftrightarrow 7(x-8)^2(x-1)^3(x-5) = 0$

$\Leftrightarrow x = 8,\ \textit{twice},\quad \textit{or}\ x = 1,\ \textit{thrice},\quad \textit{or}\ x = 5.$

x	$\rightarrow$	1	$\rightarrow$	5	$\rightarrow$	8	$\rightarrow$
$f'(x)$	-	0	-	-	-	0	+



Pt of inflexion   Pt of inflexion   Min

The points of inflexion are given by $x = 1$ and $x = 5$.

Examples for class

Find the points of inflexion of the curves and determine the shape of each curve in a neighbourhood of each of the points.

(1) $y = 3x - x^3$ (2) $y = x^4 + 4x^3$ (3) $y = x^5 + x^3 + x$

(4) $y = 3x^4 - 8x^3 + 6x^2$ (5) $y = \dfrac{x}{x^4 + 3}$ (6) $y = \dfrac{4x}{3x^2 + 1}$

Answers

(1) (0, 0) shape: +++ (2) (0, 0) shape: +0+ , (-2, -16) shape: +++

(3) (0, 0) shape: +++ (4) $(\frac{1}{3}, \frac{11}{27})$ shape: +++, (1, 0) shape +0+

(5) $(-\sqrt{2}, \frac{\sqrt{2}}{7})$ shape: ---, $(\sqrt{2}, \frac{-\sqrt{2}}{7})$ shape: ---

(6) (-1, -1) shape: ---, (0, 0) shape: +++ , (1, 1) shape: ---

Vertical Asymptotes

Consider the curve $y = \frac{1}{x}$.

Since division by 0 is not allowed, y **has no value when** x **is zero. However as** x **approaches 0 through positive values, so** y **takes increasingly large positive values e.g.**

$x = 0.1$	$y = 10$
$x = 0.01$	$y = 100$
$x = 0.001$	$y = 1000$

We say that " as $x \to 0$ through positive values, $y \to +\infty$ "

And write " $x \to 0+, y \to +\infty$ "

Similarly " $x \to 0-, y \to -\infty$ "

We can show this diagrammatically as

$x = 0$

The line $x = 0$ is called an **asymptote** to the curve.

An asymptote may be thought of as a tangent to the curve at infinity.

Example 1

Find the equation of the vertical asymptote and indicate how the curve approaches the asymptote.

(a) $y = \frac{1}{x-1}$ (b) $y = \frac{3}{x+2}$ (c) $y = \frac{x}{(x-1)(x+2)}$ (d) $y = \frac{1}{x^2-1}$

(a) $x = 1 \Rightarrow x - 1 = 0$ and so y is undefined here.

$x \to 1+ \quad , y \to +\infty$

$x \to 1- \quad , y \to -\infty$

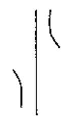
$x = 1$ is the asymptote.

$x = 1$

(b) $x = -2$ is the asymptote

$x \to (-2)+ \quad , y \to +\infty$

$x \to (-2)- \quad , y \to -\infty$

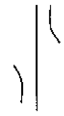


$x = -2$

(c) $x = 1$ is an asymptote

$x \to 1+ \quad , y \to +\infty$

$x \to 1- \quad , y \to -\infty$

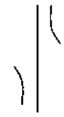


$x = 1$

$x = -2$ is also an asymptote

$x \to (-2)+ \quad , y \to +\infty$

$x \to (-2)- \quad , y \to -\infty$



$x = -2$

(d) $$y = \frac{1}{x^2 - 1} = \frac{1}{(x+1)(x-1)}$$

$x = -1$ is an asymptote

$x \to (-1)+ \quad , y \to -\infty$

$x \to (-1)- \quad , y \to +\infty$

$x = -1$

$x = 1$ is an asymptote

$x \to 1+ \quad , y \to +\infty$

$x \to 1- \quad , y \to -\infty$

$x = 1$

Rule: If $y = \dfrac{P(x)}{Q(x)}$, then the equations of the vertical asymptotes are given by factorising the denominator $Q(x)$ and equating each factor to zero.

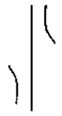
Examples for class

Find the equations of the vertical asymptotes to the following curves and show how the curve approaches each asymptote.

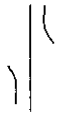
(1) $y = \dfrac{x(x-1)}{x+1}$ (2) $y = \dfrac{x^2}{4 - x^2}$ (3) $y = \dfrac{(x-1)(x+2)}{x}$

Answers

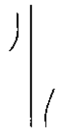
(1) $x = -1$



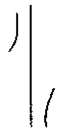
(2) $x = 2$



$x = -2$



(3) $x = 0$



Oblique asymptotes and asymptotes parallel to x-axis

The method indicated in the previous section does not find all asymptotes. In order to find other asymptotes, we must divide out the fraction and let $x \to \pm\infty$.

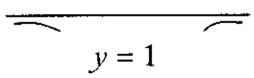
Example 1

$$y = \frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$$

As $x \to +\infty \quad , y \to 1-$

$\quad\; x \to -\infty \quad , y \to 1-$

$y = 1$ is asymptote



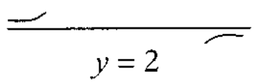
Example 2

$$y = \frac{2x}{x+3} = 2 - \frac{6}{x+3}$$

As $x \to +\infty \quad , y \to 2-$

$\quad\; x \to -\infty \quad , y \to 2+$

$y = 2$ is asymptote



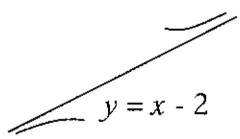
Example 3

$$y = \frac{x^2 - x}{x+1} = x - 2 + \frac{2}{x+1}$$

As $x \to +\infty \quad , y \to (x-2)+$

$\quad\; x \to -\infty \quad , y \to (x-2)-$

$y = x - 2$ is asymptote

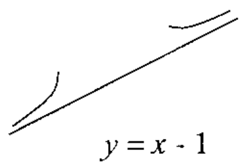


Example 4

$$y = \frac{x^3}{x^2 + x + 1} = x - 1 + \frac{1}{x^2 + x + 1}$$

As $\begin{array}{l} x \to +\infty \quad , y \to (x-1)+ \\ x \to -\infty \quad , y \to (x-1)+ \end{array}$

$y = x - 1$ is asymptote

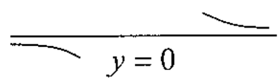


Example 5

$$y = \frac{x^2}{x^3 + 1} = \frac{\frac{x^2}{x^3}}{\frac{x^3}{x^3} + \frac{1}{x^3}} = \frac{\frac{1}{x}}{1 + \frac{1}{x^3}}$$

As $\begin{array}{l} x \to +\infty \quad , y \to 0+ \\ x \to -\infty \quad , y \to 0- \end{array}$

$y = 0$ is asymptote



We can combine all of this in the following rule:

A. Vertical asymptotes obtained from denominator, by factorising and equating each factor to zero.

B. If numerator is of degree equal to, or higher than, that of denominator, divide out. This gives asymptotes
$y = k$, if equal degree.
$y = mx + c$, if degree of numerator higher than that of denominator.

C. If numerator is of lower degree than that of denominator, divide each term by largest power of x.
This gives $y = 0$ as asymptote.

Examples for class

Find equations of asymptotes (vertical, horizontal and oblique)

(1) $y = \frac{(x+1)^3}{x^2}$ (2) $y = \frac{x^2}{x-1}$ (3) $y = \frac{x^2 - 10x + 9}{x^2 + 10x + 9}$

(4) $y = \frac{x^2 - 2x + 9}{x - 2}$ (5) $y = \frac{2x^2}{x^2 - 1}$ (6) $y = \frac{x^2 + 2x + 4}{x + 4}$

(7) $y = \frac{5}{x - 3}$ (8) $y = \frac{x + 1}{x - 1}$ (9) $y = \frac{x^2 + 1}{x}$

(10) $y = \frac{x^3 + 1}{x + 1}$ (11) $y = (x - 1) + \frac{4}{x + 1}$ (12) $y = \frac{x}{x^2 + 1}$

(13) $y = \frac{2x}{(x+1)(x-1)}$ (14) $y = \frac{2x}{x - 3}$ (15) $y = \frac{3x}{x + 2}$

(16) $y = \frac{x^3}{x^2 - 1}$ (17) $y = \frac{x^3 + 8}{x^2 - 1}$ (18) $y = \frac{4x}{3x^3 + 1}$

Answers

(1) V.A. $x = 0$
O.A. $y = x + 3$

(2) V.A. $x = 1$
O.A. $y = x + 1$

(3) H.A. $y = 1$
V.A.s $x = -1$ and $x = -9$

(4) V.A. $x = 2$
O.A. $y = x$

(5) H.A. $y = 2$
V.A.s $x = 1$ and $x = -1$

(6) O.A. $y = x - 2$
V.A. $x = -4$

(7) H.A. $y = 0$
V.A. $x = 3$

(8) V.A. $x = 1$
H.A. $y = 1$

(9) V.A. $x = 0$
O.A. $y = x$

(10) no asymptotes
(parabola)

(11) O.A. $y = x - 1$
V.A. $x = -1$

(12) H.A. $y = 0$

(13) V.A.s $x = 1$ and $x = -1$
H.A. $y = 0$

(14) V.A. $x = 3$
H.A. $y = 2$

(15) V.A. $x = -2$
H.A. $y = 3$

(16) V.A.s $x = 1$ and $x = -1$
O.A. $y = x$

(17) V.A.s $x = 1$ and $x = -1$
O.A. $y = x$

(18) V.A. $x = \frac{1}{\sqrt[3]{3}}$
H.A. $y = 0$

Curve sketching

In sketching a curve, the following should always be considered:

1) Symmetry
2) Stationary points
3) Asymptotes
4) Points of inflexion
5) Special points (interactions with axes, asymptotes etc.)

Example 1

Sketch the curve $y = \dfrac{2x}{x^2+1}$

1) If (x, y) lies on the curve, then so does $(-x, -y)$.
Hence curve possesses symmetry in the origin.

2) $y = \dfrac{2x}{x^2+1}$, $\quad \dfrac{dy}{dx} = \dfrac{2-2x^2}{(x^2+1)^2} = \dfrac{2(1-x)(1+x)}{(x^2+1)^2}$

At stationary points, $\dfrac{dy}{dx} = 0 \Leftrightarrow 2(1-x)(1+x) = 0 \Leftrightarrow x = 1 \; or \; -1$

There are two stationary points (1, 1) and (-1, -1).
(-1, -1) is a minimum stationary point

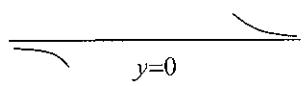
(1, 1) is a maximum stationary point.

x	$\rightarrow$	-1	$\rightarrow$	1	$\rightarrow$
$\dfrac{dy}{dx}$	$-$	0	$+$	0	$-$
	↘	Min →	↗	→ Max	↘

3) There is no vertical asymptote.

$$y = \frac{2x}{x^2+1} = \frac{\frac{2x}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \frac{\frac{2}{x}}{1+\frac{1}{x^2}}$$

as $x \rightarrow +\infty, y \rightarrow 0+$
$\quad\; x \rightarrow -\infty, y \rightarrow 0-$

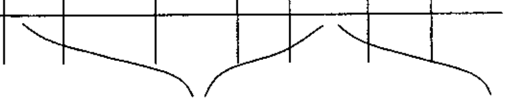


4) $\dfrac{dy}{dx} = \dfrac{2-2x^2}{(x^2+1)^2} \Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{-4x(x^2+1)(3-x^2)}{(x^2+1)^4}$

At point of inflexion, $\dfrac{d^2y}{dx^2} = 0 \Rightarrow x = 0 \; or \; \pm\sqrt{3}$

There are three points of inflexion (0, 0), $(\sqrt{3}, \frac{1}{2}\sqrt{3}$, $(-\sqrt{3}, -\frac{1}{2}\sqrt{3}$

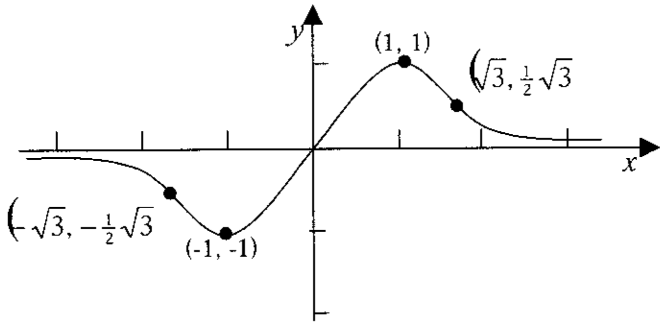
x	$\rightarrow$	$-\sqrt{3}$	$\rightarrow$	0	$\rightarrow$	$\sqrt{3}$	$\rightarrow$
$\dfrac{dy}{dx}$	$-$	$-$	$-\;+$	$+$	$+\;-$	$-$	$-$



5) When $x = 0, y = 0$

When $y = 0$, $2x = 0 \Rightarrow x = 0$

Curve meets x and y-axes at (0, 0).



Example 2

Sketch the curve $y = \dfrac{(x-1)(x-4)}{x}$.

1) No symmetry

2) $$y = \frac{(x-1)(x-4)}{x} = \frac{x^2 - 5x + 4}{x} = x - 5 + \frac{4}{x}$$

$$\frac{dy}{dx} = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2} = \frac{(x-2)(x+2)}{x^2}$$

At stationary point $\dfrac{dy}{dx} = 0 \Leftrightarrow (x-2)(x+2) = 0 \Leftrightarrow x = 2 \; or \; -2$

There are two stationary points (2, -2) and (-2, -9).

x	$\rightarrow$	-2	$\rightarrow$	$\rightarrow$	2	$\rightarrow$
$\dfrac{dy}{dx}$	+	0	–	–	0	+
	↗	→ Max	↘	↘	Min →	↗

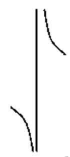
(-2, -9) is a maximum stationary point

(2, -2) is a minimum stationary point.

3) Vertical asymptote is $x = 0$

As $x \rightarrow 0+, \quad y \rightarrow +\infty$

$\quad\; x \rightarrow 0-, \quad y \rightarrow -\infty$



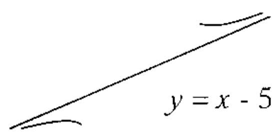
$x = 0$

For other asymptotes,

$$y = x - 5 + \frac{4}{x}$$

As $x \rightarrow +\infty, \quad y \rightarrow (x-5)+$

$\quad\; x \rightarrow -\infty, \quad y \rightarrow (x-5)-$

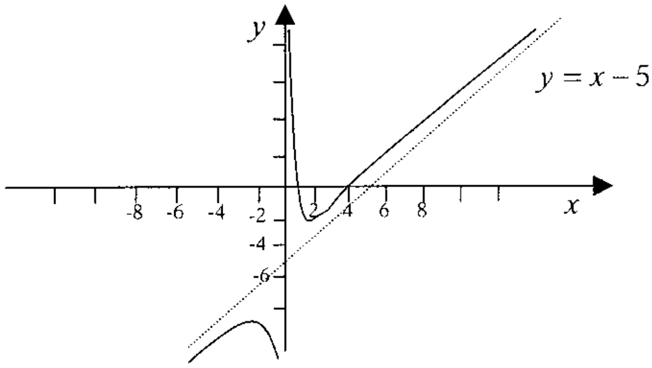


4) $\dfrac{dy}{dx} = 1 - \dfrac{4}{x^2}$

$\dfrac{d^2y}{dx^2} = \dfrac{8}{x^3}$ which is never zero.

Hence there are no points of inflexion.

5) When $y = 0$, $(x-1)(x-4) = 0 \Rightarrow x = 1$ *or* 4
Hence (1, 0) and (4, 0) are on the curve.



Example 3

Sketch the curve $y = \dfrac{x^2(x+9)}{(x+1)(x-1)}$, without discussing points of inflexion.

1) No symmetry

2) $y = \dfrac{x^2(x+9)}{(x+1)(x-1)} = \dfrac{x^3+9x^2}{x^2-1} = x + 9 + \dfrac{x+9}{x^2-1}$

$\dfrac{dy}{dx} = \dfrac{x(x-3)(x^2+3x+6)}{(x^2-1)^2}$

At stationary point, $\dfrac{dy}{dx} = 0 \Leftrightarrow x = 0$ *or* 3

There are two stationary points (0, 0) and (3, 13.5)

x	$\rightarrow$	0	$\rightarrow$	$\rightarrow$	3	$\rightarrow$
$\dfrac{dy}{dx}$	+	0	−	−	0	+
	↗	→ Max	↘	↘	Min →	↗

(0, 0) is a maximum stationary point
(3, 13.5) is a minimum stationary point.

3) Vertical asymptotes are $x = -1$ and $x = 1$

As $\begin{array}{ll} x \to (-1)+ & , y \to -\infty \\ x \to (-1)- & , y \to +\infty \end{array}$

$x = -1$

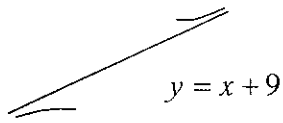
As $\begin{array}{ll} x \to 1+ & , y \to +\infty \\ x \to 1- & , y \to -\infty \end{array}$

$x = 1$

For other asymptotes

$$y = x + 9 + \frac{x+9}{x^2-1} = x + 9 + \frac{\frac{1}{x} + \frac{9}{x^2}}{1 - \frac{1}{x^2}}$$

As $\begin{array}{ll} x \to +\infty, & y \to (x+9)+ \\ x \to -\infty, & y \to (x+9)- \end{array}$



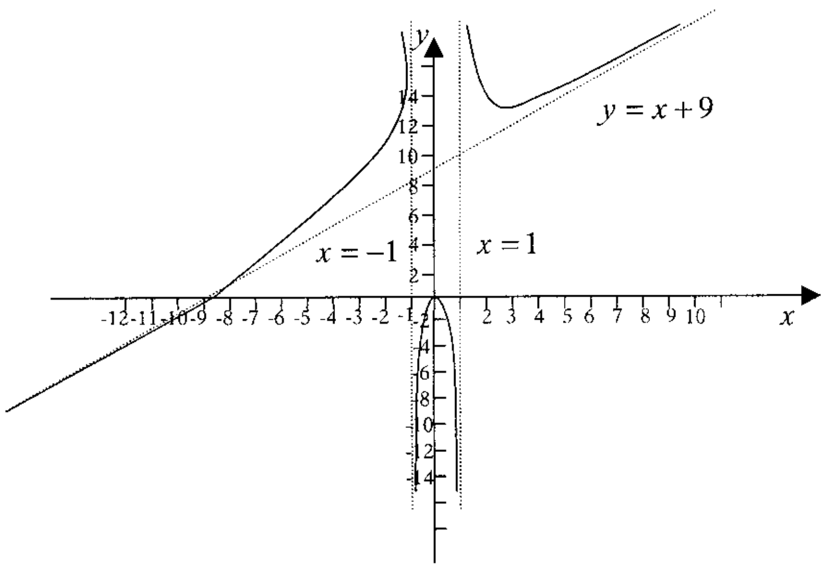
$y = x + 9$ is an asymptote.

4) When $x = 0, y = 0$

When $y = 0$, $x^2(x+9) = 0 \Rightarrow x = 0$, *twice or* $x = -9$

Hence curve crosses axes at (0, 0) and (-9, 0)

Curve crosses $y = x + 9$ at (-9, 0).



Examples for class

Sketch the curves:

(1) $y = \dfrac{x+2}{x-1}$ (2) $y = \dfrac{(x-1)(x+2)}{(x+3)(x-2)}$ (3) $y = \dfrac{x}{x^2+1}$

(4) $y = x^4 - 2x^2 - 3$ (5) $y = \dfrac{(x+1)^3}{x^2}$ (6) $y = \dfrac{x^2-2x+9}{x-2}$

(7) $y = \dfrac{2x^2}{x^2-1}$ (8) $y = \dfrac{x^2+2x+5}{x+1}$

(9) $y = \dfrac{x^2-10x+9}{x^2+10x+9}$

Answers

(1) V.A. $x = 1$, H.A. $y = 1$; (-2, 0); no stationary values

(2) V.A.s $x = -3$ and $x = 2$, H.A. $y = 1$; (1, 0), (-2, 0); max t.p. (-0.5, -0.36)

(3) No asymptotes; (0, 0); min t.p. (-1, -0.5), max t.p. (1, 0.5)

(4) No asymptotes; ($-\sqrt{3}$, 0), ($\sqrt{3}$, 0); min t.p. (-1, -4), min t.p. (1, -4), max t.p. (0, -3)

(5) V.A. $x = 0$, O.A. $y = x + 3$; (-1, 0); min t.p. (2, 6.75), pt of inflex. (-1, 0)

(6) V.A. $x = 2$, O.A. $y = x$; no zeros; min t.p. (5, 8), max t.p. (-1, -4)

(7) V.A.s $x = 1$, $x = -1$, H.A. $y = 2$; (0, 0); max t.p. (0, 0)

(8) V.A. $x = -1$, O.A. $y = x + 1$; no zeros; min t.p. (1, 4), max t.p. (-3, -4)

(9) V.A.s $x = -9$, $x = -1$, H.A. $y = 1$; (1, 0), (9, 0); min t.p. (3, -0.25), max t.p. (-3, -4)

The shapes of the graphs can be checked using a graphic calculator.

Further examples

Having sketched the curve $y = f(x)$, it is possible to sketch $y = \frac{1}{f(x)}$ easily, using the following facts.

(i) A zero of $f(x)$ gives rise to a vertical asymptote of $y = \frac{1}{f(x)}$.

(ii) A maximum (or minimum) stationary point $(a, f(a))$ gives rise to a minimum (or maximum) stationary point $\left(a, \frac{1}{f(a)}\right)$.

(iii) If $f(x) \to \pm\infty$ as $x \to \pm\infty$ then $\frac{1}{f(x)} \to 0$ as $x \to \pm\infty$.

Example 1

Deduce the sketch of the curve $y = \frac{x^2 - 1}{x^2(x+9)}$.

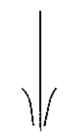
(See earlier example 3 for the graph of $y = \frac{x^2(x+9)}{(x+1)(x-1)}$)

1) No symmetry.

2) $\left(3, \frac{2}{27}\right)$ is a maximum stationary point.

3) $x = 0$ is a vertical asymptote

As $x \to 0+, \quad y \to -\infty$
   $x \to 0-, \quad y \to -\infty$



$x = 0$

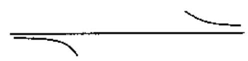
$x = -9$ is a vertical asymptote

As $x \to (-9)+, \quad y \to +\infty$
   $x \to (-9)-, \quad y \to -\infty$

$x = -9$

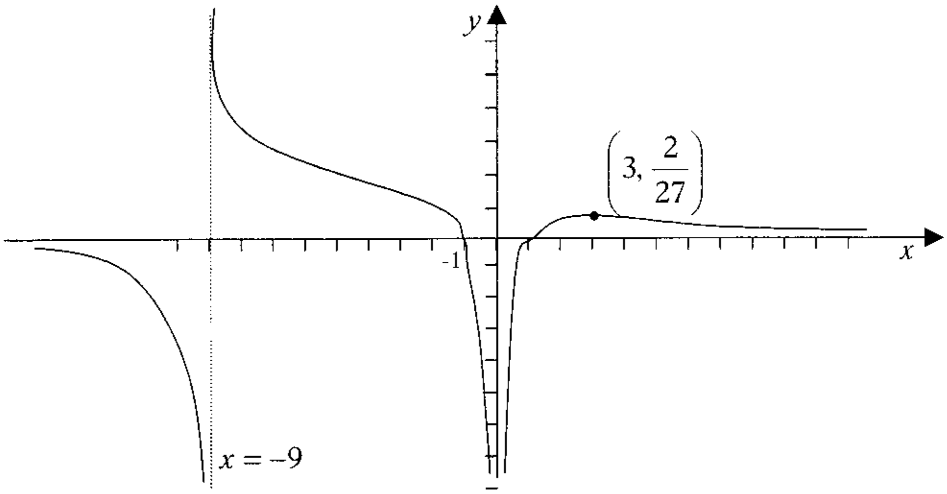
$y = 0$ is also an asymptote

As $x \to +\infty, \quad y \to 0+$
   $x \to -\infty, \quad y \to 0-$



$y = 0$

4) It meets axis at (-1, 0) and (1, 0).



Examples for class

Sketch each of the following and then deduce rough sketches of $y = \frac{1}{f(x)}$

And $y = -\frac{1}{f(x)}$

(1) $y = \frac{4x}{3x^2 + 1}$ (2) $y = \frac{x}{(1 + x)^2}$ (3) $y = \frac{x^3}{x^2 - 3}$

(4) $y = x + \frac{4}{x^2}$ (5) $y = \frac{(x-2)^2}{x(x-3)}$ (6) $y = \frac{(x-2)^2(x+3)}{3-x}$

(7) $y = \frac{2x^2 + 13x + 15}{(x-1)(x^2 - 1)}$ (8) $y = \frac{x^2}{x^3 + 1}$

Answers can be checked using a graphic calculator.

Mathematics

Mathematics 2 (AH)

MATHEMATICS 2 (AH)

Introduction

These support materials for Mathematics were developed as part of the Higher Still Development Programme in response to needs identified at needs analysis meetings and national seminars.

Advice on learning and teaching may be found in *Achievement for All* (SOEID 1996), *Effective Learning and Teaching in Mathematics* (SOEID 1993), *Improving Mathematics* (SEED 1999) and in the Mathematics Subject Guide.

These materials were originally issued to schools to support CSYS Mathematics. They have now been updated and matched to the Arrangements for Advanced Higher Mathematics to be issued as part of the support material being provided for mathematics.

It should be noted that not all of the content of Mathematics 2 (AH) is covered by this support material. The outcomes covered are Further Differentiation, Further Integration and Complex Numbers. At the start of each outcome a list of the content covered by the material has been included.

MATHEMATICS 2 (AH)

CONTENT
Further differentiation
2.1.1 know the derivatives of $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$
2.1.2 differentiate any inverse function using the technique: $y = f^{-1}(x) \Rightarrow f(y) = x \Rightarrow (f^{-1}(x))'f'(y) = 1$, etc., and know the corresponding result $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$
2.1.3 understand how an equation $f(x, y) = 0$ defines y implicitly as one (or more) functions of x
2.1.4 use implicit differentiation to find first **and second derivatives [A/B]**
2.1.5 use logarithmic differentiation, recognising when it is appropriate in extended products and quotients and indices involving a variable [A/B]
2.1.6 understand how a function can be defined parametrically
2.1.7 understand simple applications of parametrically defined functions
2.1.8 use parametric differentiation to find first **and second derivatives [A/B]**

The content listed below is **not** covered within this support material:

2.1.8 (parametric differentiation)…, and apply to motion in a plane

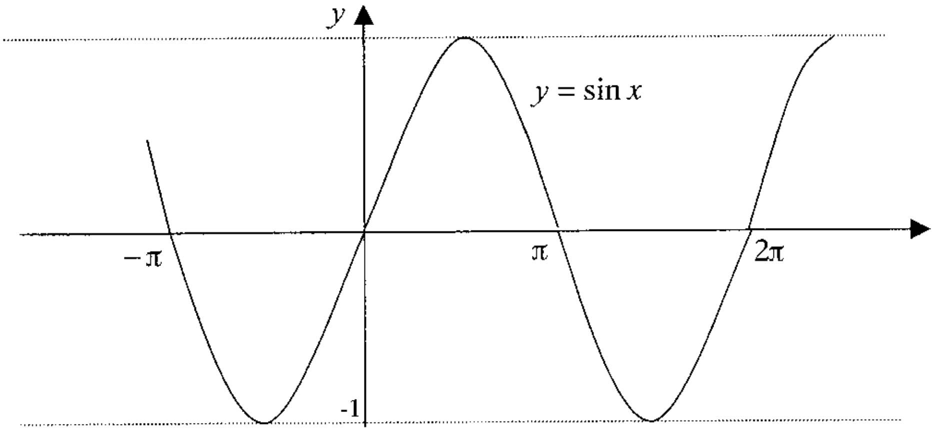
2.1.9 apply differentiation to related rates in problems where the functional relationship is given explicitly or implicitly

2.1.10 solve practical related rates by first establishing a functional relationship between appropriate variables [A/B]

MATHEMATICS 2 (AH) : FURTHER DIFFERENTIATION

The inverse sine function ($\sin^{-1}x$)

The inverse sine function, $\sin^{-1}x$, is defined to be the angle whose sine is x.



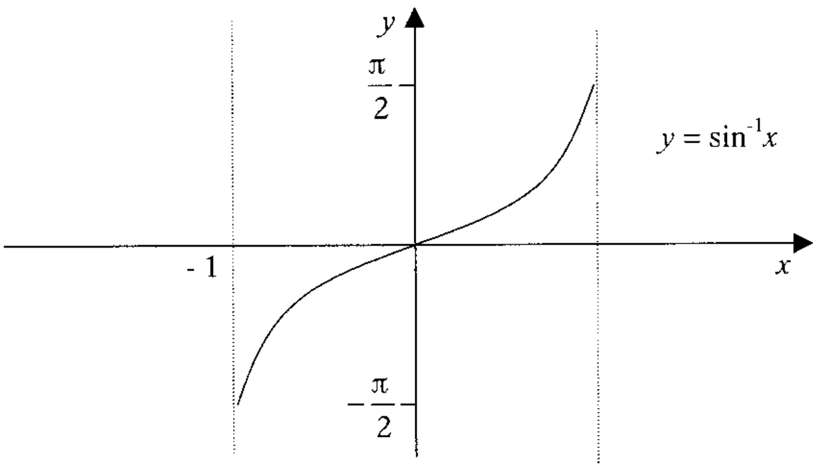
If we consider the graph of $y = \sin x$, we see that there are an infinite number of angles whose sine could be x. Consequently, in order that the inverse sine function should be a true function, we must restrict the angle concerned.

We choose the simplest possible restriction for the angle, namely the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Our definition now is:

The inverse sine function, $\sin^{-1}x$, is defined to be the angle in the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x.

N.B. $\sin^{-1}x$ is sometimes written as $\text{arc}\sin x$.

The graph of $y = \sin^{-1}x$



Example 1

$\sin^{-1}0 = 0$; $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$; $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$; $\sin^{-1}1 = \frac{\pi}{2}$

Example 2

$\sin^{-1}(\sin\pi) = \sin^{-1}(0) = 0$; $\sin^{-1}(\sin\left(\frac{5\pi}{2}\right)) = \sin^{-1}(1) = \frac{\pi}{2}$

Examples for class

Evaluate: (1) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$; (2) $\sin^{-1}(-1)$; (3) $\sin^{-1}\left(-\frac{1}{2}\right)$; (4) $\sin^{-1} 0$;

(5) $\sin^{-1}\left(\sin\left(\frac{5}{4}\pi\right)\right)$; (6) $\sin^{-1}(\sin\ (2\pi))$

Answers

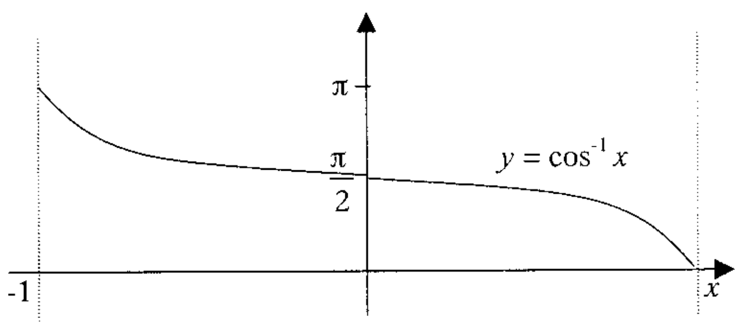
(1) $\frac{\pi}{4}$ (2) $-\frac{\pi}{2}$ (3) $-\frac{\pi}{6}$ (4) 0 (5) $-\frac{\pi}{4}$ (6) 0

The inverse cosine function ($\cos^{-1}x$)

In this case, we restrict the angle to the closed interval $[0, \pi]$.
Our definition now is as follows:

The inverse cosine function ($\cos^{-1}x$ or arc $\cos x$) is defined to be the angle in the closed interval $[0, \pi]$ whose cosine is x.

The graph of $y = \cos^{-1} x$



Example 1

$\cos^{-1} 1 = 0$; $\cos^{-1} 0 = \frac{\pi}{2}$; $\cos^{-1}(-1) = \pi$; $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$;

$\cos^{-1}(\cos\left(-\frac{\pi}{4}\right)) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

Examples for class

Evaluate

(1) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (2) $\cos^{-1}(\cos\left(-\frac{\pi}{2}\right))$ (3) $\cos^{-1}(\cos 2\pi)$ (4) $\cos^{-1}(\cos 3\pi)$

Answers

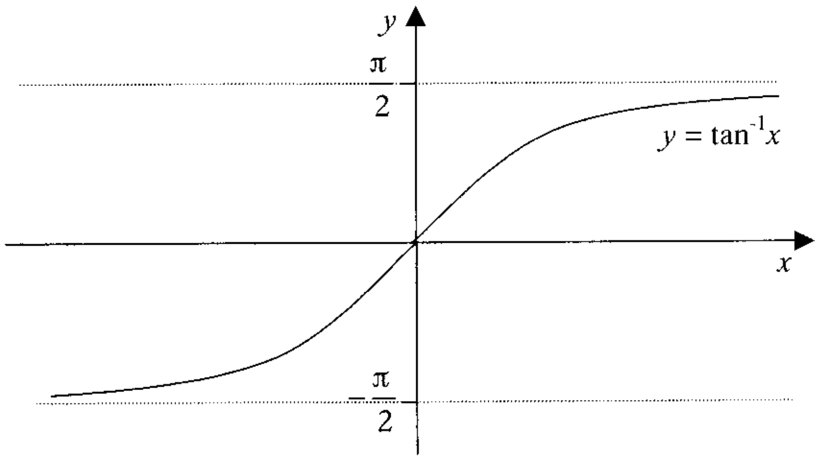
(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3) 0 (4) $\frac{\pi}{2}$

The inverse tangent function ($\tan^{-1}x$)

For this function, the angle is restricted to the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, providing us with the following definition:

The inverse tangent function, ($\tan^{-1}x$ or arc tan x) is defined to be the angle in the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose tangent is x.

The graph of y = tan^{-1}x



Example

$\tan^{-1}0 = 0$; $\tan^{-1}1 = \frac{\pi}{4}$; $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$; $\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$;

$\tan^{-1}(\tan \pi) = \tan^{-1}0 = 0$

Examples for class

Evaluate: (1) $\tan^{-1}(-1)$ (2) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (3) $\tan^{-1}0$ (4) $\tan^{-1}(\tan\frac{3\pi}{4})$

(5) $\tan^{-1}(\tan\frac{2\pi}{3})$ (6) $\tan^{-1}(\tan\frac{5\pi}{4})$

Answers

(1) $-\frac{\pi}{4}$ (2) $\frac{\pi}{6}$ (3) 0 (4) $-\frac{\pi}{4}$ (5) $-\frac{\pi}{3}$ (6) $\frac{\pi}{4}$

More difficult examples

Example 1

Evaluate $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \sin^{-1}\left(-\frac{1}{\sqrt{10}}\right)$

Let $\theta = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$ then $\sin\theta = \frac{1}{\sqrt{5}}$

Let $\phi = \sin^{-1}\left(-\frac{1}{\sqrt{10}}\right)$ then $\sin\phi = \frac{-1}{\sqrt{10}}$

We require $\theta - \phi$

$$\begin{aligned}\sin(\theta - \phi) &= \sin\theta\ \cos\phi - \cos\theta\ \sin\phi \\ &= \frac{1}{\sqrt{5}}.\frac{3}{\sqrt{10}} - \frac{2}{\sqrt{5}}.\left(-\frac{1}{\sqrt{10}}\right) \\ &= \frac{3}{\sqrt{50}} + \frac{2}{\sqrt{50}} \\ &= \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}\end{aligned}$$

Hence $\theta - \phi = \frac{\pi}{4}$

Example 2

Evaluate $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right)$

Let $\theta = \tan^{-1}\left(\frac{1}{3}\right)$ then $\tan\theta = \frac{1}{3}$

Let $\phi = \tan^{-1}\left(\frac{1}{2}\right)$ then $\tan\phi = \frac{1}{2}$

We require $\theta + \phi$.
(Students could be asked to prove the result below, first as an exercise)

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\ \tan\phi}$$

$$= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}.\frac{1}{3}}$$

$$= \frac{\frac{5}{6}}{1 - \frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

Hence $\theta + \phi = \frac{\pi}{4}$

Examples for class

Evaluate

(1) $\sin^{-1}\left(2\sqrt{\frac{3}{13}}\right) - \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{3}{13}}\right)$

(2) Prove that $\tan^{-1} 3 + \tan^{-1} 2 + \tan^{-1} 1 = 0$

(3) Prove that $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

Answers

(1) $\frac{\pi}{3}$ (2) Proof (3) Proof

Derivative of $\sin^{-1} x$

In the following, we assume that $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, provided $\frac{dy}{dx} \neq 0$

Let $y = \sin^{-1} x$ then $x = \sin y$

and $\frac{dx}{dy} = \cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}$

hence $\frac{dy}{dx} = \pm\frac{1}{\sqrt{1 - x^2}}$

Since $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos y > 0$

And so $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$

Example 1

$$\frac{d}{dx}\left[\sin^{-1}\left(\sqrt{1 - x}\right)\right] = \frac{1}{\sqrt{1 - \left(\sqrt{1 - x}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{1 - x}\right)$$

$$= \frac{1}{\sqrt{1 - (1 - x)}} \cdot \frac{1}{2\sqrt{1 - x}} \cdot (-1)$$

$$= -\frac{1}{2\sqrt{x}\sqrt{1 - x}} \qquad = -\frac{1}{2\sqrt{x - x^2}}$$

Example 2

$$\frac{d}{dx}\left[\left(\sin^{-1} x\right)^2\right] = 2\sin^{-1} x \cdot \frac{1}{\sqrt{1 - x^2}} = \frac{2\sin^{-1} x}{\sqrt{1 - x^2}}$$

Example 3

$$\frac{d}{dx}\left[\sin^{-1}\left(\sqrt{1 - x^2}\right)\right] = \frac{1}{\sqrt{1 - \left(\sqrt{1 - x^2}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{1 - x^2}\right) = \frac{1}{\sqrt{1 - \left(1 - x^2\right)}} \cdot \frac{1}{2\sqrt{1 - x^2}} \cdot -2x$$

$$= -\frac{x}{\sqrt{x^2}\sqrt{1 - x^2}}$$

$$= -\frac{x}{x\sqrt{1 - x^2}}$$

$$= -\frac{1}{\sqrt{1 - x^2}}$$

Examples for class

Differentiate with respect to x

(1) $\sin^{-1}(1-x)$ (2) $\sin^{-1}\left(\frac{x-1}{x+1}\right)$ (3) $\sin^{-1}\sqrt{1-x^2}$ (4) $x\sin^{-1}x^2$ (5) $\frac{1}{\sin^{-1}x}$

(6) $\frac{\sin^{-1}2x}{\sin x}$ (7) $\sin^{-1}4x$ (8) $\sin^{-1}(\sqrt{x})$ (9) $\sin^{-1}\left(\frac{x}{2}\right)$ (10) $\sin^{-1}\left(\frac{1}{x}\right)$

(11) $\sin^{-1}(\sin x)$ (12) $x\sin^{-1}x$ (13) $\sin^{-1}(\sqrt{\sin x})$ (14) $\sin^{-1}\frac{x}{\sqrt{1+x^2}}$

Answers

(1) $-\frac{1}{\sqrt{2x-x^2}}$ (2) $\frac{1}{\sqrt{x}}$ (3) $-\frac{1}{2x\sqrt{1-x}}$ (4) $\sin^{-1}x^2+\frac{2x^2}{\sqrt{1-x^4}}$

(5) $-\frac{\sin^{-1}x}{\sqrt{1-x^2}}$ (6) $\frac{2}{\sqrt{1-4x^2}\sin x}-\frac{\cos x\sin^{-1}2x}{\sin x}$ (7) $\frac{4}{\sqrt{1-16x^2}}$

(8) $\frac{1}{2\sqrt{x-x^2}}$ (9) $\frac{1}{\sqrt{4-x^2}}$ (10) $\frac{-1}{x\sqrt{x^2-1}}$ (11) 1

(12) $\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}$ (13) $\frac{\cos x}{2\sqrt{\sin x(1-\sin x)}}$ (14) $\frac{1}{1+x^2}$

Derivative of $\cos^{-1}x$

Let $y=\cos^{-1}x$ then $x=\cos y$

and $\frac{dx}{dy}=-\sin y=-\sqrt{1-\cos^2 y}$, since $0\le y\le\pi$ and $0\le\sin y\le 1$

$= -\sqrt{1-x^2}$

and so $\frac{dy}{dx}=\frac{-1}{\sqrt{1-x^2}}$ i.e. $\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}$

Example 1

$$\frac{d}{dx}\cos^{-1}\left(\frac{1}{x}\right)=\frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\cdot\frac{-1}{x^2}=\frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}=\frac{1}{x^2\frac{\sqrt{x^2-1}}{\sqrt{x^2}}}=\frac{1}{x\sqrt{x^2-1}}$$

Example 2

$$\frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}.\frac{1}{a} = \frac{-1}{a\sqrt{\frac{a^2-x^2}{a^2}}} = \frac{-1}{a^1\frac{\sqrt{a^2-x^2}}{a^1}} = \frac{-1}{\sqrt{a^2-x^2}}$$

Examples for class

Differentiate with respect to x:

(1) $\cos^{-1}(3x+5)$ (2) $\cos^{-1}\frac{1}{\sqrt{x}}$ (3) $\cos^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)$ (4) $b\cos^{-1}\left(\frac{x}{a}\right)$

(5) $\cos^{-1}\left(\frac{x}{3}\right)$ (6) $\cos^{-1}(2x^2)$

Answers

(1) $-\frac{3}{\sqrt{24-30x-9x^2}}$ (2) $\frac{1}{x\sqrt{x-1}}$ (3) $\frac{x^{½}}{(1+x)(2-2x)^{½}}$

(4) $-\frac{b}{\sqrt{a^2-x^2}}$ (5) $-\frac{1}{\sqrt{9-x^2}}$ (6) $-\frac{4x}{\sqrt{1-4x^4}}$

Derivative of $\tan^{-1} x$

Let $y = \tan^{-1} x$ then $x = \tan y$

and so $\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2$

Hence $\frac{dy}{dx} = \frac{1}{1+x^2}$ i.e. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

Example 1

$$\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{a}\right)\right] = \frac{1}{1+\frac{x^2}{a^2}}.\frac{1}{a} = \frac{a^2}{a^2+x^2}.\frac{1}{a} = \frac{a}{a^2+x^2}$$

Example 2

$$\frac{d}{dx}\left[\tan^{-1}\sqrt{x-1}\right]=\frac{1}{1+\left(\sqrt{x-1}\right)^2}\cdot\frac{1}{2\sqrt{x-1}}=\frac{1}{1+(x-1)}\cdot\frac{1}{2\sqrt{x-1}}=\frac{1}{2x\sqrt{x-1}}$$

Examples for class

Differentiate with respect to x

(1) $\tan^{-1}\frac{1}{x}$ (2) $\tan^{-1}(1-2x)$ (3) $\tan^{-1}\sqrt{1-x}$ (4) $\tan^{-1}(3\sin 2x)$

(5) $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ (6) $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ (7) $\tan^{-1}(a-x)$

(8) $(x^2+1)\tan^{-1}x$ (9) $\tan^{-1}\left(\frac{2x}{\sqrt{1-x^2}}\right)$ (10) $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

(11) $x\tan^{-1}x$

Answers

(1) $-\frac{1}{1+x^2}$ (2) $-\frac{1}{1-2x+2x^2}$ (3) $-\frac{1}{2x(1-x)^{½}}$

(4) $\frac{6\cos 2x}{1+9\sin^2 2x}$ (5) $\frac{1}{(1-x^2)^{½}}$ (6) $\frac{3(1+x^2)^2}{(1-3x^2)^2+(3x-x^3)^2}$

(7) $-\frac{1}{1+(a-x)^2}$ (8) $2x\tan^{-1}x+1$ (9) $\frac{2}{(1+3x^2)(1-x^2)^{½}}$

(10) $\frac{x-2}{2(1-x)^{½}(x^2-x+1)}$ (11) $\tan^{-1}x+\frac{x}{1+x^2}$

Differentiation of implicit functions

Consider the function y defined by the equation

$3x^2 + 7xy + 9y^2 = 6$

It is impossible to write y in terms of x. Such a function y is called an **implicit function.**

To find $\frac{dy}{dx}$, we take derivatives of both sides.

Example 1

$$3x^2 + 7xy + 9y^2 = 6$$

$$\Rightarrow \frac{d}{dx}(3x^2 + 7xy + 9y^2) = \frac{d}{dx}(6)$$

$$\Rightarrow 6x + 7\frac{d}{dx}(xy) + 9\frac{d}{dx}y^2 = 0$$

$$\Rightarrow 6x + 7\left(y + x\frac{dy}{dx}\right) + 9\left(2y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow 6x + 7y + 7x\frac{dy}{dx} + 18y\frac{dy}{dx} = 0$$

$$\Rightarrow 7x\frac{dy}{dx} + 18y\frac{dy}{dx} = -6x - 7y$$

$$\Rightarrow \frac{dy}{dx}(7x + 18y) = -6x - 7y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-6x - 7y}{7x + 18y}$$

Examples for class

Find $\frac{dy}{dx}$ for each of the following:

(1) $x^2 - y^2 = 4$ (2) $y^2 = 2ax + 2by$ (3) $x^2 - 4x + 6y = 0$ (4) $y^2 = 4ax$

(5) $x^2 = 4ay$ (6) $(x^2 + y^2)^2 - (x^2 - y^2) = 0$ (7) $x^3 + y^3 = 3xy$

(8) $\frac{x^2}{4} + \frac{y^2}{5} = 1$ (9) If $(x + y)^3 - 5x + y = 1$ defines y, find $\frac{dy}{dx}$

(10) If $(x + y)^4 = 4xy$, show that $\frac{dy}{dx} = \frac{y(3x - y)}{x(x - 3y)}$

Answers

(1) $\frac{x}{y}$ (2) $\frac{a}{y-b}$ (3) $\frac{2}{3}-\frac{1}{3}x$ (4) $\frac{2a}{y}$

(5) $\frac{x}{2a}$ (6) $\frac{x-2x^3-2xy}{2x^2y+2y^3+y}$ (7) $\frac{y-3x^2}{y^2-3x}$ (8) $\frac{5x}{4y}$

(9) $\frac{5-3(x+y)^2}{1+3(x+y)^2}$ (10) Proof

Logarithmic differentiation

If the independent variable occurs as the power or part of the power then **logarithmic differentiation** should be employed as follows:-

Example 1

Differentiate x^x

Let $y = x^x$

Take logs of both sides

$\ln y = \ln x^x$

$\Rightarrow \ln y = x \ln x$

$\Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln x + x.\frac{1}{x}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln x + 1$

$\Rightarrow \frac{dy}{dx} = y\,(\ln x + 1)$

$\Rightarrow \frac{d}{dx}(x^x) = x^x\,(\ln x + 1)$

Example2

Differentiate a^x

Let $y = a^x$

Take logs of both sides

$\ln y = \ln a^x$

$\Rightarrow \ln y = x \ln a$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln a$

$\Rightarrow \frac{dy}{dx} = y \ln a$

$\Rightarrow \frac{d}{dx}(a^x) = a^x \ln a$

Examples for class

Use logarithmic differentiation to differentiate the following:

(1) 10^x (2) 2^x (3) $x\,a^x$ (4) $x^{\log x}$ (5) $(\sin x)^x$ (6) $2^{\sin x}$

(7) $(\log x)^x$ (8) x^{x^2} (9) $x^2.\,2^x$ (10) $(\log x)^{\log x}$

Answers

(1) $10^x.\ln x$ (2) $2^x.\ln 2$ (3) $a^x + x.a^x.\ln a$ (4) $2\ln x.x^{\ln x - 1}$

(5) $(\sin x)^x(\ln(\sin x) + \frac{x\cos x}{\sin x})$ (6) $2^{\sin x}.\ln 2.\cos x$

(7) $(\ln x)^x[\ln(\ln x) + \frac{1}{\ln x}]$ (8) $x^{x^2}(2x\ln x + x)$

(9) $2^x(2x + x^2\ln x)$ (10) $(\ln x)^{\ln x}\frac{1}{x}(1 + \ln(\ln x))$

Logarithmic differentiation can also be used to differentiate complicated products.

Example 3

Differentiate $\sqrt{\frac{1+x}{1-x}}$

Let $y = \sqrt{\frac{1+x}{1-x}}$

Take logs of both sides

$$\log y = \log\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$$

$$\Rightarrow \log y = \frac{1}{2}[\log(1+x) - \log(1-x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{1+x} + \frac{1}{1-x}\right] = \frac{1}{2}.\frac{2}{(1+x)(1-x)} = \frac{1}{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{1-x^2}\right) = \frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}.\frac{1}{(1+x)(1-x)} = \frac{1}{(1-x)^{\frac{3}{2}}(1+x)^{\frac{1}{2}}}$$

Examples for class

Differentiate

(1) $\dfrac{x^{\frac{1}{2}}(3-x)^{\frac{1}{2}}}{(2x+1)^{\frac{3}{4}}}$ (2) $\dfrac{(2x+1)^{\frac{1}{2}}(4x+5)^{\frac{1}{4}}}{(4x-3)^{\frac{3}{4}}}$ (3) $\dfrac{x^{\frac{3}{4}}(2-x)^2(1+x)}{(5+2x)^3}$

Answers

(1) $\dfrac{x^{1/2}(3-x)^{1/2}}{(2x+1)^{3/4}}\left(\dfrac{1}{2x}-\dfrac{1}{2(3-x)}-\dfrac{3}{2(2x+1)}\right)$

(2) $\dfrac{(4x+5)^{1/4}}{(2x+1)^{1/2}(4x-3)^{3/4}}+\dfrac{(2x+1)^{1/2}}{((4x-3)(4x+5))^{3/4}}-\dfrac{3(2x+1)^{1/2}(4x+5)^{1/4}}{(4x-3)^{7/4}}$

(3) $\dfrac{x^{3/4}(2-x)^2(1+x)}{(5+2x)^3}\left(\dfrac{3}{4x}-\dfrac{2}{2-x}+\dfrac{1}{1+x}-\dfrac{6}{5+2x}\right)$

Curves by parametric equations

Parametric equations

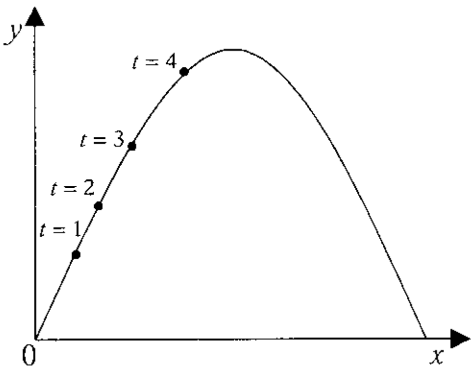
It is sometimes convenient to define the co-ordinates of a moving point by means of two equations, expressing x and y separately in terms of a third variable, say t, called a **parameter.** E.g. the position of a ball relative to the x and y may be plotted at 1 second intervals. Here the x and y axes positions of the ball depend on time t.

It would be found that the path of this ball is given by the equations:

$x = at^2, y = 2at$, where a is constant.

These equations are called the **parametric equations** of the curve.

t is the **parameter.**



By replacing t by various values in the two equations, we can obtain the co-ordinates of various points on the curve.

e.g. At point P(t), $x = at^2, y = 2at$. Hence P is the point $(at^2, 2at)$

$P_0(0), x = 0, \quad y = 0$. Hence P_0 is the point (0, 0)

$P_1(1), x = a, \quad y = 2a$ Hence P_1 is the point $(a, 2a)$

$P_2(2), x = 4a, \quad y = 4a$ Hence P_2 is the point $(4a, 4a)$ etc.

It is often useful to eliminate t from the two parametric equations to obtain an equation involving x and y but not t. This equation is called the **constraint equation.**

Here $x = at^2, y = 2at$

$$x = at^2,\ t = \frac{y}{2a}$$

$$x = a\,\frac{y^2}{4a^2}$$

$y^2 = 4ax$ - **constraint equation**

Example 1

Eliminate θ from the parametric equations $x = a\sec\theta$, $y = b\tan\theta$

$$x = a\sec\theta,\ y = b\tan\theta$$

$$\sec\theta = \frac{x}{a},\ \tan\theta = \frac{y}{b}$$

$$\sec^2\theta - \tan^2\theta = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Examples for class

(1) Eliminate t from $x = ct,\ y = \dfrac{c}{t}$

(2) Eliminate θ from $x = a\cos\theta$, $y = a\sin\theta$

Answers

(1) $y = \dfrac{c^2}{x}$ (2) $x^2 + y^2 = a^2$

Expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of a parameter t

Suppose that a curve C has parametric equations $x = x(t), y = y(t)$.

Since $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$ and $\frac{dx}{dt} = \frac{1}{\frac{dt}{dx}}$

We have $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Example 1

Find $\frac{dy}{dx}$ at P(t) on the curve $x = at^2, y = 2at$

$y = 2at \Rightarrow \frac{dy}{dt} = 2a$

$x = at^2 \Rightarrow \frac{dx}{dt} = 2at$

Thus $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

Example 2

Find a formula for the gradient of the tangent to the curve whose parametric equations are

$x = a(t - \sin t), y = a(1 - \cos t)$.

$y = a(1 - \cos t) \Rightarrow \frac{dy}{dt} = a \sin t$

$x = a(t - \sin t) \Rightarrow \frac{dx}{dt} = a(1 - \cos t)$

Thus $m_{\text{tangent}} = \frac{dy}{dx} = \frac{a \sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t}$

Example 3

Find the co-ordinates of the points on the curve $x = 1 - t^2, y = t^3 + t$, at which the gradient is 2.

$y = t^3 + t \Rightarrow \frac{dy}{dt} = 3t^2 + 1$

$x = 1 - t^2 \Rightarrow \frac{dx}{dt} = -2t$

$m_{\text{tangent}} = \frac{dy}{dx} = \frac{3t^2+1}{-2t}$

$m_{\text{tangent}} = 2 \Leftrightarrow \frac{-3t^2+1}{2t} = 2 \Leftrightarrow 3t^2+4t+1=0 \Leftrightarrow t=-1 \quad \text{or} \quad \frac{-1}{3}$

t = -1 gives the point (0, -2), t = $\frac{-1}{3}$ gives the point $\left(\frac{8}{9}, \frac{-10}{27}\right)$

and these are the points at which the gradient is 2.

Examples for class

(1) Find the equation of the tangent to each of the following curves at the given points:

(a) $x = t^2 - 2t, y = 1 - t^4$ at the point t = -1.

(b) $x = \sin t, y = \cos 2t \left(-\frac{1}{2}\pi \le t \le \frac{1}{2}\pi\right)$ at the point $t = \frac{1}{4}\pi$.

(2) Show that there are four points on the curve $x = \frac{t^3}{t^2+1}, \quad y = \frac{t}{t^2+1}$

at which the gradient is equal to $-\frac{1}{10}$.

(3) Find the equation of the tangent at the point with parameter t = 2 on the curve

$x = \frac{4t}{t^3+1}, \quad y = \frac{t^2}{t^3+1} \quad (t \ne -1)$.

Find the parameter of the other point at which the tangent meets the curve.

Answers

(1) (a) $y = -x + 3$ (b) $y = -2\sqrt{2}x + 2$

(2) Proof

(3) $y - \frac{4}{9} = \frac{1}{5}(x - \frac{8}{9})$

(4) $t = -\frac{1}{4}$

Extension examples (covers 2^{nd} order derivatives)

Example 1

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x = at^2, y = 2at$.

$y = 2at \Rightarrow \frac{dy}{dt} = 2a$

$x = at^2 \Rightarrow \frac{dx}{dt} = 2at$

Thus $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}\left(\frac{1}{t}\right).\frac{dt}{dx} = \frac{-1}{t^2}.\frac{1}{2at} = \frac{-1}{2at^3}$

Note: $\frac{d^2y}{dx^2} = \frac{d}{dt}(\frac{dy}{dx}).\frac{dt}{dx}$

Examples for class

Find $\frac{d^2y}{dx^2}$ in terms of t:

(1) $x = t^2, y = t^3$ (2) $x = at,\ y = \frac{a}{t}$ (3) $x = \frac{1}{1+t},\ \ y = \frac{t}{1-t}$

(4) $x = \frac{t-1}{t+1},\ \ y = \frac{2t-1}{t-2}$ (5) $x = 3t - t^3,\ y = 1 - t^2$

(6) $x = \tan\theta$, $y = \sin 2\theta$ (7) $x = t^2 + t,\ y = 3 - t^3$

Answers

(1) $\frac{3}{4t}$ (2) $\frac{2}{at^3}$ (3) $\frac{4(1+t)^3}{(1-t)^3}$ (4) $\frac{9(t+1)^3}{2(t-2)^3}$

(5) $\frac{-2(1-3t)(1+t)}{(1-t)^2}$ (6) $2\sin 2\theta\ (\cos^2\theta - 4\cos^4\theta)$ (7) $\frac{-6t(t+1)}{(2t+1)^3}$

MATHEMATICS 2 (AH)

CONTENT
Further integration 2.2.1 know the integrals of $\frac{1}{\sqrt{1-x^2}}, \frac{1}{1+x^2}$ use the substitution $x = at$ to integrate functions of the form $\frac{1}{\sqrt{a^2-x^2}}, \frac{1}{a^2+x^2}$ integrate rational functions, both proper and improper by means of partial fractions; the degree of the denominator being ≤ 3 the denominator may include: (i) two separate or repeated linear factors **(ii) three linear factors [A/B]** **(iii) a linear factor and an irreducible quadratic factor of the form $x^2 + a^2$ [A/B]** 2.2.2 integrate by parts with one application **2.2.3 integrate by parts involving repeated applications [A/B]** 2.2.4 know the definition of differential equation and the meaning of the terms linear, order, general solution, arbitrary constants, particular solution, initial condition 2.2.5 solve first order differential equations (variables separable)

The content listed below is **not** covered within this support material:

2.2.6 formulate a simple statement involving rate of change as a simple separable first order differential equation, including the finding of a curve in a plane, given the equation of the tangent at (x, y), which passes through a given point

2.2.7 know the laws of growth and decay : applications in practical contexts

Notes
1. Some of the examples include the integration of rational functions with denominators > 3, these have been included as extension.
2. The section on differential equations contains a short insert on homogeneous equations which are covered in Mathematics 3 (AH), as above, they have been included as extension.

MATHEMATICS 2 (AH): FURTHER INTEGRATION

Application of partial fractions to integration of rational functions

Example 1

$$\int \frac{x+35}{x^2-25}\,dx$$

Let $\dfrac{x+35}{x^2-25} = \dfrac{A}{x+5} + \dfrac{B}{x-5}$, where A, B are constants

Then $x + 35 = A(x - 5) + B(x + 5)$

Put $x = 5$ gives $40 = 10B \Rightarrow B = 4$
Put $x = -5$ gives $30 = -10A \Rightarrow A = -3$

Hence $\dfrac{x+35}{x^2-25} = \dfrac{-3}{x+5} + \dfrac{4}{x-5}$

$$\int \frac{x+35}{x^2-25}\,dx = \int \left(\frac{4}{x-5} - \frac{3}{x+5}\right)dx$$

$= 4\ln(x-5) - 3\ln(x+5) + C$, where C is a constant

Example 2

$$\int \frac{x^2+10x+6}{x^2+2x-8}\,dx$$

$$\begin{array}{r} 1 \\ x^2+2x-8 \overline{)\, x^2+10x+6} \\ \underline{x^2+\ 2x-8} \\ 8x+14 \end{array}$$

$$\frac{x^2+10x+6}{x^2+2x-8} = 1 + \frac{8x+14}{x^2+2x-8}$$

Let $\dfrac{8x+14}{x^2+2x-8} = \dfrac{A}{x+4} + \dfrac{B}{x-2}$, where A, B are constants

Then $8x + 14 = A(x - 2) + B(x + 4)$

Put $x = 2$ gives $30 = 6B \Rightarrow B = 5$
Put $x = -4$ gives $-18 = -6A \Rightarrow A = 3$

Hence $\dfrac{x^2+10x+6}{x^2+2x-8} = 1 + \dfrac{3}{x+4} + \dfrac{5}{x-2}$

Hence $\displaystyle\int \frac{x^2+10x+6}{x^2+2x-8}\,dx = \int\left(1 + \frac{3}{x+4} + \frac{5}{x-2}\right)dx$

$= x + 3\ln(x + 4) + 5\ln(x - 2) + C$, where C is a constant

Examples for class

(1) $\int \frac{dx}{x^2-1}$ (2) $\int \frac{x^2}{x^2-4}\,dx$ (3) $\int \frac{x+8}{x^2+6x+8}\,dx$

(4) $\int \frac{3x-1}{x^2+x-6}\,dx$ (5) $\int \frac{2x^3-2x^2-11x-8}{x^2-x-6}\,dx$

Answers

(1) $\frac{1}{2}\ln(x-1)-\frac{1}{2}\ln(x+1)+C$ (2) $x+\ln(x-2)-\ln(x+2)+C$

(3) $-2\ln(x+4)+3\ln(x+2)+C$ (4) $2\ln(x+3)+\ln(x-2)+C$

(5) $x^2-\ln(x-3)+2\ln(x+2)+C$

Example 3

$$\int \frac{3x+1}{(x+1)^2}\,dx$$

Let $\frac{3x+1}{(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}$, where A, B are constants

Then $3x+1=A(x+1)+B$

Put $x=-1$ gives $-2=B$
Put $x=0$ gives $1=A+B \Rightarrow A=1+2=3$

Hence $\frac{3x+1}{(x+1)^2}=\frac{3}{x+1}-\frac{2}{(x+1)^2}$

$$\int \frac{3x+1}{(x+1)^2}\,dx=\int\left[\frac{3}{x+1}-\frac{2}{(x+1)^2}\right]dx$$

$= 3\ln(x+1)+\frac{2}{x+1}+C$, where C is a constant

Examples for class

(1) $\int \frac{2x-1}{(x+2)^2}\,dx$ (2) $\int \frac{2x+1}{(x+2)(x-3)^2}\,dx$

(3) $\int \frac{x}{(x+1)^2(x-1)}\,dx$ (4) $\int \frac{x^3+1}{x(x-1)^3}\,dx$

Answers

(1) $2\ln(x+2) + \frac{5}{x+2}$

(2) $-\frac{3}{25}\ln(x+2) - \frac{19}{175}\ln(x-3) - \frac{5}{7}(x-3)^{-1} + C$

(3) $-\frac{1}{4}\ln(x+1) + \frac{1}{4}\ln(x-1) - \frac{1}{2(x+1)} + C$

(4) $\ln x + \frac{1}{3}\ln(x-1) - \frac{5}{3(x-1)} - \frac{1}{(x-1)^2} + C$

Example 4

$$\int \frac{x-1}{(x+1)(x^2+1)}\,dx$$

Let $\frac{x-1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$, where A, B, C are constants

Then $x - 1 = A(x^2 + 1) + (Bx + C)(x + 1)$

Put $x = -1$ gives $-2 = 2A \Rightarrow A = -1$
Put $x = 0$ gives $-1 = A + C \Rightarrow C = -1 + 1 = 0$
Put $x = 1$ gives $0 = 2A + 2(B + C) \Rightarrow 0 = -2 + 2B \Rightarrow B = 1$

Hence $\frac{x-1}{(x+1)(x^2+1)} = -\frac{1}{x+1} + \frac{x}{x^2+1}$

$$\int \frac{x-1}{(x+1)(x^2+1)}\,dx = \int \left[\frac{x}{x^2+1} - \frac{1}{x+1}\right]dx$$

$$= \frac{1}{2}\int \frac{2x}{(x^2+1)}\,dx - \int \frac{1}{x+1}\,dx$$

$= \frac{1}{2}\ln(x^2+1) - \ln(x+1) + C$, where C is a constant.

Special integrals

1) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

2) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

3) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

4) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Example 5

$\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$, where C is a constant

Example 6

$$\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{\sqrt{4\left(\frac{9}{4}-x^2\right)}} = \frac{1}{2}\int \frac{dx}{\sqrt{\frac{9}{4}-x^2}} = \frac{1}{2}.\sin^{-1}\left(\frac{x}{\frac{3}{2}}\right) + C = \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$$

where C is a constant

Example 7

$\int \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$, where C is a constant

Example 8

$$\int \frac{dx}{9x^2+4} = \int \frac{dx}{4+9x^2} = \int \frac{dx}{9\left(\frac{4}{9}+x^2\right)} = \frac{1}{9}\int \frac{dx}{\left(\frac{4}{9}+x^2\right)} = \frac{1}{\cancel{9}_3}.\frac{\cancel{3}^1}{2}\tan^{-1}\left(\frac{3}{2}x\right) + C$$

$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}x\right) + C$ where C is a constant

Examples for class

(1) $\int \frac{dx}{x(x^2+1)}$ (2) $\int \frac{x\,dx}{x^4-1}$ (3) $\int \frac{dx}{\sqrt{4-x^2}}$

(4) $\int \frac{dx}{9+x^2}$ (5) $\int \frac{dx}{\sqrt{16-x^2}}$ (6) $\int \frac{dx}{x^2+16}$

(7) $\int \frac{dx}{\sqrt{25-9x^2}}$ (8) $\int \frac{dx}{4x^2+9}$ (9) $\int \frac{dx}{9x^2+4}$

(10) $\int \frac{dx}{\sqrt{5-x^2}}$ (11) $\int \frac{(x+1)^2\,dx}{x^3+x}$ (12) $\int \frac{dx}{(x^2+1)(x-2)}$

(13) $\int \frac{x\,dx}{(x+1)(x^2+4)}$ (14) $\int \frac{(x+2)\,dx}{(x^2+4)(1-x)}$

Answers

(1) $\ln x - \frac{1}{2}\ln(x^2+1)+C$

(2) $\frac{1}{4}\ln(x-1)+\frac{1}{4}\ln(x+1)-\frac{1}{4}\ln(x^2+1)+C$

(3) $\sin^{-1}\frac{x}{2}+C$ (4) $\frac{1}{3}\tan^{-1}\frac{x}{3}+C$ (5) $\sin^{-1}\frac{x}{4}+C$

(6) $\frac{1}{4}\tan^{-1}\frac{x}{4}+C$ (7) $\frac{1}{3}\sin^{-1}\frac{3x}{5}+C$ (8) $\frac{1}{6}\tan^{-1}\frac{2x}{3}+C$

(9) $\frac{1}{6}\tan^{-1}\frac{3x}{2}+C$ (10) $\sin^{-1}\frac{x}{\sqrt{5}}+C$ (11) $\ln x + 2\tan^{-1}x + C$

(12) $\frac{1}{5}\ln(x-2)-\frac{1}{10}\ln(x^2+1)-\frac{2}{5}\tan^{-1}x+C$

(13) $-\frac{1}{5}\ln(x+1)+\frac{1}{10}\ln(x^2+4)+\frac{2}{5}\tan^{-1}\frac{x}{2}+C$

(14) $-\frac{3}{5}\ln(1-x)+\frac{3}{10}\ln(x^2+4)-\frac{2}{10}\tan^{-1}\frac{x}{2}+C$

Integration by parts
Indefinite integral

We have already seen that $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$
Integrating both sides of the equation gives

$$uv = \int u\frac{dv}{dx}.dx + \int v\frac{du}{dx}.dx$$

Rearranging

$$\int\left(u\frac{dv}{dx}\right)dx = uv - \int\left(v\frac{du}{dx}\right)dx$$

Example 1

$$\int x\cos x\,dx = x\sin x - \int 1.\sin x\,dx$$

$= x\sin x + \cos x + C$, where C is a constant

Example 2

$$\int xe^{ax}dx = \frac{1}{a}xe^{ax} - \int 1.\frac{1}{a}e^{ax}dx$$

$= \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax} + C$, where C is a constant

Example 3

$$\int x^2\sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$$

$$= -x^2\cos x + 2x\sin x - 2\int \sin x\,dx$$

$= -x^2\cos x + 2x\sin x + 2\cos x + C$, where C is a constant

Example 4

$$\int \ln x\,dx = \int 1.\ln x\,dx = x\ln x - \int x.\frac{1}{x}dx$$

$= x\ln x - \int 1\,dx = x\ln x - x + C$, where C is a constant

Example 5

$$\int \sin^{-1} x\, dx = \int 1.\sin^{-1} x\, dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x\sin^{-1} x - \left(-\sqrt{1-x^2}\right) + C = x\sin^{-1} x + \sqrt{1-x^2} + C$$, where C is a constant

Examples for class

(1) $\int x\sin x\, dx$ (2) $\int x\sin 3x\, dx$ (3) $\int x^2 \cos x\, dx$ (4) $\int x\ln x\, dx$

(5) $\int x^2 \ln x\, dx$ (6) $\int \sqrt{x}\, \ln x\, dx$ (7) $\int xe^x\, dx$ (8) $\int x^2 e^x\, dx$

(9) $\int e^x \cos 2x\, dx$ (10) $\int \tan^{-1} x\, dx$ (11) $\int x\tan^{-1} x\, dx$ (12) $\int e^x \sin x\, dx$

(13) $\int x\sin^2 x\, dx$

Answers

(1) $-x \cos x + \sin x + C$

(2) $-\frac{x}{3}\cos 3x + \frac{1}{9}\sin 3x + C$

(3) $x^2\sin x + 2x\cos x - 2\sin x + C$

(4) $\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

(5) $\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

(6) $\frac{2}{3}x^{3/2} \ln x - \frac{4}{9}x^{3/2} + C$

(7) $xe^x - e^x + C$

(8) $x^2e^x - 2xe^x + 2e^x + C$

(9) $\frac{1}{5}e^x \cos 2x + \frac{2}{5}e^x \sin 2x + C$

(10) $x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$

(11) $x^2 \tan^{-1} x - \frac{1}{2}\ln(1+x^2) - x\tan^{-1} x + \frac{1}{2}\ln(1+x^2) + C$

(12) $\frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x + C$

(13) $\frac{1}{2}x^2 \sin^2 x + \frac{1}{4}x^2 \cos 2x - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$

Definite Integrals

Example

$$\int_0^\infty e^{-\frac{x}{3}}\sin 2x\,dx = \left[-\frac{1}{2}e^{-\frac{x}{3}}\cos 2x\right]_0^\infty - \int_0^\infty\left(-\frac{1}{2}.-\frac{1}{3}e^{-\frac{x}{3}}\cos 2x\right)dx$$

$$= 0-\left(-\frac{1}{2}\right)-\frac{1}{6}\int_0^\infty e^{-\frac{x}{3}}\cos 2x\,dx$$

$$=\frac{1}{2}-\frac{1}{6}\left(\left[\frac{1}{2}e^{-\frac{x}{3}}\sin 2x\right]_0^\infty - \int_0^\infty\left(\frac{1}{2}.-\frac{1}{3}e^{-\frac{x}{3}}\sin 2x\right)dx\right)$$

$$=\frac{1}{2}-\frac{1}{6}\left((0)-(0)+\frac{1}{6}\int_0^\infty e^{-\frac{x}{3}}\sin 2x\,dx\right)$$

$$=\frac{1}{2}-\frac{1}{36}\int_0^\infty e^{-\frac{x}{3}}\sin 2x\,dx$$

$$\Rightarrow \frac{37}{36}\int_0^\infty e^{-\frac{x}{3}}\sin 2x\,dx = \frac{1}{2}$$

$$\Rightarrow \int_0^\infty e^{-\frac{x}{3}}\sin 2x\,dx = \frac{18}{37}$$

Examples for class

(1) $\int_1^e \frac{1}{x^2}\ln x\,dx$ (2) $\int_0^\infty x^3 e^{-x}\,dx$ (3) $\int_0^\infty e^{-x}\sin 2x\,dx$

(4) $\int_0^1 \tan^{-1}x\,dx$ (5) $\int_{\frac{1}{2}}^1 x\sin^{-1}x\,dx$

Answers

(1) $1-2e^{-1}$ (2) 6 (3) $\frac{2}{5}$ (4) $\frac{\pi}{4}-\frac{1}{2}\ln 2$ (5) $\frac{\pi}{24}-\frac{3\sqrt{3}}{4}$

Differential equations

Introduction

Suppose $y = ce^x$, then $\frac{dy}{dx} = ce^x$

So that, for any particular value of c, we can say that $y = ce^x$ is a solution of the equation $\frac{dy}{dx} - y = 0$.

We call an equation with a derivative as part of it, **a differential equation.**
The order of the highest derivative in the equation gives the **order** of the differential equation.

Here our equation is of the **first order**.

Finally the equation $\frac{dy}{dx} - y = 0$ has a **general solution** $y = ce^x$, where c is a constant.

Suppose $y = c_1 \cos x + c_2 \sin x$, where c_1 and c_2 are constants.

Then $\frac{dy}{dx} = -c_1 \sin x + c_2 \cos x$

and $\frac{d^2y}{dx^2} = -c_1 \cos x - c_2 \sin x = -y$

In this case, the second order differential equation $\frac{d^2y}{dx^2} + y = 0$ has a general solution $y = c_1 \cos x + c_2 \sin x$, where c_1 and c_2 are constants.

Differential equations of the first order

Such an equation can be expressed in the form $\frac{dy}{dx} = f(x, y)$.

Because the equation is of the first order, the general solution involves one constant.

We consider three special types

A. Simple

B. Variables separable

C. Homogeneous

A. **Simple**

Method: Solve by direct integration

Example 1

Find the general solution of the differential equation $\frac{dy}{dx} = 2x$

$$\frac{dy}{dx} = 2x$$

$$\Leftrightarrow \int \frac{dy}{dx}\,dx = \int 2x\,dx$$

$\Leftrightarrow y = x^2 + C$, where C is a constant

Hence general solution is $y = x^2 + C$

Meaning of a solution: This general solution is a system of parabolas, each with the common property that $\frac{dy}{dx} = 2x$.

Example 2

Find the general solution of $\frac{dy}{dx} = y^2$

$$\frac{dy}{dx} = y^2 \Leftrightarrow \frac{dx}{dy} = \frac{1}{y^2}$$

$$\Leftrightarrow \int \frac{dx}{dy}.dy = \int \frac{1}{y^2}\,dy$$

$\Leftrightarrow x = -\frac{1}{y} + C$, where C is a constant

$$\Leftrightarrow \frac{1}{y} = C - x$$

$$\Leftrightarrow y = \frac{1}{C - x}$$

Examples for class

Solve the following

(1) $\frac{dy}{dx} = x^3$ (2) $\frac{dy}{dx} = y^3$ (3) $\frac{dy}{dx} = ax$, where a is a constant

(4) $\frac{dy}{dx} = ay$, where a is a constant (5) $\frac{dy}{dx} = y^2 - 1$ (6) $\frac{dy}{dx} = \sec^2 x$

Answers

(1) $y = \frac{x^4}{4} + C$ (2) $y = \frac{1}{\sqrt{C - 2x}}$ (3) $y = \frac{ax^2}{2} + C$

(4) $y = Ae^{ax}$ (5) $y = \frac{2}{1 - Ae^{2x}} - 1$ (6) $y = \tan x + C$

Example 3

Find the particular solution of $\frac{dy}{dx} = y^2$ when $y = \frac{1}{2}, x = 0$

$\frac{dy}{dx} = y^2 \Leftrightarrow \frac{dx}{dy} = \frac{1}{y^2} \Leftrightarrow x = -\frac{1}{y} + C$, where C is a constant.

General solution is $x = -\frac{1}{y} + C$

When $x = 0$, $y = \frac{1}{2}$, $0 = -2 + C \Rightarrow C = 2$.

Hence particular solution is $x = -\frac{1}{y} + 2$ i.e. $y = \frac{1}{2 - x}$

Examples for class

(1) $\frac{dy}{dx} + y = 0$; $y = 2$ when $x = 0$

(2) $\frac{dy}{dx} + y = 3$; $y = 0$ when $x = 0$

(3) $\frac{dy}{dx} = \sin x$; $y = 0$ when $x = 0$

Answers

(1) $y = 2e^{-x}$ (2) $y = 3 - 3e^{-x}$ (3) $y = \cos x - 1$

B. Variables Separable

The differential equation is called separable when $\frac{dy}{dx} = g(x)\,h(y)$.

The equation becomes $\frac{dy}{dx} = g(x)\,h(y) \Leftrightarrow \frac{1}{h(y)}\frac{dy}{dx} = g(x)$

Its general solution is $\int \frac{1}{h(y)}\frac{dy}{dx}\,dx = \int g(x)\,dx + C$ where C is a constant.

i.e. $\int \frac{1}{h(y)}\,dy = \int g(x)\,dx + C$

Example 1

Find the general solution of $\frac{dy}{dx} = e^{x+y}$

$\frac{dy}{dx} = e^{x+y} \Leftrightarrow \frac{dy}{dx} = e^{x}.e^{y}$

$\Leftrightarrow \frac{1}{e^{y}}\,dy = e^{x}\,dx$

$\Leftrightarrow \int e^{-y}dy = \int e^{x}dx + C$, where C is a constant.

$\Leftrightarrow -e^{-y} = e^{x} + C$

Hence general solution is $-e^{-y} = e^{x} + C$

Example 2

Solve $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

$\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \Leftrightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} + C$, where C is a constant

$\Leftrightarrow \tan^{-1} y = \tan^{-1} x + C$

$\Leftrightarrow \tan(\tan^{-1} y) = \tan(\tan^{-1} x + C)$

$\Leftrightarrow y = \frac{\tan(\tan^{-1} x) + \tan C}{1 - \tan(\tan^{-1} x)\,\tan C}$

$\Leftrightarrow y = \frac{x + \tan C}{1 - x\,\tan C}$

$\Leftrightarrow y = \frac{x+B}{1-Bx}$, where $B = \tan C$

Hence general solution is $y = \frac{x+B}{1-Bx}$

Example 3

Find the general solution of $x\dfrac{dy}{dx} = (1+x)\,y^2$ and find the particular solution for which $y = 1$ when $x = 1$.

$x\dfrac{dy}{dx} = (1+x)\,y^2 \Leftrightarrow \int y^{-2}\,dy = \int \dfrac{(1+x)}{x}\,dx + C$, where C is a constant

$\Leftrightarrow -y^{-1} = \int \left(\dfrac{1}{x}+1\right) dx + C$

$\Leftrightarrow -\dfrac{1}{y} = \log x + x + C$

Hence general solution is $-\dfrac{1}{y} = \log x + x + C$

When $x = 1, y = 1$, $-1 = 0 + 1 + C \Rightarrow C = -2$

Hence particular solution is $-\dfrac{1}{y} = \log x + x - 2$

i.e. $\dfrac{1}{y} = 2 - \log x - x$

Examples for class

Find solution of

(1) $(1+x)\dfrac{dy}{dx} = xy$ (2) $\dfrac{dy}{dx} = x\,(1-y)^2$ (3) $(1+y)^2\dfrac{dy}{dx} = e^{x+y}$

(4) $\cos^2 x\,\dfrac{dy}{dx} = xy^2$

(5) $\dfrac{dy}{dx} = (1-y)\cos x$, and the particular solution for which $y = 2$ when $x = \dfrac{1}{2}\pi$

(6) $y - x\dfrac{dy}{dx} = 1 + x^2\dfrac{dy}{dx}$ (7) $\dfrac{dy}{dx} = \dfrac{xy}{\sqrt{x^2+C}}$, where C is a constant

(8) $\dfrac{dy}{dx} = y^{\frac{3}{2}}$ (9) $\dfrac{dy}{dx} = \dfrac{x}{y}$ (10) $\dfrac{dy}{dx} = \tan x \tan y$

Answers

(1) $y = \frac{Ae^x}{x+1}$ (2) $y = 1 - \frac{2}{x^2 + A}$ (3) $(1+y)^3 = 3e^x + C$

(4) $\frac{1}{y} = -x\tan x - \ln(\cos x) + C$ (5) $\frac{1}{y-1} = -e^{\sin x - 1}$

(6) $y = \frac{Ax}{1+x} + 1$ (7) $y = Ae^{\sqrt{x^2+C}}$ (8) $y = \frac{4}{(x+C)^2}$

(9) $y = Ae^{\frac{1}{2}x^2}$ (10) $\sin y = \frac{A}{\cos x}$

C. **Homogeneous**
(Homogeneous equations are covered within Mathematics 3 (AH))

A homogeneous differential equation is one in which the sum of the powers of x and y in each term is the same.

To deal with these equations, we must use the substitution y = vx and then treat the new differential equation as variables separable.

Example 1

Solve $\frac{dy}{dx} = \frac{2x - y}{x - y}$

Let $y = vx$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ substituting the equation becomes

$$v + x\frac{dv}{dx} = \frac{2x - vx}{x + vx}$$

$$\Leftrightarrow v + x\frac{dv}{dx} = \frac{x(2-v)}{x(1-v)}$$

$$\Leftrightarrow v + x\frac{dv}{dx} = \frac{2-v}{1+v}$$

$$\Leftrightarrow x\frac{dv}{dx} = \frac{2-v}{1+v} - v = \frac{2 - v - v - v^2}{1+v} = \frac{2 - 2v - v^2}{1+v}$$

$$\Leftrightarrow \int \frac{(1+v)}{2 - 2v - v^2}\,dv = \int \frac{1}{x}\,dx + C \text{, where } C \text{ is a constant}$$

$$\Leftrightarrow -\frac{1}{2}\int \frac{(-2-2v)}{2 - 2v - v^2}\,dv = \int \frac{1}{x}\,dx + C$$

$$\Leftrightarrow -\frac{1}{2}\ln(2 - 2v - v^2) = \ln x + C$$

$$\Leftrightarrow \ln x + \frac{1}{2}\ln\left(2 - \frac{2y}{x} - \frac{y^2}{x^2}\right) = A \text{, where } A \text{ is a constant}$$

$$\Leftrightarrow \ln\left(x.\sqrt{\frac{2x^2 - 2xy - y^2}{x^2}}\right) = A$$

$$\Leftrightarrow \sqrt{2x^2 - 2xy - y^2} = B$$

$$\Leftrightarrow 2x^2 - 2xy - y^2 = K$$

where B and K are constants.

Example 2

Find the general solution of $x\,(2x - y)\,\dfrac{dy}{dx} = (x + y)^2$ and the particular solution for which $y = 0$ when $x = 1$.

Let $y = vx$

Then $\dfrac{dy}{dx} = v + x\,\dfrac{dv}{dx}$ substituting the equation becomes

$$x\,(2x - vx)\left[v + x\frac{dv}{dx}\right] = (x + vx)^2$$

$$\Leftrightarrow x^2\,(2 - v)\left(v + x\frac{dv}{dx}\right) = x^2\,(1 + v)^2$$

$$\Leftrightarrow (2 - v)\,v + (2 - v)\,x\frac{dv}{dx} = 1 + 2v + v^2$$

$$\Leftrightarrow (2 - v)\,x\frac{dv}{dx} = 1 + 2v + v^2 - 2v + v^2$$

$$\Leftrightarrow (2 - v)\,x\frac{dv}{dx} = 1 + 2v^2$$

$$\Leftrightarrow \int \frac{2 - v}{1 + 2v^2}\,dv = \int \frac{dx}{x} + C \text{, where } C \text{ is a constant}$$

$$\Leftrightarrow \int \frac{2}{1 + 2v^2}\,dv - \int \frac{v}{1 + 2v^2}\,dv = \int \frac{dx}{x} + C$$

$$\Leftrightarrow \frac{2}{2}\int \frac{dv}{\frac{1}{2} + v^2} - \frac{1}{4}\int \frac{4v}{1 + 2v^2}\,dv = \int \frac{dx}{x} + C$$

$$\Leftrightarrow \sqrt{2}\tan^{-1}\left(\sqrt{2}\,v\right) - \frac{1}{4}\ln\left(1 + 2v^2\right) = \ln x + C$$

$$\Leftrightarrow \sqrt{2}\tan^{-1}\left(\frac{\sqrt{2}\,y}{x}\right) - \frac{1}{4}\ln\left(\frac{x^2 + 2y^2}{x^2}\right) = \ln x + C \quad \text{- General solution}$$

When $x = 1, y = 0,\ 0 - 0 = 0 + C \Rightarrow C = 0$

Particular solution is $\sqrt{2}\tan^{-1}\left(\frac{\sqrt{2}y}{x}\right) - \frac{1}{4}\ln\left(\frac{x^2 + 2y^2}{x^2}\right) = \ln x$

Examples for class

Find the general solution of each of the following

(1) $(x - y)\frac{dy}{dx} = y - 4x$ (2) $(x^2 + y^2)\frac{dy}{dx} = xy$ (3) $(x^2 - xy)\frac{dy}{dx} + 2y^2 = 0$

(4) $x\left(\tan\left(\frac{y}{x}\right)\right)\frac{dy}{dx} + x - y\tan\left(\frac{y}{x}\right) = 0$ (5) $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$

(6) $\frac{dy}{dx} = \frac{y + x}{y - x}$ (7) $x\left(\frac{dy}{dx} - \tan\left(\frac{y}{x}\right)\right) = y$

Answers

(1) $\left(\frac{x}{2y - 5x}\right)^{3/4} e^{-y/2x} = Ax$ (2) $\frac{1}{y}e^{x^2/2y^2} = A$

(3) $(x + y)^2 = Ax^2y$ (4) $\cos\frac{y}{x} = Ae^{1/2x^2}$

(5) $\frac{y^2}{2x^2} = \ln x + C$ (6) $(x^2 + 2xy - y^2)^{1/2} = Ax^2$

(7) $\sin\frac{y}{x} = \frac{A}{x}$

MATHEMATICS 2 (AH)

CONTENT	
Complex numbers	
2.3.1	know the definition of I as a solution of $x^2 + 1 = 0$, so that $i = \sqrt{-1}$
2.3.2	know the definition of the set of complex numbers as $C = \{a + ib : a, b \in \boldsymbol{R}\}$
2.3.3	know the definition of real and imaginary parts
2.3.4	know the terms complex plane, Argand diagram
2.3.5	plot complex numbers as points in the complex plane
2.3.6	perform algebraic operations on complex numbers: equality (equating real and imaginary parts), addition, subtraction, multiplication and division
2.3.7	evaluate the modulus, argument and conjugate of complex numbers
2.3.8	convert between Cartesian and polar form
2.3.9	know the fundamental theorem of algebra and conjugate roots property
2.3.10	factorise polynomials with real coefficients
2.3.11	solve simple equations involving a complex variable by equating real and imaginary parts
2.3.13	know and use de Moivre's theorem with positive integer indices
2.3.14	apply de Moivre's theorem to multiple angle trigonometric formulae [A/B]
2.3.15	**apply de Moivre's theorem to find the nth roots of unity [A/B]**

The content listed below is **not** covered within this support material:

2.3.12 interpret geometrically certain equations or inequalities in the complex plane e.g. $|z| = 1$; $|z - a| = b$; $|z - 1| = |z - i|$; $|z - a| > b$

2.3.13 know and use de Moivre's theorem with **fractional** indices **[A/B]**

MATHEMATICS 2 (AH): COMPLEX NUMBERS

So far we are unable to solve equations such as $x^2 = -1$ or $x^2 + x + 6 = 0$.

If we let $i = \sqrt{-1}$ then the solutions to these two equations are

$$x = \pm\, i \quad \text{and} \quad x = \frac{-1 \pm \sqrt{-23}}{2} = -\frac{1}{2} \pm \frac{\sqrt{23}\, i}{2}$$

The largest set of numbers that we have already met are the real numbers.

Imaginary numbers are of the form bi where b is real, and **complex numbers** are of the form $a + ib$ where $a, b \in \mathbf{R}$.

Clearly if $b = 0$ then **R** is a subset of the complex numbers, and if $a = 0$ then the imaginary numbers is also a subset of the complex numbers. We use **C** to stand for the set of complex numbers.

Basic operations on C

Addition: We link the real parts together and the imaginary parts together.

$(3 + 7i) + (4 - 8i) = (7 - i)$

Subtraction: Similarly we obtain

$(3 + 2\,i) - (2 - i) = (1 + 3i)$

Multiplication: Note that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, etc.

$(2 + i)(3 - 4i) = 6 - 8i + 3i - 4i^2 = 10 - 5i$
$(4 - 2i)(4 + 2i) = 16 - 4i^2 = 20$

In general $(a + ib)(a - ib) = a^2 + b^2$ which is real.

Division: When dividing by $a + ib$ we rationalise the denominator by multiplying the numerator and denominator by $a - ib$.

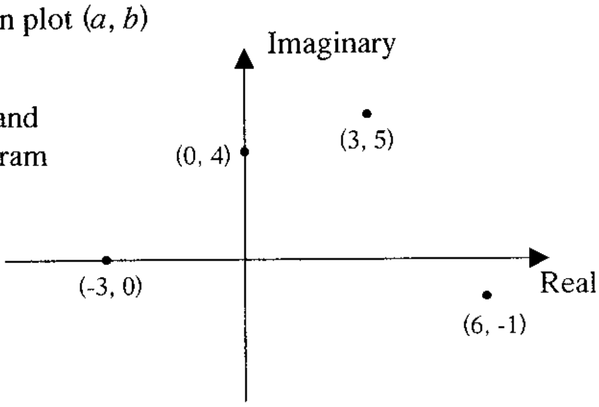
$$\frac{4-i}{1-\ 2i} = \frac{(4-i)(1+2i)}{(1-2\mathrm{i})(1+2\mathrm{i})} = \frac{6+7i}{5} = \frac{6}{5} + \frac{7}{5}i$$

Modulus and Argument – The Argand diagram

In general, **C** can be thought of as ***R*** x ***R*** where (a, b) represents $a + ib$, $a, b \in$ **R**.

We can plot (a, b)

Argand diagram



$(3, 5) \equiv 3 + 5i$
$(0, 4) \equiv 4i$
$(-3, 0) \equiv -3$
$(6, -1) \equiv 6 - i$

The modulus of a complex number $z = x + iy$ can be defined as being the distance that z is from the origin on the Argand diagram.

This is written $|z|$ and therefore is the distance between the points (x, y) and $(0, 0)$, which is $\sqrt{x^2 + y^2}$.

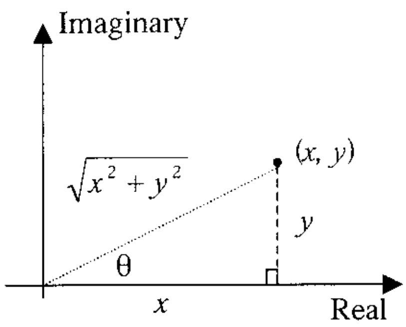
Hence $|z| = \sqrt{x^2 + y^2}$ (note: $|z| = r$ is always non-negative).

The argument (or amplitude) of a complex number $z = x + iy$ is defined as being the angle OZ makes with the positive real axis.

Obviously there are several such angles differing by multiples of 2π. If θ is the argument then the principal value for θ is $-\pi < \theta \leq \pi$

Clearly $\tan\theta = \dfrac{y}{x}$.

The argument is not defined for $z = 0 + 0i$.



Example 1

Find r and θ for $z = \dfrac{(1 + i\sqrt{3})^4}{16}$

$$z = \frac{(-2 + 2\sqrt{3}\,i)^2}{16} = \frac{-8 - 8\sqrt{3}\,i}{16} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$|z| = r = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$ and $\arg(z) = \theta = \dfrac{4}{3}\pi$ since $\tan\theta = \sqrt{3}$ and θ is in the 3rd quadrant.

Modulus and argument of products and quotients

From the diagram above it is clear that $x = r\cos\theta$ and $y = r\sin\theta$ and hence any complex number of the form $x + iy$ can be written as $r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta)$ where r is the modulus and θ the argument.

Let $z = x + iy = r(\cos\theta + i\sin\theta)$ and $w = u + iv = s(\cos\phi + i\sin\phi)$.

Product $zw = rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$
$= rs(\cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi))$
$= rs(\cos(\theta + \phi) + i\sin(\theta + \phi))$

Hence zw is a complex number whose modulus is rs and argument is $\theta + \phi$.

Quotient $\dfrac{z}{w} = \dfrac{r(\cos\theta + i\sin\theta)}{s(\cos\phi + i\sin\phi)} = \dfrac{r(\cos\theta + i\sin\theta)(\cos\phi - i\sin\phi)}{s(\cos\theta + i\sin\theta)(\cos\phi - i\sin\phi)}$

$= \dfrac{r(\cos\theta\cos\phi + \sin\theta\sin\phi + i(\sin\theta\cos\phi - \cos\theta\sin\phi))}{s(\cos^2\phi + \sin^2\phi)}$

$= \dfrac{r}{s}(\cos(\theta - \phi) + i\sin(\theta - \phi))$

Hence $\dfrac{z}{w}$ is a complex number whose modulus is $\dfrac{r}{s}$ and argument is $(\theta - \phi)$.

In both zw and $\dfrac{z}{w}$ the argument would need to be corrected to the principal value range.

These results can be generalised for the product and quotient of n complex numbers, and can be summarised as follows:-

When multiplying two complex numbers together, to obtain the product,
(i) multiply their moduli (ii) add their arguments

When finding the quotient of two complex numbers,
(i) divide their moduli (ii) subtract their arguments.

The results above easily lead to the following:

$\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ $\quad |zw| = |z|\,|w|$ $\quad |z^n| = |z|^n$

$\arg\left(\dfrac{z}{w}\right) = \arg(z) - \arg(w)$: $\arg(zw) = \arg(z) + \arg(w)$: $\arg(z^n) = n\arg(z)$

Example 2

Find r and θ for $z = (3+3i)^4$. Hence rewrite z in the form $a + ib$.

To find r: $|3+3i| = \sqrt{9+9} = \sqrt{18}$

$|z| = |(3+3i)^4| = (\sqrt{18})^4 = 324$

To find θ : $\arg(3 + 3i) = \frac{\pi}{4}$

$\arg(z) = \arg(3 + 3i)^4 = 4 \times \frac{\pi}{4} = \pi$

Hence $z = r(\cos\theta + i\sin\theta) = 324(\cos\pi + i\sin\pi) = 324(-1 + 0i) = -324$

Example 3

Find r and θ for $z = \frac{(\sqrt{3}+i)^3}{4i(1-i)^2}$ and express z in the form $a + ib$.

To find r: $\left|(\sqrt{3}+i)^3\right| = |\sqrt{3}+i|^3 = 2^3 = 8$

$|4i| = 4$ $\quad |(1-i)^2| = |1-i|^2 = (\sqrt{2})^2 = 2$ $\quad |z| = \frac{8}{4\times 2} = 1$

To find θ : $\arg(\sqrt{3}+i)^3 = 3\arg(\sqrt{3}+i) = 3 \times \frac{\pi}{6} = \frac{\pi}{2}$

$\arg(4i) = \frac{\pi}{2}$ $\quad \arg(1 - i)^2 = 2\arg(1 - i) = 2 \times -\frac{\pi}{4} = -\frac{\pi}{2}$

$\arg(z) = \frac{\pi}{2} - \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$

(which is within the PV range and requires no correction)

Hence $z = r(\cos\theta + i\sin\theta) = 1(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 1(0 + i) = i$.

Conjugates

If $z = x + iy$ then the complex conjugate of z is $\bar{z} = x - iy$.

For example: if (i) $z = 3 + 2i$ then $\bar{z} = 3 - 2i$ (ii) $z = 4 - 5i$ then $\bar{z} = 4 + 5i$
(iii) $z = -2$ then $\bar{z} = -2$ (iv) $z = 4i$ then $\bar{z} = -4i$

There are various simple results which are worth noting about conjugates.
These are stated below, for two complex numbers z and w.

(i) $\overline{z+w} = \bar{z} + \bar{w}$

(ii) $\overline{zw} = \bar{z} \times \bar{w}$

(iii) $\text{Re}\, z = \frac{1}{2}(z + \bar{z})$ NB Re z is the real part of z.

(iv) $\text{Im}\, z = \frac{1}{2i}(z - \bar{z}$ NB Im z is the imaginary part of z which is real.

(v) $\bar{\bar{z}} = z$

(vi) $z\bar{z} = |z|^2$

(vii) $|z| = |\bar{z}|$

(viii) $\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$

All of these results are proved easily by letting $z = x + iy$ and $w = u + iv$.

A few are proved below.

(i) $z + w = x + u + i(y + v)$ L.H.S. $= \overline{z + w} = x + u - i(y + v)$

$\bar{z} = x - iy$ $\bar{w} = u - iv$ R.H.S. $= \bar{z} + \bar{w} = x + u - i(y + v)$

(ii) $zw = (x + iy)(u + iv) = xu - vy + i(xv + uy)$: $\overline{zw} = xu - vy - i(xv + uy)$ = L.H.S.

R.H.S. = $\bar{z} \times \bar{w} = (x - iy)(u - iv) = xu - vy - i(xv + uy)$

(iv) R.H.S. = $\frac{1}{2i}(x + iy - x + iy) = \frac{1}{2i}(2iy) = y = \text{Im}(x + iy) = \text{Im}\, z$

More complicated problems can be solved using this technique.

Example 4

Show that Re $(z\bar{w}) = \frac{1}{2}(z\bar{w} + \bar{z}w)$

Let $z = x + iy$ and $w = u + iv$

$z\bar{w} = (x + iy)(u - iv)$

Re $(z\bar{w}) = xu + yv$ = L.H.S.

$z\bar{w} = xu + yv + i(uy - vx)$

$\bar{z}w = xu + yv - i(uy - vx)$

$\frac{1}{2}(z\bar{w} + \bar{z}w) = xu + yv = R.H.S.$

Hence result since L.H.S. = R.H.S.

Equating real and imaginary parts

Result: If $x + iy = u + iv$ then $x = u$ and $y = v$.

Proof: $x + iy = u + iv \Rightarrow x - u = i(v - y)$

But $x - u$ is real and $i(v - y)$ is imaginary.

The only number satisfying this is 0.

Hence $x - u = 0$ and $x = u$ and $i(v - y) = 0$ and $v = y$.

Example 5

Find all $z \in \mathbf{C}$ which satisfy $z = \sqrt{5-12i}$

Let $z = x + iy$

Then $(x + iy)^2 = 5 - 12i \Rightarrow x^2 - y^2 + 2ixy = 5 - 12i$

Equating real and imaginary parts we obtain

$x^2 - y^2 = 5$ and $2xy = -12 \Rightarrow y = -\frac{6}{x}$ and $x^2 - \frac{36}{x^2} = 5$

$\Rightarrow x^4 - 5x^2 - 36 = 0 \Rightarrow (x^2 - 9)(x^2 + 4) = 0$

$\Rightarrow x^2 = 9$ (NB x is real and $x^2 = -4$ has no solution)

$\Rightarrow x = +3$ or $-3 \Rightarrow y = -2$ or $+2$

and $z = x + iy$ is $3 - 2i$ or $-3 + 2i$

Note that a more general method of finding the nth root of a complex number will be discussed later.

Complex equations

There are two types of complex equation which need to be studied.

Type A involves equations in z and $\bar{z}$ with complex coefficients.

Type B involves polynomial equations to degree 4 with complex solutions, and real coefficients.

Type A

Before starting the solutions of this type it is worth noting the following

$z = x + iy$ $\qquad \bar{z} = x - iy$ $\qquad z\bar{z} = x^2 + y^2$

$z^2 = x^2 - y^2 + 2ixy$ $\qquad \bar{z}^2 = x^2 - y^2 - 2ixy$

Example 6

Find all $z \in \mathbf{C}$ which satisfy $z^2 - 2\bar{z} + z\bar{z} - 12i = 0$

Substituting $z = x + iy$ into this equation we obtain

$x^2 - y^2 - 2x + x^2 + y^2 + i(2xy + 2y - 12) = 0 + 0i$

Equating real and imaginary parts we obtain

$2x^2 - 2x = 0$ and $2xy + 2y - 12 = 0$

$\Rightarrow 2x(x - 1) = 0 \Rightarrow x = 0$ or $x = 1$ and $y = 6$ or $y = 3$

Example 7
Find all $z \in \mathbf{C}$ which satisfy $2z^2 - \bar{z}^2 + 4z - 2\bar{z} + 5 = 0$

Substituting $z = x + iy$ into this equation we obtain

$$2x^2 - 2y^2 - x^2 + y^2 + 4x - 2x + 5 + i(4xy + 2xy + 4y + 2y) = 0 + 0i$$

Equating real and imaginary parts we obtain

$x^2 - y^2 + 2x + 5 = 0$ and $6xy + 6y = 0$

$\Rightarrow 6y(x + 1) = 0 \Rightarrow y = 0$ when x is imaginary

or $x = -1$ when $y^2 = 4$ and $y = 2$ or -2

Hence $z = x + iy$ is $-1 + 2i$ or $-1 - 2i$.

Type B

The theory behind this method of solution is that if z is a solution then $\bar{z}$ is also a solution provided the equation has real coefficients.

Proof: If z is a solution of the polynomial equation

$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 = 0$ where $a_j \in R$ and $x \in C$ $\quad (0 \le j \le n)$

then $a_n z^n + a_{n-1}z^{n-1} + \ldots + a_2z^2 + a_1z + a_0 = 0$

Taking conjugates of both sides we obtain

$$\overline{a_n z^n + a_{n-1}z^{n-1} + \ldots + a_2z^2 + a_1z + a_0} = \bar{0}$$

Since $a_j \in R$, $\overline{a_j} = a_j$ and $\bar{0} = 0$ we are able to rewrite the above as

$a_n \bar{z}^n + a_{n-1}\bar{z}^{n-1} + \ldots + a_2\bar{z}^2 + a_1\bar{z} + a_0 = 0$

using previous properties of the conjugate.

But this implies that $\bar{z}$ is also a solution to the original equation.
Hence if z is a solution then $\bar{z}$ also is a solution.

Example 8

Find all the solutions to $z^4 - 4z^3 + 11z^2 - 14z + 10 = 0$ given that $1 - i$ is a root.

Since $1 + i$ is also a root if $1 - i$ is, i.e. the equation can be written:

$$\begin{aligned} z^4 - 4z^3 + 11z^2 - 14z + 10 &= (z - (1 - i))(z - (1 + i))\ Q(z) \\ &= ((z - 1) + i)((z - 1) - i)\ Q(z) \\ &= (z^2 - 2z + 2)\ Q(z) \end{aligned}$$

After dividing the original expression by $z^2 - 2z + 2$, $Q(z) = z^2 - 2z + 5$.

The remaining two solutions to the equation come from $Q(z) = 0$.

$$z^2 - 2z + 5 = 0 \Rightarrow z = \frac{+2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$$

Hence the four solutions to the equation are

$z = 1 - i,\ 1 + i,\ 1 + 2i$ and $1 - 2i$.

DeMoivre's Theorem

We know already that $z = x + iy$ can be written in the form $r(\cos\theta + i\sin\theta)$.

DeMoivre's theorem states that for $n \in \mathbf{N}$ and $\theta \in \mathbf{R}$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta .$$

Proof by induction

For $n = 1$ the result is clearly true, both sides being $\cos\theta + i\sin\theta$.

Assume that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$ for some $n \geq 1$

Then

$$\begin{aligned} (\cos\theta + i\sin\theta)^{n+1} &= (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta) \\ &= (\cos n\theta + i \sin n\theta)(\cos\theta + i\sin\theta) \\ &= \cos n\theta \cos\theta - \sin n\theta \sin\theta + i(\sin n\theta \cos\theta + \cos n\theta \sin\theta) \\ &= \cos (n+1)\theta + i\sin (n+1)\theta \end{aligned}$$

Hence the result is true for $n + 1$.
As the result is true for $n = 1$ it is true for $n = 2$.
As it is true for $n = 2$ it is true for $n = 3$, and so on.
Hence the result is true for all $n \in \mathbf{N}$.

There are various standard types of problems associated with this theorem. Some of these are given below.

Example 9

Evaluate $(1-i)^7$.

$|1-i| = \sqrt{2}$ and $\arg(1-i) = -\frac{1}{4}\pi$

Hence $1-i = \sqrt{2}\left(\cos\left(-\frac{1}{4}\pi\right) + i\sin\left(-\frac{1}{4}\pi\right)\right)$

and $(1-i)^7 = 8\sqrt{2}\left(\cos\left(-\frac{1}{4}\pi\right) + i\sin\left(-\frac{1}{4}\pi\right)\right)^7$

$= 8\sqrt{2}\left(\cos\left(-\frac{7}{4}\pi\right) + i\sin\left(-\frac{7}{4}\pi\right)\right)$

$= 8\sqrt{2}\left(\cos\left(\frac{1}{4}\pi\right) + i\sin\left(\frac{1}{4}\pi\right)\right)$

$= 8\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$= 8 + 8i$

Example 10

Show that $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$ for $n \in \mathbf{N}$ and deduce that $(\cos\theta + i\sin\theta)^{-n} = (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

Now

$$\begin{aligned}(\cos\theta - i\sin\theta)^n &= (\cos(-\theta) - i\sin(-\theta))^n \\ &= \cos(-n\theta) - i\sin(-n\theta) \\ &= \cos n\theta - i\sin n\theta\end{aligned}$$

In addition $(\cos\theta + i\sin\theta)^{-1} = \dfrac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$ on multiplying top and bottom by $\cos\theta - i\sin\theta$.

Hence $(\cos\theta + i\sin\theta)^{-n} = (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$.

Example 11

Simplify $\dfrac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta}$

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$
$\cos 2\theta - i\sin 2\theta = (\cos\theta + i\sin\theta)^{-2}$
and the expression equals $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

Example 12

Expand $(\cos\theta + i\sin\theta)^4$ and express $\cos 4\theta$ and $\frac{\sin 4\theta}{\sin\theta}$ in terms of $\cos\theta$.

If we write c for $\cos\theta$ and s for $\sin\theta$ we obtain

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= (c + is)^4 \\ &= c^4 + 4c^3 is + 6c^2 i^2 s^2 + 4c i^3 s^3 + s^4 \\ &= c^4 + i4c^3 s - 6c^2 s^2 - 4ics^3 + s^4\end{aligned}$$

Equating real parts,

$$\begin{aligned}\cos 4\theta &= c^4 - 6c^2 s^2 + s^4 \\ &= c^4 - 6c^2(1-c^2) + (1-c^2)^2 \\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

Equating imaginary parts we obtain

$\sin 4\theta = 4c^3 s - 4cs^3 = 4sc(c^2 - s^2)$

so $\frac{\sin 4\theta}{\sin\theta} = 4c(c^2 + c^2 - 1) = 8\cos^3\theta - 4\cos\theta$

Example 13

Express $\cos^4\theta$ in terms of cosine multiples of θ.

If we let $z = \cos\theta + i\sin\theta$ then $\bar{z} = z^{-1} = \cos\theta - i\sin\theta$ and $z + z^{-1} = 2\cos\theta$.

In general, by applying DeMoivre's theorem to z^n and z^{-n}
we obtain $z^n + z^{-n} = 2\cos n\theta$.

We have $\cos\theta = \frac{1}{2}(z + z^{-1})$ from above and hence

$\cos^4\theta = \frac{1}{16}(z + z^{-1})^4$

$= \frac{1}{16}(z^4 + 4z^2 + 6z + 4z^{-2} + z^{-4})$

$= \frac{1}{16}(z^4 + z^{-4}) + \frac{4}{16}(z^2 + z^{-2}) + \frac{6}{16}$

$= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ using the above statements.

Roots of complex numbers and roots of unity

We can use DeMoivre's theorem to find the n^{th} root of a complex number provided we know the modulus and argument.

Example 14

If $z = -8i$, find the cube root of z.

Here $r = 8$ and $\theta = -\frac{1}{2}\pi$ and $z = 8\left(\cos\left(-\frac{1}{2}\pi\right) + i\sin\left(-\frac{1}{2}\pi\right)\right)$

$$z^{\frac{1}{3}} = 2\left(\cos\left(-\frac{1}{2}\pi\right) + i\sin\left(-\frac{1}{2}\pi\right)\right)^{\frac{1}{3}}$$

However we expect three answers to this problem and solving the above will only give us one answer. We can obtain three by letting the argument be $(2n - \frac{1}{2})\pi$ where $n \in \mathbf{Z}$. This effectively alters the argument from the principal value range by multiples of 2π.

So $z^{\frac{1}{3}} = 2(\cos(2n - \frac{1}{2})\pi + i\sin(2n - \frac{1}{2})\pi)^{\frac{1}{3}}$

$= 2(\cos\frac{1}{3}(2n - \frac{1}{2})\pi + i\sin\frac{1}{3}(2n - \frac{1}{2})\pi$

We obtain our three answers by choosing any three consecutive values of n.

$n = 0 : z^{\frac{1}{3}} = 2\left(\cos\left(-\frac{1}{6}\pi\right) + i\sin\left(-\frac{1}{6}\pi\right)\right) = 2cis\left(-\frac{1}{6}\pi\right) = \sqrt{3} - i$

$n = 1 : z^{\frac{1}{3}} = 2cis\left(\frac{1}{2}\pi\right) = 2i$

$n = 2 : z^{\frac{1}{3}} = 2cis\left(\frac{7}{6}\pi\right) = -\sqrt{3} - i$

Example 15

If $z = (2 - 2\sqrt{3}i)^5$, find the square root of z.

$|z| = 4^5 = 2^{10}$. $\arg(z) = 5 \times -\frac{1}{3}\pi = -\frac{5}{3}\pi = \frac{1}{3}\pi$ to PV range.

So $z = 2^{10}(\cos(2n + \frac{1}{3})\pi + i\sin(2n + \frac{1}{3})\pi$

and $z^{\frac{1}{2}} = 2^5(\cos(2n + \frac{1}{3})\pi + i\sin(2n + \frac{1}{3})\pi)^{\frac{1}{2}}$

$= 32(\cos\frac{1}{2}(2n + \frac{1}{3})\pi + i\sin\frac{1}{2}(2n + \frac{1}{3})\pi$

$n = 0 : z^{\frac{1}{2}} = 32cis(\frac{1}{6})\pi = 32(\frac{1}{2}\sqrt{3} + \frac{1}{2}i) = 16\sqrt{3} + 16i$

$n = 1 : z^{\frac{1}{2}} = 32cis(\frac{7}{6})\pi = 32(-\frac{1}{2}\sqrt{3} - \frac{1}{2}i) = -16\sqrt{3} - 16i$

Example 16

Solve $x^5 = -1$ and hence factorise $x^5 + 1$ into real linear/quadratic factors.

Deduce $x^4 - x^3 + x^2 - x + 1 = (x^2 - 2x\cos\frac{1}{5}\pi + 1)(x^2 - 2x\cos\frac{3}{5}\pi + 1)$ and

hence that $\sin\left(\frac{\pi}{10}\right)\sin\left(\frac{3\pi}{10}\right) = \frac{1}{4}$.

$x^5 = -1 = \cos(2n+1)\pi + i\sin(2n+1)\pi$

$x = \cos\frac{1}{5}(2n+1)\pi + i\sin\frac{1}{5}(2n+1)\pi$

$n = 0\ :\ x = cis\frac{1}{5}\pi$

$n = 1\ :\ x = cis\frac{3}{5}\pi$

$n = 2\ :\ x = -1$

$n = 3\ :\ x = cis\frac{7}{5}\pi$

$n = 4\ :\ x = cis\frac{9}{5}\pi$

So $x^5 + 1 = (x+1)(x - cis\frac{1}{5}\pi)(x - cis\frac{3}{5}\pi)(x - cis\frac{7}{5}\pi)(x - cis\frac{9}{5}\pi)$

Using the fact that $\cos\frac{9}{5}\pi = \cos\frac{1}{5}\pi$ and $\sin\frac{9}{5}\pi = -\sin\frac{1}{5}\pi$

$(x - cis\frac{1}{5}\pi)(x - cis\frac{9}{5}\pi)$ reduces to $x^2 - 2x\cos\frac{1}{5}\pi + 1$

$x^5 + 1 = (x+1)(x^2 - 2x\cos\frac{1}{5}\pi + 1)(x^2 - 2x\cos\frac{3}{5}\pi + 1)$

on dividing both sides by $x + 1$ we obtain

$x^4 - x^3 + x^2 - x + 1 = (x^2 - 2x\cos\frac{1}{5}\pi + 1)(x^2 - 2x\cos\frac{3}{5}\pi + 1)$

Let $x = 1\ :\ 1 = 4(1 - \cos\frac{1}{5}\pi)(1 - \cos\frac{3}{5}\pi) = 16\sin^2\frac{\pi}{10}\sin^2\frac{3\pi}{10}$

giving $\frac{1}{4} = \sin\frac{\pi}{10}\sin\frac{3\pi}{10}$

(NB putting $x = -1$ gives $\frac{1}{4}\sqrt{5} = \cos\frac{\pi}{10}\cos\frac{3\pi}{10}$)

Mathematics

Mathematics 3 (AH)

MATHEMATICS 3 (AH)

Introduction

These support materials for Mathematics were developed as part of the Higher Still Development Programme in response to needs identified at needs analysis meetings and national seminars.

Advice on learning and teaching may be found in *Achievement for All* (SOEID 1996), *Effective Learning and Teaching in Mathematics* (SOEID 1993), *Improving Mathematics* (SEED 1999) and in the Mathematics Subject Guide.

These materials were originally issued to schools to support CSYS Mathematics. They have now been updated and matched to the Arrangements for Advanced Higher Mathematics to be issued as part of the support material being provided for mathematics.

It should be noted that not all of the content of Mathematics 3 (AH) is covered by this support material. The outcomes covered are Vectors, Further sequences and series and Further ordinary differential equations. At the start of each outcome a list of the content covered by the material has been included.

MATHEMATICS 3 (AH)

CONTENT	
Vectors	
3.1.1	know the meaning of the terms position vector, unit vector, scalar triple product, vector product, components, direction ratios/cosines
3.1.2	calculate scalar and vector products in three dimensions
3.1.3	know that $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$
3.1.4	find $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{a}.\boldsymbol{b} \times \boldsymbol{c}$ in component form
3.1.5	know the equation of a line in vector form, parametric and symmetric form
3.1.6	know the equation of a plane in vector form, parametric and symmetric form, Cartesian form
3.1.7	find the equations of lines and planes given suitable defining information
3.1.8	find the angles between **two planes [A/B]**, and between a line and a plane
3.1.9	find the intersection of two lines, a line and a plane and two or three planes

The content listed below is **not** covered within this support material:

3.1.8 find the angles between two lines,...

MATHEMATICS 3 (AH): VECTORS

Revision

Definitions

A **vector** is an object which has a magnitude and a direction and which is completely determined by these.

A **scalar** is an object which has only magnitude associated with it.

From the definition of the vector, it follows that a vector can be represented by a directed line segment, whose length represents the magnitude of the vector and whose direction is that of the vector.

The notation $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots$ is used for vectors.

It is convenient to introduce an object which has magnitude zero and which is not related to any one direction in space. We call this the **zero vector** and denote it by the symbol $\boldsymbol{0}$.

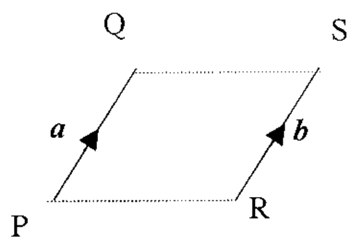
The magnitude of a vector $\boldsymbol{a}$ is denoted by $|\boldsymbol{a}|$ and it should be noted that $|\boldsymbol{a}|$ is a scalar and also $|\boldsymbol{a}| \geq 0$.

Properties

Equality

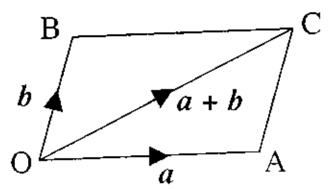
Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are defined to be equal if they have the same magnitude and the same direction.

If $\overrightarrow{PQ}$ represents $\boldsymbol{a}$ and $\overrightarrow{RS}$ represents $\boldsymbol{b}$.
Then $\boldsymbol{a} = \boldsymbol{b} \Leftrightarrow$ PQSR is a parallelogram.

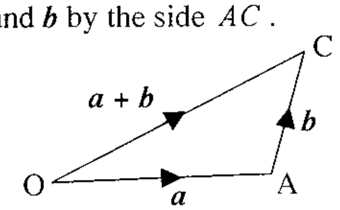


Addition of vectors

Let $\overrightarrow{OA}$ represent $\boldsymbol{a}$ and let $\overrightarrow{OB}$ represent $\boldsymbol{b}$ then, if C is as shown, we define $\boldsymbol{a} + \boldsymbol{b}$ to be the vector represented by $\overrightarrow{OC}$.

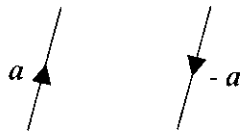


It is clear from the figure that the directed segment $\overrightarrow{AC}$ also represents $\boldsymbol{b}$ and hence we may also define the vector $\boldsymbol{a} + \boldsymbol{b}$ to be the vector represented by the side $\overrightarrow{OC}$ of the triangle OAC in which $\boldsymbol{a}$ is represented by $\overrightarrow{OA}$ and $\boldsymbol{b}$ by the side $\overrightarrow{AC}$.



Negative of a vector

The negative of a vector $\boldsymbol{a}$, denoted by $-\boldsymbol{a}$, is defined to be the vector with the same magnitude as the vector $\boldsymbol{a}$ but the opposite direction.



Subtraction of a vector

$\boldsymbol{a} - \boldsymbol{b}$ is defined to be $\boldsymbol{a} + (-\boldsymbol{b})$

Also $\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$
$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$
$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$

Multiplication by a scalar

If $\boldsymbol{a}$ is a non-zero vector and k a non-zero scalar, then the vector $k\boldsymbol{a}$ is defined by the following rules, giving, in order, its magnitude and direction.

1) $|k\boldsymbol{a}| = |k|\,|\boldsymbol{a}|$
2) the direction of $k\boldsymbol{a}$ is the same as the direction of $\boldsymbol{a}$ if k is positive and opposite if k is negative.

Also $0\boldsymbol{a} = k\boldsymbol{0} = \boldsymbol{0}$

Properties

$k(\boldsymbol{a} + \boldsymbol{b}) = k\boldsymbol{a} + k\boldsymbol{b}$
$(k + l)\,\boldsymbol{a} = k\boldsymbol{a} + l\boldsymbol{a}$
$(kl)\boldsymbol{a} = k(l\boldsymbol{a})$

Special case

$1\boldsymbol{a} = \boldsymbol{a}$
$-1\boldsymbol{a} = -\boldsymbol{a}$

Components of a vector

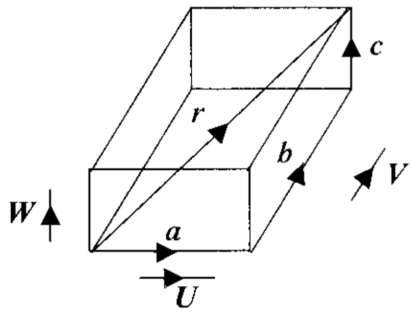
Definition

Vectors $v_1, v_2, \ldots$ are said to be **coplanar** if there exists a direction to which they are all perpendicular.

If v_1 is represented by $\overrightarrow{OP_1}$, v_2 is represented by $\overrightarrow{OP_2}$ etc, then it follows that these are coplanar $\Leftrightarrow \overrightarrow{OP_1}, \overrightarrow{OP_2}, \ldots$ etc lie in a plane.

Let $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$ be given non-coplanar vectors and let $\boldsymbol{r}$ be any vector.
About $\boldsymbol{r}$ as diagonal construct a parallelepiped with edges parallel to the vectors $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{W}$.

We have $\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$



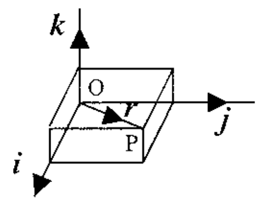
It is clear that these three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are uniquely determined by $\boldsymbol{r}$ and by $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$.

We call $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ the **components** of $\boldsymbol{r}$ in the directions of $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{W}$.

It is clear that there exists scalars λ, μ, γ such that
$\boldsymbol{a} = \lambda\, \boldsymbol{U}$, $\boldsymbol{b} = \mu\, \boldsymbol{V}$, $\boldsymbol{c} = \gamma\, \boldsymbol{W}$ and so $\boldsymbol{r} = \lambda\, \boldsymbol{U} + \mu\, \boldsymbol{V} + \gamma\, \boldsymbol{W}$.

It is often convenient to take the three given vectors to be
(i) mutually perpendicular and
(ii) of magnitude 1 (i.e. **unit** vectors)

The vectors in this case are usually denoted by $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$.



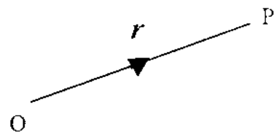
If the projections of $\boldsymbol{r}$ on $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are r_1, r_2, r_3, we have
$\boldsymbol{r} = r_1\boldsymbol{i} + r_2\boldsymbol{j} + r_3\boldsymbol{k}$

It is clear that P is a point with co-ordinates (r_1, r_2, r_3) in the co-ordinate system defined by these three mutually perpendicular unit vectors.
We also write $\boldsymbol{r} = (r_1, r_2, r_3)$

NB Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are non-parallel vectors, then any vector in the plane of $\boldsymbol{u}$ and $\boldsymbol{v}$ can be expressed in the form $\lambda\, \boldsymbol{u} + \mu\, \boldsymbol{v}$.

Definition

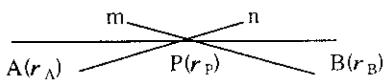
Let O be a fixed in space and let P be any point.
Then if $\overrightarrow{OP} = \boldsymbol{r}$, $\boldsymbol{r}$ is called the position vector of P with respect to O.



The point O is called the origin and has position vector $\boldsymbol{0}$

Section formula

If P is the point on the line determined by the points A and B such that AP : PB = m : n then



$$r_P = \frac{1}{m+n}(m\, r_B + n\, r_A)$$

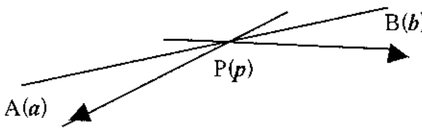
Example 1

Find the position vector of P when

(1) P divides AB internally in the ratio $\frac{\text{AP}}{\text{PB}} = \frac{2}{1}$

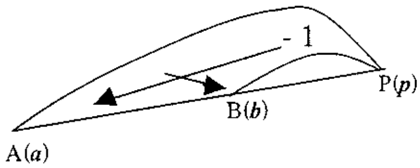
(2) P divides AB externally in the ratio $\frac{\text{AP}}{\text{PB}} = -\frac{2}{1}$

(1)



$$\boldsymbol{r}_P = \frac{1}{2+1}(2\boldsymbol{b} + \boldsymbol{a}) = \frac{1}{3}(2\boldsymbol{b} + \boldsymbol{a})$$

(2)



$$\boldsymbol{r}_P = \frac{1}{2+(-1)}(2\boldsymbol{b} - \boldsymbol{a}) = (2\boldsymbol{b} - \boldsymbol{a})$$

Example 2

If $\boldsymbol{a}$ = (2, 1, -3) and $\boldsymbol{b}$ = (-3, 5, 2), find the components of the vectors $\boldsymbol{a} + 3\boldsymbol{b}$. Find the unit vector in the direction $\boldsymbol{a} + 3\boldsymbol{b}$.

$\boldsymbol{a}$ = (2, 1, -3) $\quad$ $3\boldsymbol{b}$ = (-9, 15, 6) $\quad$ $\boldsymbol{a} + 3\boldsymbol{b}$ = (-7, 16, 3)

$$|\boldsymbol{a} + 3\boldsymbol{b}| = \sqrt{49 + 256 + 9} = \sqrt{314}$$

Hence required unit vector $\frac{1}{\sqrt{314}}(\boldsymbol{a} + 3\boldsymbol{b}) = \frac{1}{\sqrt{314}}(-7, 16, 3)$

Example 3

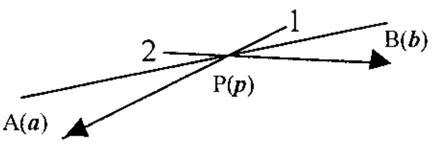
Find the lengths of the sides of the triangle whose vertices are the points A(-1, -1, -1), B(2, -1, 2), C(2, 1, 0).

A point P divides AB internally in the ratio AP : PB = 2 : 1.
Find the co-ordinates of P and show that angle BPC is a right angle.

$$\overrightarrow{\text{AB}} = b - a = (3, 0, 3) \Rightarrow \left|\overrightarrow{\text{AB}}\right| = \sqrt{9+0+9} = \sqrt{18} = 3\sqrt{2}$$

$$\overrightarrow{\text{BC}} = c - b = (0, 2, -2) \Rightarrow \left|\overrightarrow{\text{BC}}\right| = \sqrt{0+4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\overrightarrow{\text{CA}} = a - c = (-3, -2, -1) \Rightarrow \left|\overrightarrow{\text{CA}}\right| = \sqrt{9+4+1} = \sqrt{14}$$



$$p = \frac{1}{2+1}(a + 2b) = \frac{1}{3}(a + 2b)$$
$$= \frac{1}{3}(3, -3, 3)$$
$$= (1, -1, 1)$$

Hence P is the point (1, -1, 1)

$\overrightarrow{BP} = p - b = (-1, 0, -1) \Rightarrow |\overrightarrow{BP}| = \sqrt{2}$

$\overrightarrow{PC} = c - p = (1, 2, -1) \Rightarrow |\overrightarrow{PC}| = \sqrt{6}$

$\overrightarrow{BC} = c - b = (0, 2, -2) \Rightarrow |\overrightarrow{BC}| = 2\sqrt{2}$

Hence $|\overrightarrow{BP}|^2 + |\overrightarrow{PC}|^2 = 8 = |\overrightarrow{BC}|^2$

and so angle BPC is a right angle, by the Converse of Pythagoras.

Example 4

Are the following points collinear? A(1, 2, 3), B(3, 3, 2), C(7, 5, 0).

$\overrightarrow{AB} = b - a = (2, 1, -1)$

$\overrightarrow{AC} = c - a = (6, 3, -3)$

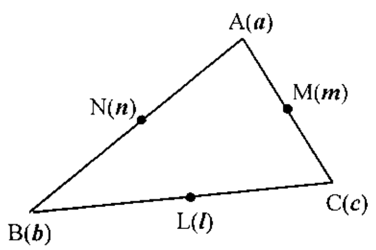
Here $\overrightarrow{AC} = 3\overrightarrow{AB}$

$\Rightarrow$ AC || AB

$\Rightarrow$A, B, C are collinear.

Example 5

Prove that the medians of a triangle are concurrent and that this point is a point of trisection of each median.



Let A, B, C be the vertices of a triangle, with position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ respectively.
The mid-points L, M, N of the sides BC, CA and AB in that order have position vectors.

$$l = \frac{1}{2}(b + c); \quad m = \frac{1}{2}(a + c); \quad n = \frac{1}{2}(a + b)$$

Consider G on AL such that $\frac{AG}{GL} = \frac{2}{1}$, then

$$\boldsymbol{g} = \frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{l}) = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$$

Consider H on BM such that $\frac{BH}{HM} = \frac{2}{1}$, then

$$\boldsymbol{h} = \frac{1}{3}(\boldsymbol{b} + 2\boldsymbol{m}) = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$$

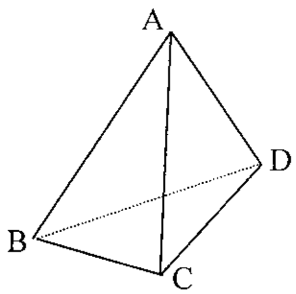
and if I on CN such that $\frac{CI}{IN} = \frac{2}{1}$, then

$$\boldsymbol{i} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$$

Thus G is the point on all three medians, this point being the point of trisection.

Example 6

Show that the three line segments joining the mid-points of pairs of opposite edges of a tetrahedron bisect each other.



Let A, B, C, D have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ respectively.

If L, M are the mid-points of AB and CD then

$$\boldsymbol{l} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}); \quad \boldsymbol{m} = \frac{1}{2}(\boldsymbol{c} + \boldsymbol{d})$$

and the mid-point P of LM has position vector

$$\boldsymbol{p} = \frac{1}{2}(\boldsymbol{l} + \boldsymbol{m}) = \frac{1}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$$

Now this expression is symmetrical in $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$.
Hence P is also the mid-point of segment joining mid-point of AC to that of BD and also the mid-point of segment joining mid-point of AD to that of BC, and all three segments bisect each other as required.

Example 7

ABCD is a parallelogram and E is the mid-point of AB. Prove that DE and AC trisect one another.

Choose A as origin, so that $\boldsymbol{b} = \overrightarrow{AB}$ is the position vector of B and $\boldsymbol{d} = \overrightarrow{AD}$ is the position vector of D.

Since ABCD is a parallelogram, $\boldsymbol{c} = \overrightarrow{AC} = \boldsymbol{b} + \boldsymbol{d}$ The position vector of E is $\frac{1}{2}\boldsymbol{b}$

There are two points of trisection of DE and AC. Working out their position vectors, we find

trisecting AC : $\frac{1}{3}(\boldsymbol{b} + \boldsymbol{d})$ and $\frac{2}{3}(\boldsymbol{b} + \boldsymbol{d})$ trisecting DE : $\frac{2}{3}\boldsymbol{d} + \frac{1}{6}\boldsymbol{b}$ and $\frac{1}{3}\boldsymbol{b} + \frac{1}{3}\boldsymbol{d}$

Hence result.

Examples for class

(1) The position vectors of the points A, B, C, D are respectively. ***a***. ***b***. $3\boldsymbol{a} + \boldsymbol{b}$. $-\boldsymbol{a} + 2\boldsymbol{b}$.

Express in terms of ***a*** and ***b***, the vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}, \overrightarrow{BC}, \overrightarrow{BD}, \overrightarrow{CD}$. Find also the position vectors of the mid-points of the segments AB, BC, CD, DA and the centroids of the triangles ABC, ACD, BCD.

(2) ABCD is a parallelogram and ***a***, ***b***, ***c*** are the position vectors of A, B, C. What is the position vector of D ?

(3) Show that the points P, Q and R with position vectors ***p***, ***q*** and ***r*** respectively are collinear
when (a) $\boldsymbol{r} = 5\boldsymbol{p} - 4\boldsymbol{q}$; (b) $r = (n+1)p - nq$ for any constant n.

(4) If L, M, N are the mid-points of BC, CA, AB respectively and O is any point, show that
(i) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OL} + \overrightarrow{OM} + \overrightarrow{ON}$ (ii) $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN} = \boldsymbol{O}$

(5) ABC, $A_1B_1C_1$ are two triangles and G and G_1 their centroids, respectively.
Prove that $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = 3\overrightarrow{GG_1}$.

(6) Six points A, B, C, D, E, F are given in space. P, Q, R, S are centroids of the triangles ABC, ABD, DEF, CEF, respectively. Show that P, Q, R, S are the vertices of a parallelogram.

(7) Suppose O is the centre of a circle $A_1A_2A_3$ of unit radius in a plane π.
If the point B_1 is the point defined by the vector relation $\overrightarrow{OB_1} = \overrightarrow{OA_2} + \overrightarrow{OA_3}$, show that B_1 is the centre of the other circle of unit radius in π which passes through A_2 and A_3.

If, further, two other circles of unit radius and centres B_2 and B_3 are drawn through A_3, A_1 and A_1 and A_2 respectively, prove that the three circles with centres B_1, B_2, B_3 meet at C where $\overrightarrow{OC} = \overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3}$.

(8) If the vectors ***a***, ***b***, ***c*** have components (2, 1, -1), (-3, 1, 0) and (0, 1, -2) respectively, find the components of the following vectors: $3\boldsymbol{a} + 2\boldsymbol{b} - 7\boldsymbol{c}$, $4\boldsymbol{a} + 5\boldsymbol{b}$, $3\boldsymbol{b} + 7\boldsymbol{c}$, $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}$.

(9) Points A, B, C have co-ordinates (1, 1, -1) , (4, 1, 2) , (-2, 1, 2).
Find the co-ordinates of the following points:
(a) the point P on AB between A and B such that AB = 3AP
(b) the point Q on BA produced such that AB = 3AQ
(c) the mid-point of CQ
(d) the centroid of the triangle ABC

(10) The points A, B, C have co-ordinates A(-1, 4, 3) , B(2, -1, 0) , C(5, 2, -3). A point P is chosen on BC between B and C so that 2BP = PC. A point Q is chosen on AP between A and P so that AQ = $\frac{1}{4}$AP.

Find the co-ordinates of P and Q and prove that PBQ is a right angle.

(11) If $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \boldsymbol{c} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, find numbers p, q, r such that

$$p\boldsymbol{a} + q\boldsymbol{b} + r\boldsymbol{c} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(12) Which of the following sets of points are collinear ?
(a) (1, -2, 5) , (2, -4, 4) , (-1, 2, 7)
(b) (3, 6, 1) , (9, 9, 2) , (1, 5, $\frac{2}{3}$)
(c) (5, 1, 7) , (3, -1, 1) , (6, 2, 11)
In those cases where the points are collinear, find the ratio in which the second point divides the segment defined by the first and third.

(13) The vertices of a triangle are A(3, 1, 1) , B(1, 0, -1) , C(4, -3, 2). M is the point dividing BC internally so that BM = $\frac{1}{2}$ MC. Show that AM is perpendicular to BC.

Answers

(1) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$, $\overrightarrow{AC} = 2\boldsymbol{a} + \boldsymbol{b}$, $\overrightarrow{AD} = -2\boldsymbol{a} + 2\boldsymbol{b}$, $\overrightarrow{BC} = 3\boldsymbol{a}$, $\overrightarrow{BD} = -\boldsymbol{a} + \boldsymbol{b}$, $\overrightarrow{CD} = -4\boldsymbol{a} + \boldsymbol{b}$

$\text{mid}_{AB} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$, $\text{mid}_{BC} = \frac{1}{2}(4\boldsymbol{a} + \boldsymbol{b})$, $\text{mid}_{CD} = \frac{1}{2}(2\boldsymbol{a} + 3\boldsymbol{b})$, $\text{mid}_{DA} = \boldsymbol{b}$

centroids: ABC $\frac{4}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$; ACD $\boldsymbol{a} + \boldsymbol{b}$; BCD $\frac{2}{3}\boldsymbol{a} + \frac{4}{3}\boldsymbol{b}$

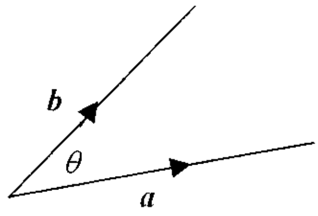
(2) $\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c}$
(3) Proof
(4) Proof
(5) Proof
(6) Proof
(7) Proof
(8) (0, -2, 11) , (-7, 9, -4) , (-9, 10, -14) , (-1, 1, 1)
(9) (2, 1, 0) , (0, 1, -2) , (-1, 1, 0) , (1, 1, 1)
(10)P(3, 0, -1) , Q(0, 3, 2)
(11) p = 0, q = r = 1
(12) (a) collinear, $-\frac{1}{3}$; (b) collinear, $-\frac{3}{4}$; (c) not collinear
(13) M is (2, -1, 0). Hence $\overrightarrow{AM} = (-1, -2, -1), |\overrightarrow{AM}| = \sqrt{6}$

$\overrightarrow{BM} = (1, 1, 1), |\overrightarrow{BM}| = \sqrt{3}$, $\overrightarrow{AB} = (-2, -1, -2), |\overrightarrow{AB}| = 3$]

Scalar product

Definition

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two non-zero vectors and θ is the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined to be the scalar $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$.



The scalar product is denoted by $\boldsymbol{a.b}$
i.e. $\boldsymbol{a.b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$

Also $\boldsymbol{a}\,.0 = 0.\boldsymbol{a} = 0$

Properties

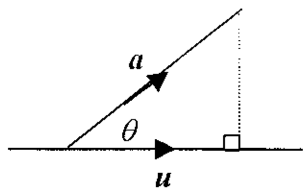
(a) The non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular $\Leftrightarrow$ $\boldsymbol{a.b} = 0$

(b) $\boldsymbol{a.b} = \boldsymbol{b.\ a}$

(c) $\boldsymbol{a}\,.\boldsymbol{a} = \boldsymbol{a}^2$ $\quad$ $\boldsymbol{a}^2$ is usually written $\boldsymbol{a}\,.\boldsymbol{a}$ and so $\boldsymbol{a}^2 = |\boldsymbol{a}|^2$

(d) $\boldsymbol{a}.(\lambda\boldsymbol{b}) = \lambda\ (\boldsymbol{a.b}) = (\lambda\boldsymbol{a}).\boldsymbol{b}$

(e) If $\boldsymbol{u}$ is a unit vector and $\boldsymbol{a}$ is any vector, then
$\boldsymbol{a.u}$ = the projection of vector $\boldsymbol{a}$ on the directed line defined by the vector $\boldsymbol{u}$.



(f) $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a.b} + \boldsymbol{a.c}$ and $(\boldsymbol{a} + \boldsymbol{b}).\boldsymbol{c} = \boldsymbol{a.c} + \boldsymbol{b.c}$

(g) If $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ then $\boldsymbol{a.b} = a_1b_1 + a_2b_2 + a_3b_3$

Example 1

$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = 1$

$\boldsymbol{i.j} = \boldsymbol{j.k} = \boldsymbol{k.i} = 0$

$\boldsymbol{j.i} = \boldsymbol{k.j} = \boldsymbol{i.k} = 0$

Example 2

Show that $\boldsymbol{r} = (\boldsymbol{r.i})\boldsymbol{i} + (\boldsymbol{r.j})\boldsymbol{j} + (\boldsymbol{r.k})\boldsymbol{k}$

Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

Then $\boldsymbol{r.i} = (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}).\boldsymbol{i} = (x(\boldsymbol{i.i}) + y(\boldsymbol{j.i}) + z(\boldsymbol{k.i}) = x$

Similarly $\boldsymbol{r.j} = y$, $\boldsymbol{r.k} = z$

Hence $\boldsymbol{r} = (\boldsymbol{r.i})\boldsymbol{i} + (\boldsymbol{r.i})\boldsymbol{j} + (\boldsymbol{r.k})\boldsymbol{k}$

Example 3
If $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors of equal magnitude, show that $(\boldsymbol{a} - \boldsymbol{b}).(\boldsymbol{a} + \boldsymbol{b}) = 0$.
Interpret this result geometrically.

$(\boldsymbol{a} - \boldsymbol{b}).(\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a} + \boldsymbol{a}.\boldsymbol{b} - \boldsymbol{b}.\boldsymbol{a} - \boldsymbol{b}.\boldsymbol{b} = \boldsymbol{a}^2 - \boldsymbol{b}^2 = |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2 = 0$ since $|\boldsymbol{a}| = |\boldsymbol{b}|$

$(\boldsymbol{a} - \boldsymbol{b})$ and $(\boldsymbol{a} + \boldsymbol{b})$ are perpendicular.

Example 4
$\boldsymbol{a}$ and $\boldsymbol{b}$ are non-parallel unit vectors and $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}\sqrt{3}$ and $\boldsymbol{d} = \boldsymbol{a} - \sqrt{3}\,\boldsymbol{b}$.

If the angle θ between the directions of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the same as the angle between the directions of vectors $\boldsymbol{c}$ and $\boldsymbol{d}$, show that $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$.

$|\boldsymbol{c}|^2 = \boldsymbol{c}^2 = (\boldsymbol{a} + \sqrt{3}\boldsymbol{b})^2 = \boldsymbol{a}^2 + 2\sqrt{3}\boldsymbol{a}.\boldsymbol{b} + 3\boldsymbol{b}^2$
We have $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$ and $\boldsymbol{a}.\boldsymbol{b} = \cos\theta$
Hence $|\boldsymbol{c}|^2 = 4 + 2\sqrt{3}\cos\theta$
$|\boldsymbol{d}| = \boldsymbol{a}^2 - 2\sqrt{3}\boldsymbol{a}.\boldsymbol{b} + 3\boldsymbol{b}^2 = 4 - 2\sqrt{3}\cos\theta$
Now $\boldsymbol{c}.\boldsymbol{d} = |\boldsymbol{c}||\boldsymbol{d}|\cos\theta$ and $\boldsymbol{c}.\boldsymbol{d} = (\boldsymbol{a} + \sqrt{3}\,\boldsymbol{b}).(\boldsymbol{a} - \sqrt{3}\,\boldsymbol{b}) = \boldsymbol{a}^2 - 3\boldsymbol{b}^2 = |\boldsymbol{a}|^2 - 3|\boldsymbol{b}|^2 = -$
2
Hence $\boldsymbol{c}.\boldsymbol{d} = |\boldsymbol{c}||\boldsymbol{d}|\cos\theta = -2$

$\Rightarrow |\boldsymbol{c}|^2|\boldsymbol{d}|^2\cos^2\theta = 4 \Rightarrow (16 - 12\cos^2\theta)\cos^2\theta = 4 \Rightarrow 4\cos^2\theta - 3\cos^4\theta = 1$
$\Rightarrow 3\cos^4\theta - 4\cos^2\theta + 1 = 0 \Rightarrow (3\cos^2\theta - 1)(\cos^2\theta - 1) = 0$
$\Rightarrow \cos^2\theta = \frac{1}{3}$ or 1
$\Rightarrow \cos\theta = \pm\frac{1}{\sqrt{3}}$ or ± 1
$\cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow$ vectors parallel. **contradiction**

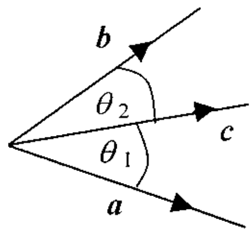
$\cos\theta = -1 \Rightarrow \theta = \pi \Rightarrow$ vectors parallel. **contradiction**

$\cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \boldsymbol{c}.\boldsymbol{d}$ is positive. **contradiction**

Hence $\cos\theta = -\frac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Example 5

Prove that the vector $\boldsymbol{c} = |\boldsymbol{b}|\boldsymbol{a} + |\boldsymbol{a}|\boldsymbol{b}$ bisects the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Find this vector when $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i} + 3\boldsymbol{k}$.



Let angle between directions of vectors $\boldsymbol{a}$ and $\boldsymbol{c}$ be θ_1
Let angle between directions of vectors $\boldsymbol{c}$ and $\boldsymbol{b}$ be θ_2

Then $\boldsymbol{a.c} = \boldsymbol{a}.\,(|\boldsymbol{b}|\boldsymbol{a} + |\boldsymbol{a}|\boldsymbol{b}) = |\boldsymbol{b}|\boldsymbol{a}^2 + |\boldsymbol{a}|\boldsymbol{a.b}$
$= |\boldsymbol{b}||\boldsymbol{a}|^2 + |\boldsymbol{a}||\boldsymbol{a}||\boldsymbol{b}| \cos(\theta_1 + \theta_2) = |\boldsymbol{a}|^2|\boldsymbol{b}|[1 + \cos(\theta_1 + \theta_2)]$

Also $\boldsymbol{a.c} = |\boldsymbol{a}||\boldsymbol{c}| \cos\theta_1$

Hence $|\boldsymbol{a}||\boldsymbol{c}| \cos\theta_1 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 [1 + \cos(\theta_1 + \theta_2)]$

$$\Rightarrow \cos\theta_1 = \frac{|\boldsymbol{a}||\boldsymbol{b}|[1 + \cos(\theta_1 + \theta_2)]}{|\boldsymbol{c}|}$$

$\boldsymbol{b.c} = \boldsymbol{b}.\,(|\boldsymbol{b}|\boldsymbol{a} + |\boldsymbol{a}|\boldsymbol{b}) = |\boldsymbol{b}|\boldsymbol{a.b} + |\boldsymbol{a}|\boldsymbol{b}^2 = |\boldsymbol{b}||\boldsymbol{a}||\boldsymbol{b}| \cos(\theta_1 + \theta_2) + |\boldsymbol{a}||\boldsymbol{b}|^2$
$= |\boldsymbol{a}||\boldsymbol{b}|^2[1 + \cos(\theta_1 + \theta_2)]$

Also $\boldsymbol{b.c} = |\boldsymbol{b}||\boldsymbol{c}| \cos\theta_2$
Hence $|\boldsymbol{b}||\boldsymbol{c}| \cos\theta_2 = |\boldsymbol{a}||\boldsymbol{b}|^2[1 + \cos(\theta_1 + \theta_2)]$

$$\Rightarrow \cos\theta_2 = \frac{|\boldsymbol{a}||\boldsymbol{b}|[1 + \cos(\theta_1 + \theta_2)]}{|\boldsymbol{c}|}$$

Hence $\cos\theta_1 = \cos\theta_2 \Rightarrow \theta_1 = \theta_2$

When $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i} + 3\boldsymbol{k}$.
$|\boldsymbol{a}| = \sqrt{9} = 3$ and $|\boldsymbol{b}| = \sqrt{25} = 5$

and so $\boldsymbol{c} = 5(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) + 3(4\boldsymbol{i} + 3\boldsymbol{k}) = 22\boldsymbol{i} + 10\boldsymbol{j} + 14\boldsymbol{k}$

Examples for class

(1) Find the scalar product $\boldsymbol{a}.\boldsymbol{b}$ where

(a) $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k}$

(b) $\boldsymbol{a} = 2\boldsymbol{i} + \frac{3}{2}\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \frac{1}{2}\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$

(c) $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

(2) Find two vectors of unit length which make an angle of 45° with $\begin{pmatrix}1\\0\\0\end{pmatrix}$

and are perpendicular to the vector $\begin{pmatrix}0\\0\\1\end{pmatrix}$

(3) Find two unit vectors which make an angle of 60° with both the vectors $\begin{pmatrix}1\\-1\\0\end{pmatrix}$

and $\begin{pmatrix}1\\0\\-1\end{pmatrix}$. Find also the vectors which make angles of 45° with both the above vectors.

(4) Find the components of the two unit vectors which make an angle of 45° with $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ and an angle of 60° with $\begin{pmatrix}-2\\2\\1\end{pmatrix}$.

Show that these two unit vectors are perpendicular.

(5) The position vectors of the vertices of a triangle are $\boldsymbol{0}$, $\boldsymbol{a}$, $\boldsymbol{b}$.

Show that its area A is given by the formula $4A^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}.\boldsymbol{b})^2$

(6) Which of the following expressions represent vectors and which represent numbers ?

(i) $\boldsymbol{b}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{a} + \boldsymbol{a}.\boldsymbol{b}$; (ii) $(\boldsymbol{b}.\boldsymbol{c}).\boldsymbol{a} + (\boldsymbol{c}.\boldsymbol{a}).\boldsymbol{b} + (\boldsymbol{a}.\boldsymbol{b}).\boldsymbol{c}$;

(iii) $[(\boldsymbol{b}.\boldsymbol{c}).\boldsymbol{c} + (\boldsymbol{b}.\boldsymbol{a}).\boldsymbol{a}] . (\boldsymbol{b} + 2\boldsymbol{a})$; (iv) $(\boldsymbol{b}.\boldsymbol{c})(\boldsymbol{c}.\boldsymbol{a}).\boldsymbol{a}$

Evaluate these when $\boldsymbol{a} = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix}1\\1\\2\end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix}0\\2\\1\end{pmatrix}$.

Answers

(1) (a) -11 (b) 6 (c) 2

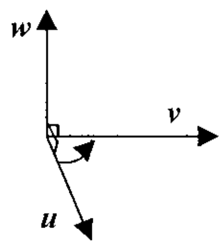
(2) $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$ (3) $\begin{pmatrix} 2\sqrt{2}/3 \\ \sqrt{2}/6 \\ \sqrt{2}/6 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1/3 \\ -2/3 \\ -2/3 \end{pmatrix}$

(4) $\begin{pmatrix} \frac{-3+2\sqrt{2}}{6} \\ \sqrt{2}/6 \\ \frac{3+2\sqrt{2}}{6} \end{pmatrix}$ and $\begin{pmatrix} \frac{-3-2\sqrt{2}}{6} \\ \sqrt{2}/6 \\ \frac{3-2\sqrt{2}}{6} \end{pmatrix}$, Proof (5) Proof

(6) (i) number (ii) vector (iii) number (iv) vector
(i) 9 (ii) $4\boldsymbol{a} + \boldsymbol{b} + 3\boldsymbol{c}$ (iii) 45 (iv) $4\boldsymbol{a}$

Vector product

Right handed systems



Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be three non-zero, non-coplanar, vectors. Then the system $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ is called a **right-handed system.**

The direction of $\boldsymbol{w}$ is that of a right-handed screw from $\boldsymbol{u}$ to $\boldsymbol{v}$.

If $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ is right-handed, then so are $(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v})$ and $(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u})$, obtained by cyclic interchange.

From now on, we assume that the base-vectors $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ form a right-handed system.
N.B. $(\boldsymbol{v}, \boldsymbol{u}, -\boldsymbol{w})$ is a right-handed system.

Definition

(a) Let $\boldsymbol{a}$, $\boldsymbol{b}$ be given vectors and let $\boldsymbol{c}$ be a unit vector such that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a right-handed system.
Then if θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, the **vector product** is defined to be the vector $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta\, \boldsymbol{c}$

We denote this by $\boldsymbol{a} \times \boldsymbol{b}$, so that $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\, \boldsymbol{c}$

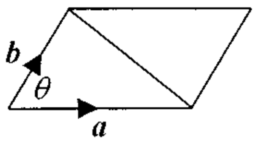
(b) If $\boldsymbol{a}$ or $\boldsymbol{b}$ is $\boldsymbol{0}$ or if $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$ then $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$

The vector product $\boldsymbol{a} \times \boldsymbol{b}$ is a vector with
Magnitude : $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$
Direction : perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$
Sense : so that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b})$ forms a right-handed system.

Properties

(a) $\boldsymbol{a} \times \boldsymbol{b}$ = area of a parallelogram with sides determined by $\boldsymbol{a}$ and $\boldsymbol{b}$.

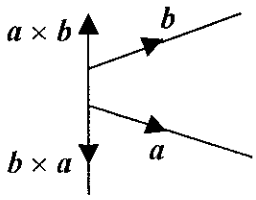
Proof : (1) trivial if $\boldsymbol{a} \times \boldsymbol{b} = 0$
(2) $\boldsymbol{a} \times \boldsymbol{b} \neq 0$



Area of parallelogram = 2(area of one triangle)
$= 2(\frac{1}{2}|\boldsymbol{a}||\boldsymbol{b}|\sin\theta)$
$= |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$

Also $|\boldsymbol{a} \times \boldsymbol{b}| = \left||\boldsymbol{a}||\boldsymbol{b}|\sin\theta\, \boldsymbol{c}\right| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\, |\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$. Hence result.

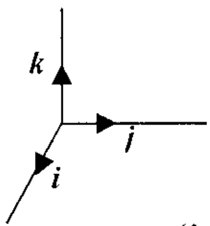
(b) $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$
We say that $\boldsymbol{a}$ and $\boldsymbol{b}$ are **Anti-commute.**



(c) $k\boldsymbol{a} \times \boldsymbol{b} = k(\boldsymbol{a} \times \boldsymbol{b})$
$k\boldsymbol{a} \times l\boldsymbol{b} = kl(\boldsymbol{a} \times \boldsymbol{b})$

(d)

X	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\boldsymbol{i}$	$\boldsymbol{0}$	$\boldsymbol{k}$	$-\boldsymbol{j}$
$\boldsymbol{j}$	$-\boldsymbol{k}$	$\boldsymbol{0}$	$\boldsymbol{i}$
$\boldsymbol{k}$	$\boldsymbol{j}$	$-\boldsymbol{i}$	$\boldsymbol{0}$



$(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ forms an **orthonormal** set of vectors

(e) Vector product is distributive over vector addition

$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{c}) \qquad (\boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{a} = (\boldsymbol{b} \times \boldsymbol{a}) + (\boldsymbol{c} \times \boldsymbol{a})$

(f) Vector product in component form

Suppose $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

then $\boldsymbol{a} \times \boldsymbol{b} = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \times (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k})$
$= a_1 b_1(\boldsymbol{i} \times \boldsymbol{i}) + a_1 b_2(\boldsymbol{i} \times \boldsymbol{j}) + a_1 b_3(\boldsymbol{i} \times \boldsymbol{k})$
$= a_2 b_1(\boldsymbol{j} \times \boldsymbol{i}) + a_2 b_2(\boldsymbol{j} \times \boldsymbol{j}) + a_2 b_3(\boldsymbol{j} \times \boldsymbol{k})$
$= a_3 b_1(\boldsymbol{k} \times \boldsymbol{i}) + a_3 b_2(\boldsymbol{k} \times \boldsymbol{j}) + a_3 b_3(\boldsymbol{k} \times \boldsymbol{k})$
$= a_1 b_2\boldsymbol{k} - a_1 b_3\boldsymbol{j} - a_2 b_1 \boldsymbol{k} + a_2 b_3\boldsymbol{i} + a_3 b_1 \boldsymbol{j} - a_3 b_2 \boldsymbol{I}$

$$= (a_2 b_3 - a_3 b_2)\boldsymbol{i} - (a_1 b_3 - a_3 b_1)\boldsymbol{j} + (a_2 b_1 - a_1 b_2)\boldsymbol{k}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix}\boldsymbol{k}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \end{vmatrix}$$

Example 1

If $\boldsymbol{a} = (3, 5, 7)$, $\boldsymbol{b} = (1, 0, 1)$, $\boldsymbol{c} = (2, -1, 3)$, find $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$

$$\boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \end{vmatrix} = (0 + 1)\boldsymbol{i} - (3 - 2)\boldsymbol{j} + (-1 - 0)\boldsymbol{k} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} 3 & 5 & 7 \\ 1 & -1 & -1 \\ \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \end{vmatrix} = (-5 + 7)\boldsymbol{i} - (-3 - 7)\boldsymbol{j} + (-3 - 5)\boldsymbol{k} = 2\boldsymbol{i} + 10\boldsymbol{j} - 8\boldsymbol{k}$$

(g) If $\boldsymbol{a} \times \boldsymbol{b} \neq 0$, then $\boldsymbol{u} = \pm \dfrac{(\boldsymbol{a} \times \boldsymbol{b})}{|\boldsymbol{a} \times \boldsymbol{b}|}$ is a unit vector perpendicular to (a plane determined by) $\boldsymbol{a}$ and $\boldsymbol{b}$.

Example 2

Find the unit vectors perpendicular to both of the vectors $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$.

The required vectors are $\pm \dfrac{(\boldsymbol{a} \times \boldsymbol{b})}{|\boldsymbol{a} \times \boldsymbol{b}|}$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \\ \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \end{vmatrix} = \boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$$

$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{1 + 9 + 25} = \sqrt{35}$. Hence unit vectors are $\pm \dfrac{1}{\sqrt{35}}(\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k})$

(h) $\boldsymbol{a} \times \boldsymbol{b} = 0 \Leftrightarrow \boldsymbol{a}$ or $\boldsymbol{b} = 0$ or $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel i.e. $\boldsymbol{b} = k\boldsymbol{a}$.

Example 3
Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be unit vectors such that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, and $\mathbf{b}$ makes an angle of 60° with $\mathbf{c}$. Show that $\mathbf{b} - \mathbf{c} = \pm\, \mathbf{a}$.

$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \Leftrightarrow \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} = \mathbf{0} \Leftrightarrow \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0} \Leftrightarrow$ ~~$\mathbf{a} = \mathbf{0}$~~ or ~~$\mathbf{b} - \mathbf{c} = \mathbf{0}$~~
or $\mathbf{b} - \mathbf{c} = k\mathbf{a}$ for some k.

Now $|\mathbf{b} - \mathbf{c}|^2 = (\mathbf{b} - \mathbf{c}) \times (\mathbf{b} - \mathbf{c}) = \mathbf{b}^2 - 2\mathbf{b}.\mathbf{c} + \mathbf{c}^2 = |\mathbf{b}|^2 - 2|\mathbf{b}||\mathbf{c}|\cos 60° + |\mathbf{c}|^2$
$= 1 - 2 . \frac{1}{2} + 1 = 1$
Hence $\mathbf{b} - \mathbf{c} = k\mathbf{a} \Leftrightarrow |\mathbf{b} - \mathbf{c}|^2 = k^2\, |\mathbf{a}|^2 \Leftrightarrow 1 = k^2 \Leftrightarrow k = \pm\ 1$

Hence $\mathbf{b} - \mathbf{c} = \pm\ \mathbf{a}$

Example 4
If $\mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} = \mathbf{a} \times \mathbf{b} \neq 0$, show that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$.

$\mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} \Leftrightarrow \mathbf{b} \times \mathbf{c} = -\mathbf{a} \times \mathbf{c} \Leftrightarrow (\mathbf{b} + \mathbf{a}) \times \mathbf{c} = \mathbf{0}$

Thus $\mathbf{b} + \mathbf{a} = k\mathbf{c}$, for some k.

Take vector products of both sides with $\mathbf{a}$.
$\mathbf{a} \times (\mathbf{b} + \mathbf{a}) = \mathbf{a} \times k\mathbf{c} \Leftrightarrow (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{a}) = k(\mathbf{a} \times \mathbf{c}) \Leftrightarrow\ \mathbf{a} \times \mathbf{b} = k(\mathbf{a} \times \mathbf{c})$

But we are given that $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a} = -\mathbf{a} \times \mathbf{c}$

Hence $k = -1$ and so $\mathbf{b} + \mathbf{a} = -\mathbf{c}$, i.e. $\mathbf{b} + \mathbf{a} + \mathbf{c} = \mathbf{0}$, i.e. $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$.

Examples for class

(1) Find the vector product $\boldsymbol{a} \times \boldsymbol{b}$ where
(a) $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}, \boldsymbol{b} = 3\boldsymbol{i} + 4\boldsymbol{j}$ (b) $\boldsymbol{a} = 3\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}, \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$
(c) $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}, \boldsymbol{b} = 3\boldsymbol{i} + \boldsymbol{j}$

(2) Simplify (a) $\boldsymbol{a} \times (2\boldsymbol{a} + \boldsymbol{b})$; (b) $(\boldsymbol{a} + \boldsymbol{b}) \times (2\boldsymbol{a} - \boldsymbol{b})$; (c) $(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) \times (\boldsymbol{b} - \boldsymbol{a} - 2\boldsymbol{c})$;
(d) $\boldsymbol{a} \,.\, (\boldsymbol{a} \times \boldsymbol{b})$ [use components if necessary]

(3) If $\boldsymbol{a} = (-2, 4, 3)$, $\boldsymbol{b} = (3, -6, -\frac{9}{2})$, $\boldsymbol{c} = (\frac{3}{2}, 9, 3)$, $\boldsymbol{d} = (1, 6, 2)$
Show that $(\boldsymbol{a} \times \boldsymbol{c}) + (\boldsymbol{b} \times \boldsymbol{d}) = \boldsymbol{0}$.

(4) Show that $(r\boldsymbol{a} + s\boldsymbol{b})$ x $(t\boldsymbol{a} + u\boldsymbol{b}) = (ru - st)(\boldsymbol{a} \times \boldsymbol{b})$

(5) Let $\boldsymbol{a} = (3, 2, -1)$, $\boldsymbol{b} = (1, -1, -2)$, $\boldsymbol{c} = (4, -3, 4)$. Evaluate the following expressions:
(i) $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ (ii) $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ (iii) $(\boldsymbol{a} \times \boldsymbol{b}) \,.\, (\boldsymbol{a} \times \boldsymbol{c})$ (iv) $(\boldsymbol{a} \times \boldsymbol{b}) \,.\, (\boldsymbol{b} \times \boldsymbol{c})$
(v) $[\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b})] \,.\, \boldsymbol{c}$.

(6) If $\boldsymbol{a} = (3, 1, 2)$, $\boldsymbol{b} = (0, 2, -1)$, $\boldsymbol{c} = (1, 1, 1)$ and $\boldsymbol{d} = \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + (\boldsymbol{a} \,.\, \boldsymbol{c})\boldsymbol{a}$,
Show that $\boldsymbol{b}$ is perpendicular to $\boldsymbol{d}$.

(7) Let $\boldsymbol{v}$ be a unit vector and $\boldsymbol{a}$ a vector such that $\boldsymbol{a} \,.\, \boldsymbol{v} = 0$.
Show directly from the definition, that if $\boldsymbol{b} = \boldsymbol{v} \times \boldsymbol{a}$ then $\boldsymbol{v} \times \boldsymbol{b} = -\boldsymbol{a}$.
Verify this when $\boldsymbol{a} = (2, -2, 1)$, $\boldsymbol{v} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

(8) Suppose that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are distinct unit vectors such that $\boldsymbol{a} \times \boldsymbol{c} = \boldsymbol{b} \times \boldsymbol{c}$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{c}$ is 45°. Show that $\boldsymbol{a}$ is at right angles to $\boldsymbol{b}$.

Answers

(1) (a) $11\boldsymbol{k}$ (b) $2\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$ (c) $-4\boldsymbol{i} + 12\boldsymbol{j} + 7\boldsymbol{k}$

(2) (a) $\boldsymbol{a} \times \boldsymbol{b}$ (b) $3\boldsymbol{b} \times \boldsymbol{a}$ (c) $2\boldsymbol{a} \times \boldsymbol{b} + 3\boldsymbol{c} \times \boldsymbol{b} + \boldsymbol{c} \times \boldsymbol{a}$ (d) 0

(3) Proof (4) Proof

(5) (i) $-10\boldsymbol{i} + 7\boldsymbol{j} + 16\boldsymbol{k}$ (ii) $5\boldsymbol{i} - 5\boldsymbol{k}$ (iii) -20 (iv) -15 (v) 20

(6) Proof (7) Proof (8) Proof

Geometry of line and plane
Geometry of the straight line and the plane

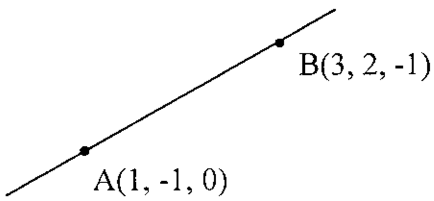
Straight line
In order to obtain the equation of a straight line in two dimensions, it is necessary to know two facts about the line:-
(i) a point on the line
(ii) direction of the line

In three dimensions, exactly the same conditions hold for finding the equation of a straight line.

Any vector, whose direction is the direction of the line itself, is called a **direction vector** of the line.

The components of a direction vector of a line are called the **direction ratios** of the line, and if the direction vector is a unit vector, then the components are called the **direction cosines** of the line.

Example 1
Find the direction cosines for the line determined by the points A(1, -1, 0) and B(3,2,-1).



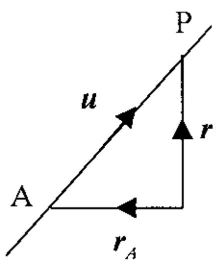
$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, $\left|\overrightarrow{AB}\right| = \sqrt{14}$

$\overrightarrow{AB}$ is a direction vector of the line.
2, 3, -1 are direction ratios

Direction cosines of the line are $\pm\frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

The line has direction ratios ($2t$, $3t$, $-t$) for any $t \neq 0$.

Vector equation of the line through the point A in the direction of unit vector
If P is any point on the line with position vector $\boldsymbol{r}$,



then P($\boldsymbol{r}$) lies on the line $\Leftrightarrow$ $\overrightarrow{AP} = |AP|\boldsymbol{u}$

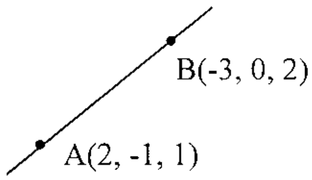
$\Leftrightarrow \boldsymbol{r} - \boldsymbol{r}_A = t\boldsymbol{u}$, for some scalar t

. $\Leftrightarrow \boldsymbol{r} = \boldsymbol{r}_A + t\boldsymbol{u}$

Hence vector equation of line is $\boldsymbol{r} = \boldsymbol{r}_A + t\boldsymbol{u}$,
t being a parameter.

Example 1

Find a vector equation for the line joining the points whose position vectors are $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $-3\boldsymbol{i} + 2\boldsymbol{k}$.



$$\overrightarrow{AB} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}, \quad \left|\overrightarrow{AB}\right| = \sqrt{27} = 3\sqrt{3}$$

Unit vector in direction of line = $\dfrac{1}{3\sqrt{3}}\begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$

Vector equation of line is $\boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k} + t\,.\,\dfrac{1}{3\sqrt{3}}(-5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

i.e. $\boldsymbol{r} = \left(2 - \dfrac{5t}{3\sqrt{3}}\right)\boldsymbol{i} + \left(-1 + \dfrac{t}{3\sqrt{3}}\right)\boldsymbol{j} + \left(1 + \dfrac{t}{3\sqrt{3}}\right)\boldsymbol{k}$

Suppose with respect to $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

$\boldsymbol{r} = (x, y, z)$, $\boldsymbol{r}_A = (a, b, c)$, $\boldsymbol{u} = (l, m, n)$

$\boldsymbol{r} = \boldsymbol{r}_A + t\boldsymbol{u} \Leftrightarrow (x, y, z) = (a + tl, b + tm, c + tn)$

$\Leftrightarrow x = a + tl, y = b + tm, z = c + tn$ ------------ (1)

$\Leftrightarrow \dfrac{x-a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n} \quad (= t)$ -------------- (2)

Equations (1) are called the **parametric equations** of the line.

Equations (2) are called the **symmetric equations** of the line.

Notes:

(1) These equations have been shown using direction cosines, but we could have used direction ratios instead of the direction cosines.

(2) These equations are not unique for any one line, because direction ratios are not unique.

(3) The line through the point (a, b, c) parallel to the z-axis has, according to the above, the following equations.

$$\frac{x-a}{0} = \frac{y-b}{0} = \frac{z-c}{1}$$

(4) When finding points of intersection, always convert symmetric equations into parametric equations, using a different parameter for each line.

Example 2

Find the equations of the line joining the points A(1, 0, 2) and B(2, 1, 0).

Direction vector of line = $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

Hence equations of line are $\frac{x-1}{1} = \frac{y}{1} = \frac{z-2}{-2} = t$, where t is a parameter.

Example 3

Show that the lines $\frac{x+3}{4} = \frac{y-4}{-7} = \frac{z+1}{3} \quad (= t)$, $\frac{x-5}{2} = \frac{y-3}{3} = \frac{z+6}{-4} \quad (= \tau)$ are coplanar.

The given lines will be coplanar $\Leftrightarrow$ either (i) they are parallel or (ii) they intersect.

First line has direction vector $\begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix}$, second line has direction vector $\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$

Since $\begin{pmatrix} 4 \\ -7 \\ 3 \end{pmatrix} \not\propto \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, the lines are not parallel. Hence we must show that they intersect. Any point on first line is $(-3 + 4t, 4 - 7t, -1 + 3t)$. Any point on second line is $(5 + 2\tau, 3 + 3\tau, -6 - 4\tau)$.

The lines intersect $\Leftrightarrow \exists\ t, \tau$ such that

$$\begin{aligned} 4t - 3 &= 2\tau + 5 & \qquad 2t - \tau &= 4 \\ -7t + 4 &= 3\tau + 3 \;\Leftrightarrow & 7t - 3\tau &= -1 \\ 3t - 1 &= -4\tau - 6 & 3t + 4\tau &= -5 \end{aligned}$$

Now $\begin{matrix} 2t - \tau = 4 \\ -7t - 3\tau = -1 \end{matrix} \Leftrightarrow t = 1\ ,\ \tau = -2$

Since the third equation is also satisfied by these values the lines intersect (at the point (1, -3, 2)).
Hence lines are coplanar.

Examples for class

(1) Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ $(=t)$, $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ $(=\tau)$ are coplanar.

(2) Find the symmetric equations and parametric equations for each of the following lines:

(a) the line through the point A(2, 5, 3) and having direction vector (-2, 4, -3).

(b) the line through the points (-2, -1, 4) and (3, 2, 2).

Find the point in which each of these lines meets the x, y-plane.

(3) Find the value of c such that the line with parametric equations $x = 1 + 7t$, $y = 2 + 3t$, $z = 1 + ct$ is parallel to the line $x = 3 + 21k$, $y = 2 + 9k$, $z = 5 + 8k$. Show also there is no value of c for which the lines intersect.

(4) Find the co-ordinates of the points in which the line $\frac{x}{-1}=\frac{y-3}{2}=\frac{z-4}{3}$ meets the surface $xy = z$.

Answers

(1) Proof

(2) (a) $x = 2 - 2t$, $y = 5 + 4t$, $z = 3 - 3t$;

$\frac{x-2}{-2}=\frac{y-5}{4}=\frac{z-3}{-3}$ meets x-y plane at (0, 9, 0);

(b) $x = -2 + 5t$, $y = -1 + 3t$, $z = 4 - 2t$;

$\frac{x+2}{5}=\frac{y+1}{3}=\frac{z-4}{-2}$ meets x-y plane at (8, 5, 0)

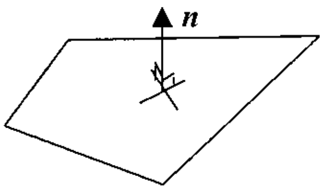
(3) $c = \frac{8}{3}$

(4) (1, 1, 1), (2, -1, -2)

The plane

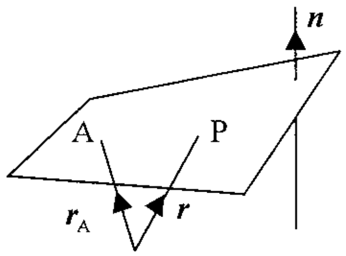
Introduction

We assume that, given any plane, there is a unique direction , said to be **perpendicular** to the plane, which is at right angles to the direction of any vector which lies in the plane.



Any vector with direction perpendicular to the plane is called a **normal vector** of the plane.

Vector equation of a plane



We consider the plane containing the point A($\boldsymbol{r}_A$) and with normal in the direction $\boldsymbol{n}$.
P($\boldsymbol{r}$) lies on the plane
$\Leftrightarrow \overrightarrow{\mathrm{AP}}$ is perpendicular to $\boldsymbol{n}$.
$\Leftrightarrow \overrightarrow{\mathrm{AP}} - \boldsymbol{n} = 0 \Leftrightarrow (\boldsymbol{r} - \boldsymbol{r}_A) \,.\, \boldsymbol{n} = 0$

If $\boldsymbol{r} = (x, y, z)$, $\boldsymbol{r}_A = (a, b, c)$, $\boldsymbol{n} = (l, m, n)$, then the equation takes the form
$(x - a, y - b, z - c) \,.\, (l, m, n) = 0$

i.e. $l(x - a) + m(y - b) + n(z - c) = 0$ i.e. $lx + my + nz = la + mb + nc$

i.e. $lx + my + nz = k$, where $k = la + mb + nc$.

Hence the **equation of the plane** is $lx + my + nz = k$

Notes:
(1) Given the equation of the plane, the coefficients of x, y and z are the components of the normal vector to the plane.

(2) If the plane passes through the origin, $k = 0$.

Example 1
Find the equation of the plane through the points A(0, 1, 1), B(1, 2, 0), C(1, 3, 0).

A normal vector = $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}}$

$$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \overrightarrow{\mathrm{AC}} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}. \quad \overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \end{vmatrix} = \boldsymbol{i} + \boldsymbol{k} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Hence the equation of plane is $1.x + 0.y + 1.z = 0 + 0 + 1$ i.e. $x + z = 1$

Examples for class

(1) Find the equation of the plane containing the three points (1, 0, 1) , (1, 1, 1) , (2, 1, -1).

(2) A plane is parallel to both lines (i) $\frac{x-1}{2} = \frac{y}{3} = \frac{z-1}{4}$ (ii) $\frac{x+1}{-1} = \frac{y}{2} = \frac{z}{1}$
It also passes through the point (1, 0, -1). Find its equation.

(3) If l is the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1}$, and m is the line through the point (5, 4, 2) which cuts l at right angles, find equations for m and the co-ordinates of the point of intersection of l and m.

Answers

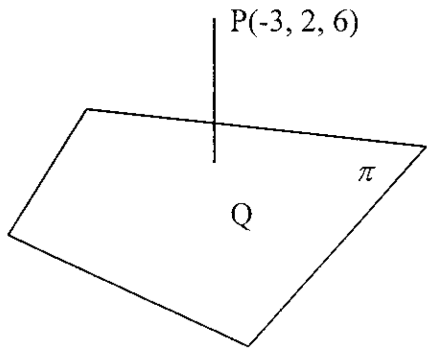
(1) $-2x - z = -3$
(2) $5x + 6y - 7z = 12$
(3) $\frac{x-1}{2} = \frac{y-6}{-1} = z$; (1, 6, 0)

Worked examples on line and plane

Intersection of line and plane

Example

Find the co-ordinates of the projection Q of the point P(-3, 2, 6) on the plane π with equation $2x - 3y - z = 10$.



Normal vector = (2, 3, -1)
PQ is normal to plane

$\Rightarrow$ direction vector of PQ = $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

Hence equations of line PQ are
$\frac{x+3}{2} = \frac{y-2}{3} = \frac{z-6}{-1}$ (= t, say)

Hence parametric equations are $x = -3 + 2t,\ y = 2 - 3t,\ z = 6 - t$

Hence Q lies on plane π $\Leftrightarrow 2(-3 + 2t) - 3(2 - 3t) - (6 - t) = 10$

$\Leftrightarrow -6 + 4t - 6 + 9t - 6 + t = 10$

$\Leftrightarrow 14t = 28$

$\Leftrightarrow t = 2$

Hence Q is the point (1, -4, 4)

Examples for class

(1) A plane passes through the points (0, 1, 2) and (1, -1, 0). It is parallel to the direction (1, -1, 1). Find the equation of the plane and the foot of the perpendicular on it from the point (3, 0, 3).

(2) Find the point of intersection of the line with parametric equations $x = 4 + t$, $y = 1 - t$, $z = 3t$ and the plane $2x + 4y + z = 9$.

(3) Find equations for the line through the point (3, 5, 2) perpendicular to the plane $5x - 7y + 4z = -2$ and obtain the co-ordinates of the foot of the perpendicular on the plane from the point .

Answers

(1) $4x + 3y - z = 1$; $\left(\frac{23}{13}, \frac{-12}{13}, \frac{43}{13}\right)$

(2) (1, 4, -9)

(3) $\dfrac{x-3}{5} = \dfrac{y-5}{-7} = \dfrac{z-2}{4}$; $\left(\frac{32}{9}, \frac{38}{9}, \frac{22}{9}\right)$

Angle between two planes

Definition: - The angle between two planes is defined to be the angle between their normals.

The cosine of this angle is given by the formula

$$\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$$

Example

Find the acute angle between the planes $x - 2y + z = 0$ and $x - y = 1$

Normal vector to plane $x - 2y + z = 0 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Normal vector to plane $x - y = 1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

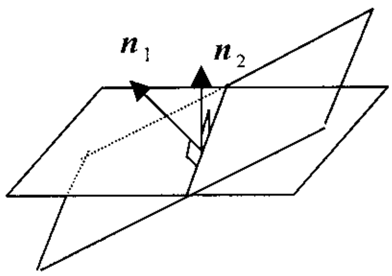
Let θ be the angle between two direction vectors, then $\cos\theta = \dfrac{1+2+0}{\sqrt{6}.\sqrt{2}} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$

Thus $\theta = \dfrac{\pi}{6}$.

The line of intersection of two planes

Example

Find the equations, in symmetric form, for the line of intersection of the planes $x - y + 3z = 2$ and $3x + y + z = 2$.



We require (i) direction vector of line
(ii) a point on line .

$$\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \ \boldsymbol{n}_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$n_1 \times n_2 = \begin{vmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \\ i & j & k \end{vmatrix} = -4\boldsymbol{i} + 8\boldsymbol{j} + 4\boldsymbol{k} = \begin{pmatrix} -4 \\ 8 \\ 4 \end{pmatrix} = 4\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Hence direction vector for line = $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

We now look for a point on the line at which $z = 0$.

In this case $\left.\begin{aligned} x - y &= 2 \\ 3x + y &= 2 \end{aligned}\right\} \Rightarrow x = 1\ , y = -1$

Hence a point on line is (1, -1, 0)

Hence equation of line is $\dfrac{x-1}{-1} = \dfrac{y+1}{2} = \dfrac{z}{1}$

Examples for class

(1) Two planes (A), (B) each pass through the origin.
The plane (A) contains the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{2}$ and the plane (B) contains the line of intersection of the planes $x + y + z = 1$. $2x - y + 3z = 2$.

Find the cosine of the angle between the planes (A) and (B).

(2) A line L is the intersection of the two planes $x + y + z = 1$, $x - 2y + 3z = 2$.
Find the equation of the plane containing L and passing through the origin.

Show that the plane found makes an angle of 60° with the plane $y + z = 0$.

(3) Find the angle between the planes $3x - 4y + 5z = 10$, $x - 4y + 3z = 15$.
Find also the equation of the plane which passes through their line of intersection and is parallel to the line $x = 2y = 3z$.

(4) Find the equation of the plane which contains the first of the following two lines and also contains their common perpendicular.

$$\frac{x-1}{2} = -y = -z-1 \; ; \; x+1 = \frac{-y}{2} = z-3$$

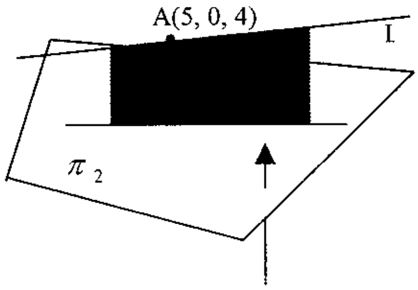
Answers

(1) $2\frac{13}{\sqrt{105}}$ (2) $x + 4y - z = 0$ (3) $\cos^{-1}\left(\frac{17}{5\sqrt{13}}\right)$; $x - 4y + 3z = 15$

(4) $y - z = 1$

Projection of a line on a plane
Example

Find the equations of the projection of the line $\frac{x - 5}{3} = y = z - 4$ on the plane $x + 2y + z = 3$.



The projection is the intersection of the planes π_1 and π_2.

We first find the equation of the plane π_1. The normal vector, $\boldsymbol{n}_1$ to the plane π_1 is perpendicular to $\boldsymbol{n}_2 = \begin{pmatrix}1\\2\\1\end{pmatrix}$, vector $\boldsymbol{a} = \begin{pmatrix}3\\1\\1\end{pmatrix}$ and to the vector $\boldsymbol{n}_2 = \begin{pmatrix}1\\2\\1\end{pmatrix}$.

$$\boldsymbol{a} \times \boldsymbol{n}_2 = \begin{vmatrix}3 & 1 & 1\\1 & 2 & 1\\ \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\end{vmatrix} = -\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k} = \begin{pmatrix}-1\\-2\\5\end{pmatrix}. \quad \text{Hence } \boldsymbol{n}_1 = \begin{pmatrix}-1\\-2\\5\end{pmatrix}.$$

Since the point A(5, 0, 4) lies on line L, A is a point on π_1.
Hence the equation of π_1 is $-x - y + 5z = -5 - 0 + 20$
i.e. $-x - y + 5z = 15$.
Hence equations for projection of L on π_2 are $\begin{cases} x + 2y + z = 3 \\ -x - y + 5z = 15 \end{cases}$.

Example for class

(1) Find equations of the projections on the plane $6x - 3y + 2z = 1$ of the line of intersection of the planes $x + y + 2z = 3$, $3x + y + 3z = 4$.

Answers

(1) $\frac{6x-2}{13} = \frac{y-1}{3} = \frac{z-1}{-2}$

The plane passing through the line of intersection through two planes

Example

Find the equation of the plane passing through the line of intersection of the two planes $x - y + 3z = 2$ and $3x + y + z = 2$ and also passing through the origin.

Any equation passing through the line of intersection has equation

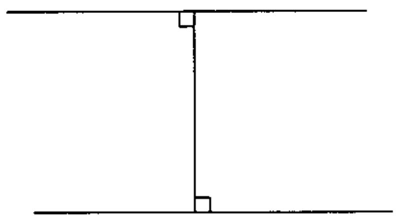
$x - y + 3z - 2 + k(3x + y + z - 2) = 0$, for some k

i.e. $(1 + 3k)x + (-1 + k)y + (3 + k)z = 2 + 2k$

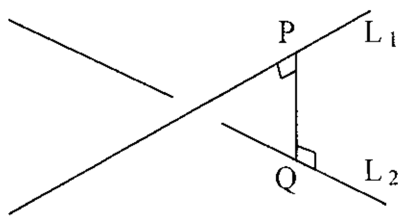
0 lies on the plane $\Leftrightarrow 2 + 2k = 0 \Leftrightarrow k = -1$

Hence the equation of the required plane is $-2x - 2y + 2z = 0$ i.e. $x + y - z = 0$.

The shortest distance between two non-intersecting lines



Parallel lines



Skew lines

Line of shortest distance is the **common perpendicular**.

Example

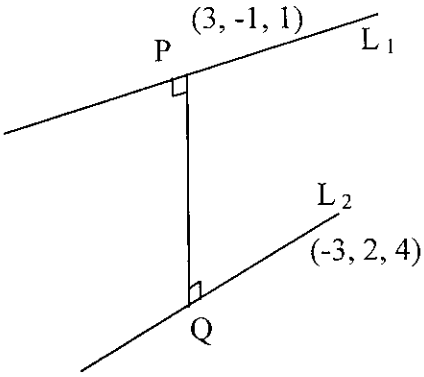
Find the length and equations of the line of shortest distance between the lines

$$\frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1} = s \qquad -L_1$$

$$\frac{x-6}{-3} = \frac{y+5}{2} = \frac{z-10}{4} = t \qquad -L_2$$

Any point on L_1 has co-ordinates $(6 + 3s, 7 - s, 4 + s)$

Any point on L_2 has co-ordinates $(-6 - 3t, -5 + 2t, 10 + 4t)$



$$\overrightarrow{PQ} = \begin{pmatrix} -12-3t-3s \\ -12+2t+s \\ 6+4t-s \end{pmatrix}$$

$\overrightarrow{PQ} \perp L_1 \Leftrightarrow \overrightarrow{PQ}\,.\,(3, -1, 1) = 0$

$\Leftrightarrow -36 - 9t - 9s + 12 - 2t - s + 6 + 4t - s = 0$

$\Leftrightarrow -7t - 11s = 18$ .... (1)

$\overrightarrow{PQ} \perp L_2 \Leftrightarrow \overrightarrow{PQ}\,.\,(-3, 2, 4) = 0$

$\Leftrightarrow 36 + 9t + 9s - 24 + 4t + 2s + 24 + 16t - 4s = 0$

$\Leftrightarrow 29t + 7s = -36$ .... (2)

Hence $\left.\begin{aligned} -7t - 11s &= 18 \\ 29t + 7s &= -36 \end{aligned}\right\} \Rightarrow t = -1, s = -1$

and as P is point (3, 8, 3) and Q is the point (-3, -7, 6)

$\overrightarrow{PQ} = \begin{pmatrix} -6 \\ -15 \\ 3 \end{pmatrix}$ and $\left|\overrightarrow{PQ}\right| = \sqrt{270} = 3\sqrt{30}$

Hence equations of PQ are $\dfrac{x-3}{2} = \dfrac{y-8}{5} = \dfrac{z-3}{-1}$

Examples for class

(1) Find the shortest distance between the following lines and the equation of this line of shortest distance:

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}\ ;\ \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$$

(2) Find the length of and the equations, in symmetric form, for the shortest distance between the lines $\frac{x-3}{0}=\frac{y}{-2}=\frac{z-5}{1}\ ;\ \frac{x+2}{3}=\frac{y+1}{2}=\frac{z+1}{-2}$

and the coordinates of the points at which it meets the lines.

(3) Find the feet of the common perpendicular of the two skew lines

$$x=\frac{y+1}{2}=z\ ;\ \frac{x}{2}=\frac{y-2}{-1}=\frac{z+2}{2}$$

(4) Find the equations of the common perpendicular of the skew lines

$$\frac{x+5}{1}=-y=\frac{z}{2}\ ;\ \frac{x+2}{1}=\frac{y}{2}=\frac{z-6}{-1}$$

and show that the shortest distance between them is $\sqrt{3}$.

(5) A line L is the intersection of the two planes $x+y+z=3$, $x-2y+3z=2$.
Another line M is given by the equations $x-1=y-2=z-3$.
Find the direction ratios of the line of intersection of the planes joining L and M to the origin.

(6) Four points in space have co-ordinates A(1, 1, 0), B(3, 0, 1), C(1, 0, 2), D(1, 1, 3).
Find the equations of two parallel planes, of which one contains A and B and the other contains C and D. Deduce the shortest distance between the lines AB, CD.

(7) Find the equation of the plane through the intersection of the planes $2x-y-z=2$, $3x-2z=5$, which is parallel to the line $x=\frac{y}{2}=\frac{z}{3}$. Hence deduce the shortest distance between the line and the line of intersection of the first two planes.

(8) Find the equations of a line with direction vector (1, 2, -3) which meets both the lines $\frac{x-2}{5}=\frac{y+1}{3}=\frac{z-3}{2}\ ;\ x=y=z$

(9) Show that there are two planes which pass through the line $\frac{x-5}{1}=\frac{y-1}{-1}=\frac{z+3}{3}$

and makes an angle of 60° with the plane $y=z$.
Find their equations.

(10) Find the equation of a line through the origin which meets both the lines with equations:

$$x=\frac{y-1}{3}=z\ ;\ x+1=y-2=\frac{z+4}{2}$$

Answers

(1) $2\sqrt{29}$; $\dfrac{x-3}{2} = \dfrac{y-5}{3} = \dfrac{z-7}{4}$

(2) 7; (3, 4, 3), (1, 1, -3); $\dfrac{x-3}{2} = \dfrac{y-4}{3} = \dfrac{z-3}{6}$

(3) (1, 1, 1) & (2, 1, 0)

(4) $\dfrac{x+3}{-1} = \dfrac{y+2}{1} = \dfrac{z-7}{1}$

(5) (1, 1, 1)

(6) $x + y - z = 2, x + y - z = -1, \sqrt{3}$

(7) $x + y - z = 3$; $\sqrt{3}$

(8) $x + 2 = \dfrac{y+2}{2} = \dfrac{z+2}{-3}$

(9) $x + y = 6$ and $x + 4y + z = 6$

(10) $\dfrac{x}{4} = \dfrac{y}{13} = \dfrac{z}{2}$

MATHEMATICS 3 (AH)

CONTENT	
Further sequences and series	
3.3.1	know the term power series
3.3.2	understand and use the Maclaurin series: $f(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(0)$
3.3.3	find the Maclaurin series of simple functions: e^x, $\sin x$, $\cos x$, $\tan^{-1}x$, $(1+x)^{\alpha}$, $\ln(1+x)$, knowing their range of validity
3.3.4	find the Maclaurin expansions for simple composites, such as e^{2x}
3.3.5	use the Maclaurin series expansion to find power series for simple functions to a stated number of terms

The content listed below is **not** covered within this support material:

3.3.6 use iterative schemes of the form $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \ldots$ to solve equations where $x = g(x)$ is a rearrangement of the original equation

3.3.7 use graphical techniques to locate approximate solution x_0

3.3.8 know the condition for convergence of the sequence $\{x_n\}$ given by $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \ldots.$ and the meaning of the terms first and second order of convergence

MATHEMATICS 3 (AH): FURTHER SEQUENCES AND SERIES

Polynomial approximations to functions/power series

Introduction

Any series of the form $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots\dots\dots$ is called a **power series**.

In many cases the sum of such a series gets larger and larger as more terms are added on. In such a case the series is said to **diverge**.

On the other hand, some series are such that, as more terms are added on, the sum approaches more and more closely a particular limit. In this case the series is said to **converge** to this limit.

Maclaurin's theorem

Consider $f(x) = ax^3 + bx^2 + cx + d \qquad f(0) = d$

$f'(x) = 3ax^2 + 2bx + c \qquad f'(0) = c$

$f''(x) = 6ax + 2b \qquad f''(0) = 2b \Rightarrow b = \frac{1}{2} f''(0) \Rightarrow b = \frac{f''(0)}{2!}$

$f'''(x) = 6a \qquad f'''(0) = 6a \Rightarrow a = \frac{1}{6} f'''(0) \Rightarrow a = \frac{f'''(0)}{3!}$

$f''''(x) = 0 \qquad$ etc.

Hence $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \dots$

This is a particular example of a result known as Maclaurin's theorem.

Maclaurin's theorem

Under certain circumstances, a function $f(x)$ may be expanded as follows:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots\dots$$

Such functions are $\ln(1 + x)$, e^x, $(1 + x)^\alpha$, $\sin x$, $\cos x$, $\tan^{-1} x$.

Example 1

Let $f(x) = \ln(1 + x) \qquad f(0) = \ln 1 = 0$

$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \qquad f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3} \qquad f'''(0) = 2$

$$f''''(x) = \frac{-6}{(1+x)^4} \qquad f''''(0) = -6$$

Substituting in Maclaurin's expansion: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$

Examples for class

Find the Maclaurin's expansion for each of the following

(1) e^x (2) $\sin x$ (3) $\cos x$ (4) $(1+x)^\alpha$ (5) $\tan^{-1} x$

MacLaurin's expansions of elementary functions

Listed below are the expansions of the above six functions together with their **domain of validity**.

The **domain of validity** of a series is that interval in which the series converges and should always be considered with the series.

(1) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots \qquad (-1 < x \leq 1)$

(2) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots \qquad \forall x \in \boldsymbol{R}$

(3) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots \qquad \forall x \in \boldsymbol{R}$

(4) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots \qquad \forall x \in \boldsymbol{R}$

(5) $(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots\ldots\ldots \qquad (-1 < x < 1)$

(6) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots\ldots \qquad (-1 \leq x \leq 1)$

From these elementary expansions we can form expansions for other functions.

Example 1

$\ln(1-x)$:

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots..$ $\qquad (-1 < x \le 1)$

replace x by $-x$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots..$ $\qquad (-1 < -x \le 1)$ i.e. $(-1 \le x < 1)$

Example 2

$\ln\left(\frac{1+x}{1-x}\right)$:

$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$ $\qquad (-1 < x < 1)$

$= (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots..) - (-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots..)$

$= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \ldots\ldots..$

$= 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots.)$ $\qquad (-1 < x < 1)$

Example 3

Expand in powers of x, up to x^3, $e^{3x} - 2e^x + e^{-x}$

$e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{6} + \ldots\ldots..$ $\qquad \forall x \in \boldsymbol{R}$

$2e^x = 2 + 2x + x^2 + \frac{1}{3}x^3 + \ldots\ldots..$ $\qquad \forall x \in \boldsymbol{R}$

$e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\ldots..$ $\qquad \forall x \in \boldsymbol{R}$

Hence $e^{3x} - 2e^x + e^{-x} = 4x^2 + 4x^3 + \ldots\ldots..$ $\qquad \forall x \in \boldsymbol{R}$

Example 4

Show that $\ln\left(\frac{1+e^x}{2}\right) = \frac{1}{2}x + \frac{1}{8}x^2 + \ldots.$

$\ln\left(\frac{1+e^x}{2}\right) = \ln(1 + e^x) - \ln 2$

$= \ln(1 + 1 + x + \frac{x^2}{2} + \ldots.) - \ln 2$

$= \ln(2 + x + \frac{x^2}{2} + \ldots.) - \ln 2$

$= \ln 2(1 + \frac{x}{2} + \frac{x^2}{4} + \ldots.) - \ln 2$

$= \ln 2 + \ln(1 + (\frac{x}{2} + \frac{x^2}{4})) - \ln 2$

$= \ln(1 + (\frac{x}{2} + \frac{x^2}{4}))$

$= (\frac{x}{2} + \frac{x^2}{4}) - \frac{\left(\frac{x}{2}+\frac{x^2}{4}\right)^2}{2} + \ldots.$

$= \frac{x}{2} + \frac{x^2}{4} - \frac{x^2}{8} + \ldots\ldots..$

$= \frac{x}{2} + \frac{x^2}{8} + \ldots\ldots\ldots.$

Example 5

Find the coefficient of x^5 in the expansion of $\frac{1-3x+x^2}{e^x}$.

$\frac{1-3x+x^2}{e^x} = (1 - 3x + x^2)\,e^{-x} = (1 - 3x + x^2)(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \ldots.)$

$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} - 3x + 3x^2 - \frac{3x^3}{2} + \frac{x^4}{2} - \frac{x^5}{8} + x^2 - x^3 + \frac{x^4}{2} - \frac{x^5}{6} + \ldots\ldots$

$= 1 - 4x + \frac{9x^2}{2} - \frac{8x^3}{3} + \frac{25x^4}{24} - \frac{3x^5}{10} + \ldots\ldots..$

Coefficient of $x^5 = -\frac{3}{10}$.

Example 6

Show that $(1 + x)^x = 1 + x^2 - \frac{1}{2}x^3 + \frac{5}{6}x^4 + \ldots\ldots$ for $-1 < x \leq 1$.

We use the fact that $f(x) = e^{\ln f(x)} = \ln e^{f(x)}$

$$(1 + x)^x = e^{\ln(1+x)^x} = e^{x\ln(1+x)} = e^{x(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots)}$$

$$= e^{\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+\ldots\right)} \qquad -1 < x \leq 1$$

$$= 1 + (x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \ldots) + \frac{1}{2}(x^2 - \frac{x^3}{2})^2 + \frac{1}{6}(x^2)^3 + \ldots$$

$$= 1 + x^2 - \frac{x^3}{2} + \frac{5}{6}x^4 + \ldots \qquad \text{for } -1 < x \leq 1$$

Example 7

Expand $\ln(1 + \sin x)$

$$\ln(1 + \sin x) = \ln(1 + (x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots))$$

$$= (x - \frac{x^3}{6} + \frac{x^5}{120}) - \frac{1}{2}(x - \frac{x^3}{6})^2 + \frac{1}{3}(x - \frac{x^3}{6})^3 + \ldots$$

$$= x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \ldots\ldots$$

Example 8

Expand $\frac{1}{1+x}$ and $\frac{1}{1-x}$

$$\frac{1}{1+x} = (1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots \qquad -1 < x < 1$$

$$\frac{1}{1-x} = (1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots \qquad -1 < x < 1$$

Examples for class

(1) Find the term in x^4 in the expansion of $e^x(1 + 2x)$.

(2) Find the coefficient of x^6 in the expansion of $\dfrac{3-4x-x^2}{e^x}$.

(3) Find the term in x^5 in the expansion of $e^x(1 - 3x)$.

(4) Find the coefficient of x^4 in the expansion of $(e^x - 1)(x^2 - 2x - 1)$.

(5) Write down the power series for $\ln(1 - x)$.
By writing $1 + x + x^2$ in the form $\dfrac{1-x^3}{1-x}$, show that for $-1 \le x \le 1$,
$\ln(1 + x + x^2) = x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{3}x^6 + \ldots\ldots$

(6) Expand $\ln(1 + \sin x)$ as far as the term in x^4 .

(7) a) If $f(x) = \cos^2 x$, show that $f'(x) = -\sin 2x$. By finding higher derivatives obtain the Maclaurin series for $\cos^2 x$ up to the term x^6.
b) By writing $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ and replacing $\cos 2x$ by a suitable power series obtain the expansion of $\cos^2 x$ up to the term x^6.

(8) Expand $e^{\sin x}$ as far as the term in x^3 .

(9) Expand e^{2x+3} as far as the term in x^4 .

(10) Express $(1 + x)^{1+x}$ in the form $e^{f(x)}$ and deduce the result
$(1 + x)^{1+x} = 1 + x + x^2 + \frac{1}{2}x^3 + \ldots.$

(11) Expand $\dfrac{x}{e^x - 1}$ as far as the term in x^4 .

(12) Show that $\ln(1 + e^x) = \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \ldots.$

(13) Find the power series for $\ln(1 - x - 2x^2)$ as far as x^5. Give the domain of validity.

(14) By expanding $(1 + x)^{-1}$ and integrating the resulting series, show that
$\ln(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \ldots.$

Answers

(1) $\frac{9x^4}{24}$ (2) $\frac{19x^6}{240}$ (3) $-\frac{7x^5}{60}$ (4) $\frac{x^4}{8}$

(5) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$

(6) $x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{6}$

(7) (a) $\cos^2 x = 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45}$

(b) $\cos^2 x = \frac{1}{2}(1+\cos 2x)\ \ 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45}$

(8) $1 + x + \frac{1}{12}x^2 - \frac{5}{36}x^3$

(9) $38\frac{7}{8} + 55x + 27x^2 + \frac{16}{3}x^3 + \frac{2}{3}x^4$

(10) Proof

(11) $\frac{120}{120 + 60x + 20x^2 + 5x^3 + x^4}$

(12) Proof

(13) $-1 < x < -0.375$; $-x - \frac{5}{2}x^2 - \frac{7}{3}x^3 - 4x^4 - 4x^5$

(14) Proof

MATHEMATICS 3 (AH)

CONTENT
Further ordinary differential equations
3.4.1 solve first order linear differential equations using the integrating factor method
3.4.3 know the meaning of the terms: second order linear differential equation with constant coefficients, homogeneous, non-homogeneous, auxiliary equation, complementary function and particular integral
3.4.4 solve second order homogeneous ordinary differential equations with constant coefficients $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$
3.4.5 find the general solution in the three cases where the roots of the auxiliary equation: (i) are real and distinct (ii) **coincide (are equal) [A/B]** (iii) **are complex conjugates [A/B]**
3.4.6 solve initial value problems
3.4.7 **solve second order non-homogeneous ordinary differential equations with constant coefficients** $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ **using the auxiliary equation and particular integral method [A/B]**

The content listed below is **not** covered within this support material:

3.4.2 find general solutions and solve initial value problems

MATHEMATICS 3 (AH): FURTHER ORDINARY DIFFERENTIAL EQUATIONS

Linear

In this case, the equation takes the form $\frac{dy}{dx}$ + P(x)y = Q(x), where P(x) and Q(x) are expressions in x.

To solve this type of equation, we have to multiply all terms by a factor called the **integrating factor**, detailed by $\mu(x)$, which converts the whole of the left-hand side into the derivative of one function.

Hence $\frac{dy}{dx} + P(x)y = Q(x)$,

$\Leftrightarrow \mu(x)\frac{dy}{dx} + P(x)\ \mu(x)y = \mu(x)\ Q(x)$

We choose $\mu(x)$ such that $\frac{d}{dx}(\mu(x)) = P(x)\ \mu(x)$

i.e. $\frac{1}{\mu(x)}\frac{d}{dx}(\mu(x)) = P(x)$

i.e. $\frac{d}{dx}(\ln(\mu(x))) = P(x)$

i.e. $\ln(\mu(x)) = \int P(x)\,dx + C$ where C is a constant.

i.e. $\mu(x) = e^{\int P(x)\,dx + C} = e^{\int P(x)\,dx}\,.\,e^{C} = A\,e^{\int P(x)\,dx}$, where $A = e^{C}$

We choose A = 1 and obtain as integrating factor $\mu(x) = e^{\int P(x)\,dx}$.

Hence equation becomes $\mu(x)\frac{dy}{dx} + \frac{d}{dx}[\mu(x)]\,y = \mu(x)\ Q(x)$

i.e. $\frac{d}{dx}[\mu(x)\,y] = \mu(x)\ Q(x)$

i.e. $\mu(x)\,y = \int \mu(x)\ Q(x)\,dx + C$ where C is a constant

i.e. $y = e^{-\int P(x)\,dx}\int e^{\int P(x)\,dx}\ Q(x)\,dx + C.$

Example 1

Find the general solution of $\frac{dy}{dx} + \frac{y}{x} = 1$

Here $P = \frac{1}{x} \Rightarrow \mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

Multiply both sides of the equation by x.

$x\dfrac{dy}{dx} + y = x \quad \Leftrightarrow \dfrac{d}{dx}(xy) = x$

$\Leftrightarrow xy = \dfrac{x^2}{2} + C$, where C is a constant

$\Leftrightarrow y = \dfrac{x}{2} + \dfrac{C}{x}$ - General Solution

Example 2

Find the general solution of $x\dfrac{dy}{dx} + (x-2)\,y = x^3$

$x\dfrac{dy}{dx} + (x-2)\,y = x^3 \Leftrightarrow \dfrac{dy}{dx} + \left(\dfrac{x-2}{x}\right) y = x^2$

Hence P = $\dfrac{x-2}{x} = 1 - \dfrac{2}{x} \Rightarrow \mu(x) = e^{\int\left(1-\frac{2}{x}\right)dx} = e^{x-2\ln x} = e^{x-\ln x^2}$

$= e^{x+\ln\frac{1}{x^2}} = e^x . e^{\ln\frac{1}{x^2}} = \dfrac{1}{x^2}\, e^x$

Multiply both sides of the equation by $\dfrac{1}{x^2}\, e^x$

$\dfrac{1}{x^2}\, e^x \dfrac{dy}{dx} + e^x \dfrac{x-2}{x^3}\, y = e^x \quad \Leftrightarrow \dfrac{d}{dx}\left[\dfrac{1}{x^2} e^x\, y\right] = e^x$

$\Leftrightarrow \dfrac{1}{x^2}\, e^x y = e^x + C$, where C is a constant

$\Leftrightarrow y = x^2 + C\,x^2\, e^{-x}$ - General Solution

Example 3

Find the general solution of $\dfrac{dy}{dx} - 2y = 6\, e^{-x}$

Hence P = - 2 $\Rightarrow \mu(x) = e^{\int -2\,dx} = e^{-2x}$

Multiply both sides of the equation by e^{-2x}

$e^{-2x}\dfrac{dy}{dx} - 2\, e^{-2x} y = 6\, e^{-3x}$

$\Leftrightarrow \dfrac{d}{dx}[\, e^{-2x} y] = 6\, e^{-3x} + C$, where C is a constant

$\Leftrightarrow e^{-2x} y = -2\, e^{-3x} + C$

$\Leftrightarrow y = -2\, e^{-x} + C\, e^{2x}$

Examples for class

(1) $(x + 1)\,\dfrac{dy}{dx} - y = (x + 1)^2$

(2) $\dfrac{dy}{dx} - y \tan x = \sin x \cos x$

(3) $\tan x\,\dfrac{dy}{dx} + 2y = x \operatorname{cosec} x$

(4) $\dfrac{dy}{dx} + \dfrac{2y}{1-x^2} = 1 - x$

(5) $(1 + x^2)\,\dfrac{dy}{dx} + xy = 1$

(6) $\dfrac{dy}{dx} + \dfrac{x+1}{x}\,y = e^{-x}$

(7) $(1 - x)\,\dfrac{dy}{dx} + xy = (1 - x)^2 e^{-x}$

(8) $\dfrac{dy}{dx} + y = 5 \cos 2x$

Answers

(1) $y = x^2 + x + Cx + C$ (2) $y = -\dfrac{1}{3}\cos^2 x + \dfrac{C}{\cos x}$

(3) $y = \dfrac{x}{\sin x} + \dfrac{\cos x}{\sin^2 x} + \dfrac{C}{\sin^2 x}$ (4) $y = \left(\dfrac{1-x}{1+x}\right)(x + \dfrac{1}{2}x^2 + C)$

(5) $y = \dfrac{x+C}{1+x^2}$ (6) $y = \dfrac{1}{2}xe^{-x} + Cxe^{-x}$

(7) $y = -\dfrac{1}{2}(1-x)e^{-x} + \dfrac{Ce^{-x}}{1-x}$ (8) $y = \cos 2x + 2\sin 2x + C$

Further differential equations of first order

Example 1

Find by using the substitution $z = y^2$, the general solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{y}$

$z = y^2$

$\Rightarrow \frac{dz}{dx} = \frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}\frac{dz}{dx}$

Substituting equation becomes

$\frac{1}{2y}\frac{dz}{dx} + \frac{y}{x} = \frac{1}{y}$

$\Leftrightarrow \frac{dz}{dx} + \frac{2y^2}{x} = 2$

$\Leftrightarrow \frac{dz}{dx} + \frac{2}{x}z = 2$

Here P = $\frac{2}{x} \Rightarrow \mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

Multiplying by x^2, equation becomes

$x^2\frac{dz}{dx} + 2xz = 2x^2$

$\Leftrightarrow \frac{d}{dx}(x^2 z) = 2x^2$

$\Leftrightarrow x^2 z = \frac{2}{3}x^3 + C$, where C is a constant.

$\Leftrightarrow z = \frac{2}{3}x + \frac{C}{x^2}$

$\Leftrightarrow y^2 = \frac{2}{3}x + \frac{C}{x^2}$ - General Solution

Examples for class

(1) $x\frac{dy}{dx} + y = y^2 \ln x$ (put $z = \frac{1}{y}$) and the particular solution of which $y = 1$ when $x = 2$.

(2) $x\frac{dy}{dx} + y = x^6 y^4$ (put $v = y^{-3}$)

(3) $\frac{dy}{dx} - \frac{6x}{x^2+1}y = 2x^3\sqrt{y}$ (put $z = \sqrt{y}$) and the particular solution for which $y = 4$ when $x = 0$.

Answers

(1) $\frac{1}{y} = x^2 \ln x - x^2 + Cx$, $\frac{1}{y} = x^2 \ln x - x^2 + \frac{5}{2}x - 2x \ln 2$

(2) $\frac{1}{y^3} = -x^6 + Cx^3$

(3) $\sqrt{y} = 2(x^2+1)^2 + 2(x^2+1) + C(x^2+1)^{\frac{1}{2}}$, $\sqrt{y} = 2(x^2+1)^2 + 2(x^2+1) - 2(x^2+1)^{\frac{1}{2}}$

Differential equations of the second order

Homogeneous linear equations

A homogeneous linear equation of order 2 is one of the form

$\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}\,y = 0$, where P , Q are real numbers.

Suppose $y = u$ and $y = v$ are particular solutions.

Then $\frac{d^2u}{dx^2} + \mathrm{P}\frac{du}{dx} + \mathrm{Q}\,u = 0$ and $\frac{d^2v}{dx^2} + \mathrm{P}\frac{dv}{dx} + \mathrm{Q}\,v = 0$

It can easily be shown that $y = \mathrm{A}u + \mathrm{B}v$, where A, B are constants, is also a solution.

We now look for solutions of the form $y = e^{\mathrm{D}x}$ then $\frac{dy}{dx} = \mathrm{D}e^{\mathrm{D}x}$, $\frac{d^2y}{dx^2} = \mathrm{D}^2 e^{\mathrm{D}x}$

and so $\mathrm{D}^2 e^{\mathrm{D}x} + \mathrm{P}\,\mathrm{D}\,e^{\mathrm{D}x} + \mathrm{Q}\,e^{\mathrm{D}x} = 0$

i.e. $e^{\mathrm{D}x}(\mathrm{D}^2 + \mathrm{PD} + \mathrm{Q}) = 0$

i.e. $\mathrm{D}^2 + \mathrm{PD} + \mathrm{Q} = 0$ - (1)

$\therefore$ If α_1 and α_2 are solutions of the equation $\mathrm{D}^2 + \mathrm{PD} + \mathrm{Q} = 0$ then $e^{\alpha_1 x}$ and $e^{\alpha_2 x}$ are solutions of the differential equation and $y = \mathrm{A}e^{\alpha_1 x} + \mathrm{B}e^{\alpha_2 x}$, where A and B are constants.

Equation (1) formed from the differential equation by replacing the derivatives by Ds of the appropriate orders is called the **Auxiliary Equation**.

Using the formula for roots of the quadratic equation.

$\mathrm{D} = \frac{-P \pm \sqrt{P^2 - 4Q}}{2}$ and we obtain the following three cases:

I. $P^2 > 4Q$, the roots are real and unequal .
The general solution is $y = A e^{D_1 x} + B e^{D_2 x}$

II $P^2 = 4Q$ and the roots are equal $D_1 = D_2 = D$, say
The general solution is $y = e^{Dx} (A + Bx)$

III $P^2 < 4Q$, the roots are imaginary and unequal
i.e. $D = \frac{P}{2} \pm i\, \frac{B}{2}$, where $B = \sqrt{4Q - P^2}$
then the general solution is $y = e^{\frac{P}{2}x} \{A \cos \frac{B}{2} x + B \sin \frac{B}{2} x\}$

Example 1

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

Auxiliary equation is $D^2 + D - 6 = 0$
$\Leftrightarrow (D + 3)(D - 2) = 0$
$\Leftrightarrow D = -3$ or 2

$\Leftrightarrow \therefore$ General solution is $y = c_1 e^{-3x} + c_2 e^{2x}$, where c_1 and c_2 are constants.

Example 2

Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

Auxiliary equation is $D^2 + 4D + 4 = 0$
$\Leftrightarrow D = -2$, twice
$\therefore$ General solution is $y = e^{-2x} (c_1 + c_2 x)$ where c_1 and c_2 are constants.

Example 3

Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

Auxiliary equation is $D^2 + 2D + 5 = 0$

$\Leftrightarrow D = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

$\therefore$ General solution is $y = e^{-x} \{A \cos 2x + B \sin 2x\}$, where A, B are constants.

Examples for class

Type 1

(1) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ (2) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$ (3) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

(4) $2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 9y = 0$ (5) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 0$

Type 2

(1) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ (2) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$ (3) $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$

(4) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ (5) $\frac{d^2y}{dx^2} + 2n\frac{dy}{dx} + n^2y = 0$

Type 3

(1) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ (2) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 0$ (3) $8\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$

(4) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$ (5) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Answers

Type 1:

(1) $y = Ae^{2x} + Be^{x}$ (2) $y = Ae^{3x} + Be^{x}$ (3) $y = Ae^{-3x} + Be^{-2x}$

(4) $y = Ae^{3x} + Be^{\frac{3}{2}x}$ (5) $y = Ae^{3x} + B$

Type 2:

(1) $y = e^{2x}(A + Bx)$ (2) $y = e^{3x}(A + Bx)$ (3) $y = e^{-3x}(A + Bx)$

(4) $y = e^{-nx}(A + Bx)$

Type 3:

(1) $y = e^{x}(A\cos x + B\sin x)$ (2) $y = e^{2x}(A\cos 2x + B\sin 2x)$

(3) $y = e^{\frac{1}{4}x}(A\cos\frac{1}{4}x + B\sin\frac{1}{4}x)$ (4) $y = e^{3x}(A\cos 4x + B\sin 4x)$

Non-homogeneous equations of the second order

These equations are of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ - (1)

In most cases $f(x)$ will be a simple polynomial, or an exponential, or a sine or cosine or some combination of these.

We shall make use of the following theorem.

Theorem
If $y = f_1(x)$ is a particular solution of (1) and $y = G(x)$ is the general solution of the corresponding homogeneous equation, then $y = G(x) + f_1(x)$ is the general solution of (1).

Proof

$y = f_1(x)$ is a particular solution of (1) $\Leftrightarrow a\frac{d^2}{dx^2}f_1(x) + b\frac{d}{dx}f_1(x) + cf_1(x) = f(x)$

$y = G(x)$ is general solution of homogeneous

$\Leftrightarrow a\frac{d^2}{dx^2}G(x) + b\frac{d}{dx}G(x) + c\,G(x) = 0$

For $y = G(x) + f_1(x)$

$$\frac{dy}{dx} = \frac{d}{dx}G(x) + \frac{d}{dx}f_1(x)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{d}{dx}G(x) + \frac{d}{dx}f_1(x)\right) = \frac{d^2}{dx^2}G(x) + \frac{d^2}{dx^2}f_1(x)$$

$$\therefore\ a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy$$

$$= a\frac{d^2}{dx^2}G(x) + a\frac{d^2}{dx^2}f_1(x) + b\frac{d}{dx}G(x) + b\frac{d}{dx}f_1(x) + cG(x) + cf_1(x) = f(x)$$

Hence result.

We call G(x), the general solution of the corresponding homogeneous equation, the **complementary function** (C.F.) and $f_1(x)$, the particular solution of the non-homogeneous equation, the **particular integral** (P.I.)

Then y = C.F. + P.I. is the general solution of the equation.

Example 1

Find the general solution of $\frac{d^2y}{dx^2} + 4y = -4x^2 + 2$

Consider homogeneous equation first.
Auxiliary equation is $D^2 + 4 = 0 \Leftrightarrow D = \pm 2\,\boldsymbol{i} = \mathbf{0} \pm 2\,\boldsymbol{i}$

$\therefore$ C.F. is $y = e^{0.x}$ (A cos $2x$ + B sin $2x$) = A cos $2x$ + B sin $2x$, where A, B are constants.

For particular integral, we try a function of the same form as R.H.S. of equation.
Here we try $y = c_1x^2 + c_2x + c_3$ as solution of equation.

Then $y' = 2c_1x + c_2$ and $y'' = 2c_1$

Substitute in equation, $2c_1 + 4c_1x^2 + 4c_2x + 4c_3 = -4x^2 + 2$

$\Leftrightarrow 4c_1x^2 + 4c_2x + (4c_3 + 2c_1) = -4x^2 + 2$

$\Leftrightarrow 4c_1 = -4,\ 4c_2 = 0,\ 4c_3 + 2c_1 = 2$

$\Leftrightarrow c_1 = -1,\ c_2 = 0,\ c_3 = 1$ P.I. is $y = -x^2 + 1$

Hence general solution is y = A cos $2x$ + B sin $2x - x^2 + 1$

Example 2

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 2e^x$

Auxiliary equation is $D^2 + D - 6 = 0$

$\Leftrightarrow (D + 3)(D - 2) = 0 \Leftrightarrow D = -3$ or 2

$\therefore$ C.F. is $y = c_1e^{-3x} + c_2e^{2x}$, where c_1, c_2 are constants.

For P.I. try $y = Ae^x$, then

$y' = Ae^x$ and $y'' = Ae^x$

Substitute in equation, $Ae^x + Ae^x - 6Ae^x = 2e^x$

$\Leftrightarrow -4Ae^x = 2e^x$

$\Leftrightarrow -4A = 2$

$\Leftrightarrow A = -\frac{1}{2}$

$\therefore$ P.I. is $y = -\frac{1}{2}e^x$

General solution is $y = c_1e^{-3x} + c_2e^{2x} - \frac{1}{2}e^x$

Example 3

Solve $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 10\cos x$

Auxiliary equation is $D^2 + 2D + 5 = 0$

$\Leftrightarrow D = \dfrac{-2 \pm \sqrt{4-20}}{2} = \dfrac{-2 \pm \sqrt{-16}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$

$\therefore$ C.F. is $y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$

For Particular integral try y = A cos x + B sin x , then

$y' = -A\sin x + B\cos x$ and $y'' = -A\cos x - B\sin x$

Substituting in equation gives

$-A\cos x - B\sin x - 2A\sin x + 2B\cos x + 5A\cos x + 5B\sin x = 10\cos x$

$\Leftrightarrow \cos x(-A + 2B + 5A) + \sin x(-B - 2A + 5B) = 10\cos x$

$\Leftrightarrow \cos x(4A + 2B) + \sin x(4B - 2A) = 10\cos x$

$\Leftrightarrow \begin{matrix} 4A + 2B = 10 & \times 1 \\ -2A + 4B = 0 & \times 2 \end{matrix} \Leftrightarrow \begin{matrix} 4A + 2B = 10 \\ -4A + 8B = 0 \end{matrix} \Rightarrow 10B = 10$, i.e. B = 1 and A = 2

$\therefore$ P.I. is $y = 2\cos x + \sin x$

General solution is $y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + 2\cos x + \sin x$

Example 4

Find the general solution of $\dfrac{d^2y}{dx^2} + 4y = e^x - 2\sin x + x$

Auxiliary equation is $D^2 + 4 = 0$

$\Leftrightarrow D = \pm 2i$

$\therefore$ C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$

For P.I. try $y = Ae^x + B\cos x + C\sin x + Dx + E$, then

$y' = Ae^x - B\sin x + C\cos x + D$

and $y'' = Ae^x - B\cos x - C\sin x$

Substituting in equation gives

$Ae^x - B\cos x - C\sin x + 4Ae^x + 4B\cos x + 4C\sin x + 4Dx + 4E = e^x - 2\sin x + x$

$\Leftrightarrow 5Ae^x + \cos x(-B + 4B) + \sin x(-C + 4C) + 4Dx + 4E = e^x - 2\sin x + x$

$\Leftrightarrow 5A = 1$, $3B = 0$, $3C = -2$, $4D = 1$, $4E = 0$

$\Leftrightarrow$ A = $\frac{1}{5}$, B = 0 , C = - $\frac{2}{3}$, D = $\frac{1}{4}$, E = 0

$\therefore$ P.I. is y = $\frac{1}{5}e^x - \frac{2}{3}\sin x + \frac{1}{4}x$

General solution is $y = c_1\cos 2x + c_2\sin 2x + \frac{1}{5}e^x - \frac{2}{3}\sin x + \frac{1}{4}x$

Examples for class

Find the general solution of the following differential equations

Type 1

(1) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = x^3$

(2) $\frac{d^2y}{dx^2} - y = 2 - 5x$

(3) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4(x+1)$

(4) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 32x^2$

(5) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 1 + x^2$

Type 2

(6) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 10e^{2x}$

(7) $4\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 9y = 7e^{-2x}$

(8) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x$

(9) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2e^{-2x}$

Type 3

(10) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin x$

(11) $8\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = \sin x - 2\cos x$

(12) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 10\cos 2x$

(13) $4\frac{d^2y}{dx^2} + y = 4\sin x$

(14) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 26\cos 3x$

Type 4

(15) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = x - e^{2x}$

(16) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^x + e^{-x}$

(17) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = x - e^x$

(18) $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = e^x - 2\cos 2x$

(19) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2 + e^{-x}$

(20) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^x + \sin x$

Answers

(1) $y = Ae^{3x} + Be^{x} + \frac{1}{3}x^3 + \frac{4}{3}x^2 + \frac{26}{9}x + \frac{80}{27}$

(2) $y = Ae^{-x} + Be^{x} + 5x - 2$

(3) $y = Ae^{-2x} + Be^{-x} + 2x - 2$

(4) $y = Ae^{-4x} + Be^{-x} + 8x^2 - 20x + 21$

(5) $y = e^{-x}(A\cos x + B\sin x) + \frac{1}{2}x^2 - x + 1$

(6) $y = Ae^{-3x} + Be^{x} + 2e^{2x}$

(7) $y = e^{-\frac{3}{2}x}(A + Bx) + 7e^{-2x}$

(8) $y = Ae^{2x} + Be^{-x} + \frac{1}{2}e^{x}$

(9) $y = e^{2x}(A + Bx) + \frac{1}{8}e^{2x}$

(10) $y = Ae^{2x} + Be^{2x} + \frac{3}{10}\cos x + \frac{1}{10}\sin x$

(11) $y = e^{-x}(A\cos 2x + B\sin 2x) + \frac{2}{13}\cos x - \frac{3}{13}\sin x$

(12) $y = e^{x}(A + Bx) - \frac{6}{5}\cos 2x - \frac{8}{5}\sin 2x$

(13) $y = A\cos\frac{1}{4}x + B\sin\frac{1}{4}x - \frac{4}{3}\sin x$

(14) $y = e^{4x}(A\cos 3x + B\sin 3x) + \frac{1}{2}\cos 3x + \frac{3}{4}\sin 3x$

(15) $y = e^{-2x}(A + Bx) - \frac{1}{16}e^{2x} + \frac{1}{4}x$

(16) $y = Ae^{-3x} + Be^{2x} - \frac{1}{4}e^{x} - \frac{1}{6}e^{-x}$

(17) $y = e^{2x}(A\cos 2x + B\sin 2x) + \frac{1}{8}x - \frac{1}{13}e^{x} - \frac{1}{16}$

(18) $y = e^{-\frac{1}{2}x}(A + Bx) + \frac{1}{9}e^{x} + \frac{30}{289}\cos 2x - \frac{16}{289}\sin 2x$

(19) $y = Ae^{2x} + Be^{x} + \frac{1}{6}e^{-x} + 1$

(20) $y = e^{3x}(A + Bx) + \frac{1}{4}e^{x} + \frac{3}{50}\cos x + \frac{2}{25}\sin x$

If terms in particular integral already appear in C.F. we throw in an x.

Example 5

Find the general solution of $\frac{d^2y}{dx^2} - y = 2e^x$

Auxiliary equation is $D^2 - 1 = 0 \Leftrightarrow (D + 1)(D - 1) = 0 \Leftrightarrow D = \pm 1$

$\therefore$ C.F. is $y = c_1e^x + c_2e^{-x}$, where c_1, c_2 are constants.

For P.I. try $y = Axe^x$, then
$y' = Ae^x + Axe^x$ and $y'' = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x$

Substituting, equation becomes
$2Ae^x + Axe^x - Axe^x = 2e^x$
$\Leftrightarrow 2Ae^x = 2e^x$
$\Leftrightarrow A = 1$

$\therefore$ P.I. is $y = xe^x$

Hence general solution is $y = c_1e^x + c_2e^{-x} + xe^x$

Example 6

Find the general solution of the equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 5x + \cos 2x$.

Auxiliary equation is $D^2 + 2D + 5 = 0 \Leftrightarrow D = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

For P.I. try $y = (A\cos 2x + B\sin 2x) + Cx + D$, then
$y' = -2A\sin 2x + 2B\cos 2x + C$ and $y'' = -4A\cos 2x - 4B\sin 2x$

Substituting, equation becomes
$2C + 5Cx + 5D - 4A\cos 2x - 4B\sin 2x - 4A\sin 2x + 4B\cos 2x$
$\quad + 5A\cos 2x + 5B\sin 2x = 5x + \cos 2x$

$\Leftrightarrow (A + 4B)\cos 2x + \sin 2x(B - 4A) + 5Cx + 2C + 5D = 5x + \cos 2x$

$\Leftrightarrow A + 4B = 1, B - 4A = 0, 5C = 5$ and $2C + 5D = 0$

$\Leftrightarrow C = 1, D = -\frac{2}{5}, A = \frac{1}{17}, B = \frac{4}{17}$

P.I. is $y = \frac{1}{17}\cos 2x + \frac{4}{17}\sin 2x + x - \frac{2}{5}$

General solution is $y = e^{-x}(A\cos 2x + B\sin 2x) + \frac{1}{17}\cos 2x + \frac{4}{17}\sin 2x + x - \frac{2}{5}$

Examples for class

Find the general solutions and any particular solutions mentioned.

(1) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2x^2 + 1$

(2) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 3\,e^{-2x}$

(3) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 2\,e^{x}$

(4) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x$

(5) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \sin x$, and the particular solution for which $y = 0$, $\frac{dy}{dx} = 1$ when $x = 0$.

(6) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 30\sin 5x$, and the particular solution for which $y = 0$, $\frac{dy}{dx} = 1$ when $x = 0$

(7) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 5\,e^{3x}$, and the particular solution for which $y = 1$, $\frac{dy}{dx} = -6$ when $x = 0$.

(8) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 5\sin x$

(9) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = x + \sin x$

(10) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = x + \cos 2x$

(11) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = x + e^{2x}$

(12) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 8e^{x} - 50\sin x$, and the particular solution for which $y = 0$, $\frac{dy}{dx} = 1$ when $x = 0$.

Answers

(1) $y = Ae^{-2x} + Be^{-x} + x^2 - 3x + 4$

(2) $y = Ae^{-3x} + Be^{-2x} + 3xe^{-2x}$

(3) $y = Ae^{3x} + Be^{x} - xe^{x}$

(4) $y = Ae^{-2x} + Be^{x} - \frac{1}{2}x - \frac{1}{4}$

(5) $y = e^{x}(A\cos x + B\sin x) - \frac{2}{5}\cos x + \frac{1}{5}\sin x$;

$y = e^{x}(\frac{2}{5}\cos x - \frac{3}{5}\sin x) - \frac{2}{5}\cos x + \frac{1}{5}\sin x$

(6) $y = e^{3x}(A\cos 4x + B\sin 4x) - \cos 5x$; $y = e^{3x}(\cos 4x - \frac{1}{2}\sin 4x) - \cos 5x$

(7) $y = Ae^{3x} + Be^{-2x} + xe^{3x}$; $y = -e^{3x} + 2e^{-2x} + xe^{3x}$

(8) $y = Ae^{2x} + Be^{x} + \frac{3}{2}\cos x + \frac{1}{2}\sin x$

(9) $y = e^{-x}(A\cos x + B\sin x) + \frac{2}{5}\cos x + \frac{1}{5}\sin x + \frac{1}{2}x + \frac{1}{2}$

(10) $y = e^{x}(A\cos 2x + B\sin 2x) + \frac{1}{17}\cos 2x + \frac{4}{17}\sin 2x + \frac{1}{5}x - \frac{2}{25}$

(11) $y = Ae^{4x} + Be^{2x} - \frac{1}{2}xe^{x} + \frac{1}{8}x + \frac{3}{32}$

(12) $y = Ae^{-3x} + Be^{2x} + \cos x + 7\sin x - e^{x}$

Further types of second order differential equations

Example 1

Find the general solution of the differential equation $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \sin(\ln x)$

We must transform this equation into one with constant coefficients.
We do this by the change $x = e^t$.

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{e^t}\frac{dy}{dt} = e^{-t}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx} = \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)e^{-t} = \left[-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right]e^{-t}$$

$$= e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}$$

Substituting, equation becomes

$$e^{2t}\left[e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}\right] + 3e^t . e^{-t}\frac{dy}{dt} + y = \sin(\ln e^t)$$

$$\Leftrightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} + 3\frac{dy}{dt} + y = \sin t$$

$$\Leftrightarrow \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \sin t$$

Auxiliary equation is $D^2 + 2D + 1 = 0 \Leftrightarrow (D+1)^2 = 0 \Leftrightarrow D = -1$, twice
C.F. is $y = c_1 e^{-t} + c_2 t e^{-t}$, where c_1 and c_2 are constants.

For P.I. try $y = A\cos t + B\sin t$, then
$y' = -A\sin t + B\cos t$ and $y'' = -A\cos t - B\sin t$

Substituting, equation becomes
$-A\cos t - B\sin t + 2(-A\sin t + B\cos t) + A\cos t + B\sin t = \sin t$
$\Leftrightarrow -2A\sin t + 2B\cos t = \sin t$
$\Leftrightarrow -2A = 1,\ 2B = 0$
$\Leftrightarrow A = -\frac{1}{2},\ B = 0$
$\therefore$ P.I is $y = -\frac{1}{2}\cos t$

General solution is $y = c_1 e^{-t} + c_2 t e^{-t} - \frac{1}{2}\cos t$

$x = e^t$
then $t = \ln x$
and $\frac{1}{x} = e^{-t}$

i.e. $y = \frac{c_1}{x} + \frac{c_2}{x}\ln x - \frac{1}{2}\cos(\ln x)$

Examples for class

Find the general solution and particular solution where required of

(1) $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \frac{1}{x}$

(2) $x^2\frac{d^2y}{dx^2} + y = 4x$

(3) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x + (\ln x)^2$

(4) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2$ and the particular solution for which $y = -\frac{1}{8}$, $\frac{dy}{dx} = 0$ when $x = 1$.

(5) $(1 - x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + n^2y = 0$ (Put $x = \cos t$)

(6) $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x^2$ (Put $x = \sin t$)

Answers

(1) $y = \frac{A}{x^2} + \frac{B}{x} + \frac{1}{x}\ln x$

(2) $y = x^{-\frac{1}{2}}(A\cos(\frac{\sqrt{3}}{2}\ln x) + B\sin(\frac{\sqrt{3}}{2}\ln x)) + 4x$

(3) $y = Ax^2 + Bx - \frac{1}{3}x\ln x + \frac{1}{2}(\ln x)^2 + \frac{3}{2}\ln x + \frac{7}{4}$

(4) $y = Ax^2 + \frac{B}{x^2} + \frac{1}{4}x^2\ln x$; $y = -\frac{1}{8}x^2 + \frac{1}{4}x^2\ln x$

(5) $y = A\cos 2n(\cos^{-1}x) + B\sin 2n(\cos^{-1}x)$

(6) y = $A\cos(\sin^{-1} x) + Bx - x^2 - 2$