Mathematics
Mathematics 2
Advanced Higher

7533

Summer 2000

HIGHER STILL

Mathematics

Mathematics 2

Advanced Higher

Support Materials

MATHEMATICS 2 (ADVANCED HIGHER)

Introduction

These support materials for Mathematics were developed as part of the Higher Still Development Programme in response to needs identified at needs analysis meetings and national seminars.

Advice on learning and teaching may be found in *Achievement for All* (SOEID 1996), *Effective Learning and Teaching in Mathematics* (SOEID 1993), *Improving Mathematics* (SEED 1999) and in the Mathematics Subject Guide.

These notes are intended to support teachers/lecturers in the teaching of Mathematics 2 (AH). The resources referred to within the material are:

Understanding Pure Mathematics,
AJ Sadler and DWS Thorning, O.U.P., 1987 ISBN 0-19-914243-2

The complete A Level Mathematics,
Orlando Gough, Heinemann Educational Books, 1987, ISBN 0-435-51345-1

Mathematics In Action 6S,
John Hunter, Nelson Blackie, ISBN 0-17-441027-1

In these notes these texts are referred to as Sadler & Thorning, Orlando Gough and Hunter respectively.

FURTHER DIFFERENTIATION

CONTENT
know the derivatives of $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ differentiate any inverse function using the technique : $y = f^{-1}(x) \Rightarrow f(y) = x \Rightarrow (f^{-1}(x))' f'(y) = 1$ etc., and know the corresponding result $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Comments

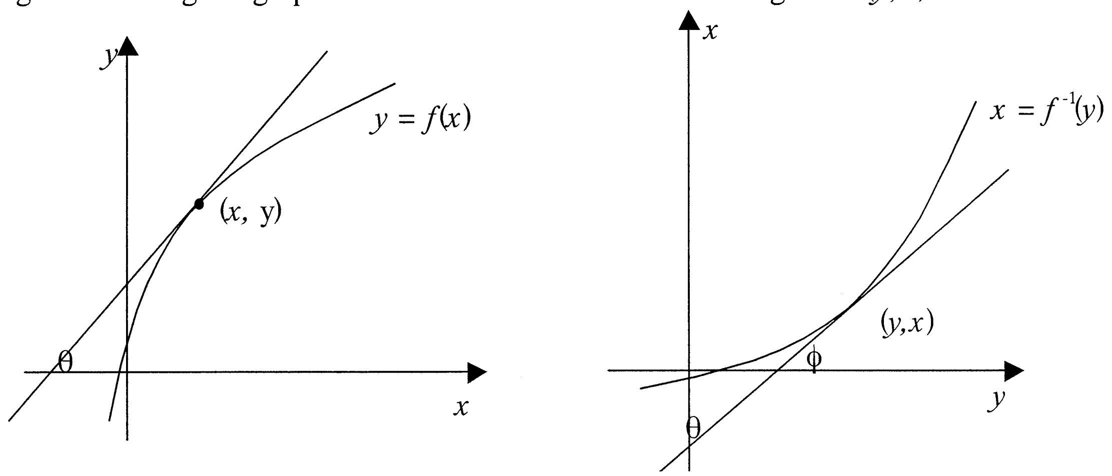
The derivatives of $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ were contained in CSYS Paper 1. The rest of the content is new although the techniques may have been used to find the derivatives of inverse trig functions.

Teaching notes

FINDING THE DERIVATIVE OF AN INVERSE FUNCTION

In Mathematics 1(AH) the relationship between the graph of a function and the graph of its inverse (i.e. the graph of $f^{-1}(x)$ is the image of the graph of $f(x)$ under reflection in the line $y = x$) was established. This provides a means of investigating the derivative of an inverse function.

The diagram below left shows the graph of some function $y = f(x)$ and the tangent at the point (x, y). The diagram below right is a reflection in the line $y = x$ of this diagram showing the graph of the inverse function and the tangent at (y, x).



For $y = f(x)$, the angle between the tangent and the positive direction of the x-axis is θ, i.e. the gradient of the tangent is $\tan\theta$, i.e. $\frac{dy}{dx} = \tan\theta$

For $y = f^{-1}(x)$, the angle between the tangent and the positive direction of the x-axis is ϕ, i.e. the gradient of the tangent is $\tan\phi$, i.e. $\frac{dx}{dy} = \tan\phi$

Now, from the diagram, $\phi = \frac{\pi}{2} - \theta$ so $\tan\theta = \frac{1}{\tan\phi}$ i.e $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

This result can now be applied to finding the derivative of an inverse function when the derivative of the function is known.

Procedure

Step 1: Express x in terms of y

Step 2: Find $\frac{dx}{dy}$

Step 3: Use $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ to find $\frac{dy}{dx}$

Derivative of $\sin^{-1}x$

Let $y = \sin^{-1}x$ $\quad$ ($-1 < x < 1$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$)

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$: $\quad y = \sin^{-1}x \Rightarrow x = \sin y \Rightarrow \frac{dx}{dy} = \cos y \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

Using $\cos^2 y + \sin^2 y = 1$ $\quad \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

Thus $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

Derivative of $\cos^{-1}x$

Let $y = \cos^{-1}x$ $\quad$ ($-1 < x < 1$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$)

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$: $\quad y = \cos^{-1}x \Rightarrow x = \cos y \Rightarrow \frac{dx}{dy} = -\sin y \Rightarrow \frac{dy}{dx} = \frac{1}{-\sin y}$

Using $\cos^2 y + \sin^2 y = 1$ $\quad \sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$

Thus $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

Derivative of $\tan^{-1}x$

Let $y = \tan^{-1}x$

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$: $\quad y = \tan^{-1}x \Rightarrow x = \tan y \Rightarrow \frac{dx}{dy} = \sec^2 y \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$

Using $\cos^2 y + \sin^2 y = 1$ $\quad \sec^2 y = \frac{1}{\cos^2 y} = \frac{\cos^2 y + \sin^2 y}{\cos^2 y} = 1 + \tan^2 y = 1 + x^2$

Thus $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

Examples

Differentiate : a) $y = \sin^{-1}3x$ b) $y = x^2\cos^{-1}x$ c) $y = \tan^{-1}\sqrt{x}$

Solutions

a) $$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x)^2}} \times 3 = \frac{3}{\sqrt{1-9x^2}}$$

b) Product rule : $f(x) = x^2$ $f'(x) = 2x$ $g(x) = \cos^{-1}x$ $g'(x) = -\frac{1}{\sqrt{1-x^2}}$

$$\frac{dy}{dx} = 2x \times \cos^{-1}x + x^2 \times \left(-\frac{1}{\sqrt{1-x^2}}\right) = 2x\cos^{-1}x - \frac{x^2}{\sqrt{1-x^2}}$$

c) $$\frac{dy}{dx} = \frac{1}{1+(\sqrt{x})^2} \times \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}(1+x)}$$

Note

The technique of finding the derivative of an inverse function links the derivatives of e^x and $\ln x$.

We can establish the result $\frac{d}{dx}e^x = e^x$ from first principles and then evaluate $\frac{d}{dx}\ln x$ as follows

$$y = \ln x \quad \Rightarrow x = e^y$$

$$\Rightarrow \frac{dx}{dy} = e^y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Exercises

1.	Orlando Gough	Page 164 Ex. 3.5:2 Q.3,4
2.	Sadler & Thorning	Page 376 Ex. 15H Q.1-12
3.	Hunter	Page 69 Ex. 6.2 Q. 1n,o,p,q,r,s

CONTENT
understand how an equation $f(x, y) = 0$ defines y implicitly as one (or more) function(s) of x use implicit differentiation to find first and **second derivatives [A/B]**

Comments

e.g. $x^2 + y^2 = 1 \Leftrightarrow y = \pm\sqrt{1-x^2}$, i.e. two functions defined on [-1, 1].

Implicit differentiation was contained in CSYS Paper 2 (Unrevised).

Teaching notes

IMPLICIT DIFFERENTIATION

In a relation such as $3x^3 + y^3 + 2y = 5$, which cannot directly be expressed in the form $y = f(x)$, y is said to be defined ***implicitly*** as a function of x.

An implicit function can be differentiated using the chain/product rule where necessary as illustrated below.

Example 1

Find $\frac{dy}{dx}$ in terms of x and y for : $3x^2 + y^2 + 2y = 5$

Differentiating with respect to x :

$$\frac{d}{dx}(3x^2) + \frac{d}{dx}(y^2) + \frac{d}{dx}(2y) = \frac{d}{dx}(5)$$

$$6x + 2y\frac{dy}{dx} + 2\frac{dy}{dx} = 0 \qquad \text{(using the chain rule as appropriate)}$$

$$\frac{dy}{dx}(2y+2) = -6x \qquad \Rightarrow \frac{dy}{dx} = \frac{-6x}{2y+2} = \frac{-3x}{y+1}$$

The second derivative $\frac{d^2y}{dx^2}$ can be found by differentiating $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-3x}{y+1)}\right) = \frac{-3(y+1) - (-3x)\times 1 \times \frac{dy}{dx}}{(y+1)^2}$$

$$= \frac{-3y - 3 + 3x\left(\frac{-3x}{y+1}\right)}{(y+1)^2}$$

$$= \frac{(-3y-3)(y+1) - 9x^2}{(y+1)^3} = \frac{-9x^2 - 3y^2 - 6y - 3}{(y+1)^3}$$

Example 2

Find $\frac{dy}{dx}$ in terms of x and y for : $x^2 + 7xy + 9y^2 = 6$

Differentiating with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(7xy) + \frac{d}{dx}(9y^2) = \frac{d}{dx}(6)$$

$$2x + 7x\frac{dy}{dx} + 7y + 18y\frac{dy}{dx} = 0 \qquad \text{(using product rule for the middle term)}$$

$$\frac{dy}{dx}(7x + 18y) = -2x - 7y \qquad \Rightarrow \frac{dy}{dx} = \frac{-2x - 7y}{7x + 18y}$$

Example 3

Find the gradient of the curve $x^2 + xy + y^2 = 1$ at $x = 1$ $(y \neq 0)$

Differentiating with respect to x:

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x + 2y) = -2x - y \qquad \Rightarrow \frac{dy}{dx} = \frac{-2x - y}{x + 2y}$$

To find y when $x = 1$, substitute into the equation of the curve:

$$1 + y + y^2 = 1$$

$$y + y^2 = 0$$

$$y(y + 1) = 0 \qquad \Rightarrow y = 0 \text{ or } y = -1, \quad \text{i.e. } y = -1$$

$$\text{Gradient} = \frac{-2 + 1}{1 - 2} = 1$$

Note Consider $y = \sin^{-1}x$. This can be expressed as $\sin y = x$, where y is now expressed implicitly as a function of x. Differentiation of an implicit function provides an alternative means of differentiating $y = \sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ and $\ln x$

$$y = \sin^{-1}x \Rightarrow \sin y = x \quad \Rightarrow \cos y\frac{dy}{dx} = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

Exercises

1.	Orlando Gough	Page 69	Ex. 1.3:8 Q. 1-12
2.	Sadler & Thorning	Page 332	Ex. 13F Q. 1-14

CONTENT
use logarithmic differentiation, recognising when it is appropriate in extended products and quotients and indices involving the variable [A/B]   e.g. Find the derivatives of $y = \dfrac{x^2\sqrt{7x-3}}{1+x}$, $y = 2^x$ **[A/B]**

Teaching notes

Logarithmic differentiation is particularly useful for differentiating functions of the form: $y = f(x)^{g(x)}$. The procedure is illustrated in the following examples.

1. To find $\dfrac{dy}{dx}$ where $y = 4^x$

Step 1 Taking the log of both sides: $\ln y = \ln 4^x = x \ln 4$

Step 2 Differentiating implicitly w.r.t. x: $\dfrac{1}{y}\dfrac{dy}{dx} = \ln 4$

Step 3 Expressing $\dfrac{dy}{dx}$ as a function of x: $\dfrac{dy}{dx} = y \ln 4 = 4^x \ln 4$

2. To find $\dfrac{dy}{dx}$ where $y = x^x$

Step 1 Taking the log of both sides: $\ln y = \ln x^x = x \ln x$

Step 2 Differentiating implicitly w.r.t. x: $\dfrac{1}{y}\dfrac{dy}{dx} = x\dfrac{1}{x} + 1.\ln x = 1 + \ln x$

Step 3 Expressing $\dfrac{dy}{dx}$ as a function of x: $\dfrac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x)$

3. To find $\dfrac{dy}{dx}$ where $y = \dfrac{x^2\sqrt{7x-3}}{1+x}$

Step 1 $\ln y = \ln \dfrac{x^2(7x-3)^{\frac{1}{2}}}{1+x} = 2\ln x + \dfrac{1}{2}\ln(7x-3) - \ln(1+x)$

Step 2 $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2}{x} + \dfrac{1}{2}\dfrac{7}{(7x-3)} - \dfrac{1}{1+x} = \dfrac{21x^2 + 29x - 12}{2x(7x-3)(1+x)}$

Step 3 $\dfrac{dy}{dx} = y\dfrac{21x^2 + 29x - 12}{2x(7x-3)(1+x)} = \dfrac{x(21x^2 + 29x - 12)}{2(1+x)^2\sqrt{7x-3}}$ (Working has been abbreviated.)

Exercises

1.	Orlando Gough	Page 109	Ex. 2.5:5 Q.1,3,4,5
2.	Sadler & Thorning	Page 487	Ex. 19B Q. 21-24

CONTENT
understand how a function can be defined parametrically *eg* $x^2 + y^2 = r^2$, $x = r\cos\theta, y = r\sin\theta$ understand simple applications of parametrically defined functions use parametric differentiation to find first **and second derivatives [A/B], and apply to motion in a plane** eg *If the position of a particle is given by* $x = f(t), y = g(t)$ *then the velocity components are given by* $\frac{dx}{dt}$ *and* $\frac{dy}{dt}$ *and the speed by* $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. *The instantaneous direction of motion is given by* : $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Comments

Parametrically defined functions were contained in CSYS Paper 2 (unrevised).

Teaching notes

The Cartesian co-ordinates of a general point on a curve may be given in terms of a third variable, often t or θ. This variable is called a **parameter** and the resultant equations are called the **parametric equations** of the curve. It is often possible to eliminate the parameter and obtain the Cartesian equation of the curve.

Examples

1. Consider the parametric equations $x = at^2$, $y = 2at$

   Using $y = 2at$ we have $t = \frac{y}{2a}$

   $x = at^2 = a\frac{y^2}{4a^2}$ thus $x = \frac{y^2}{4a}$ and so $y^2 = 4ax$

   $y^2 = 4ax$ is the Cartesian equation of a parabola.

   Thus $x = at^2$, $y = 2at$ are the parametric equations of the parabola $y^2 = 4ax$.

2. Consider the parametric equations $x = r\cos\theta$, $y = r\sin\theta$

   Here $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$

   $x^2 + y^2 = r^2$ is the Cartesian equation of the circle centre the origin and radius r.

   Thus $x = r\cos\theta$, $y = r\sin\theta$ are the parametric equations of the circle $x^2 + y^2 = r^2$.

PARAMETRIC DIFFERENTIATION

Parametric equations are differentiated as 'functions of a function' using the chain rule as illustrated in the following examples.

Example 1

If $y = 2at$ and $x = at^2$, find (i) $\frac{dy}{dx}$ and (ii) $\frac{d^2y}{dx^2}$ **[A/B]**

(i) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$$\frac{dy}{dt} = 2a \quad \frac{dx}{dt} = 2at \quad \Rightarrow \quad \frac{dy}{dx} = 2a \div 2at = \frac{1}{t}$$

(ii) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}\left(\frac{1}{t}\right) \times \frac{dt}{dx} = \frac{d}{dt}\left(\frac{1}{t}\right) \div \frac{dx}{dt} = \left(\frac{-1}{t^2}\right) \div 2at = \frac{-1}{2at^3}$

Example 2

A stone is thrown out to sea from the top of a cliff. After t seconds the horizontal and vertical distances of the stone from the point of projection are x metres and y metres, respectively, where $x = 10t$ and $y = -5t^2$. The point of projection is 80 m above sea level.

Find: (i) the position of the stone after 1 second; (ii) the time taken for the stone to reach sea level; (iii) the speed at which the stone enters the sea and (iv) the angle at which the stone enters the sea.

(i) When $t = 1$, $x = 10$ and $y = -5$. Hence, the stone is 10 m horizontally from the cliff and 5 m below the level of projection.

(ii) When $y = -80$, $-5t^2 = -80$, so $t = 4$. Hence, stone reaches sea level after 4 seconds.

(iii) The horizontal component of the velocity of the stone at time t is $\frac{dx}{dt} = 10$

The vertical component of the velocity of the stone at time t is $\frac{dy}{dt} = -10t$

$\therefore$ speed of stone after 4 second is $\sqrt{10^2 + (-40)^2} = 10\sqrt{17}$ ms^{-1}

(iv) $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ and so after 4 seconds $\frac{dy}{dx} = (-40) \div 10 = -4$

Thus the acute angle at which stone enters the sea is $\tan^{-1} 4$ to the horizontal.

Exercises

1.	Orlando Gough	Page 319	Ex. 6.3:1 Q. 9,10,15,16
		Page 322	Ex. 6.3:2 Q. 1-10 (note that most of the questions include reference to the normal to the curve.)
2.	Sadler & Thorning	Page 327	Ex. 13D Q. 1-26 (select)

CONTENT
apply differentiation to related rates in problems where the functional relationship is given explicitly or implicitly **solve practical related rates by first establishing a functional relationship between appropriate variables [A/B]**

Comments

Explicitly: $V = \frac{1}{3}\pi r^2 h$; given $\frac{dh}{dt}$, find $\frac{dV}{dt}$.

Implicitly: $x^2 + y^2 = r^2$ where x, y are functions of t; given $\frac{dx}{dt}$, find $\frac{dy}{dt}$ using $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$.

Teaching notes

We are dealing here with problems in which there is a function defined either explicitly or implicitly in terms of a third variable. In such cases we can use the chain rule to find the rate of change with respect to this third variable.

e.g. If $y = f(x)$ and $x = g(t)$ then for $\frac{dy}{dt}$ we have $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

Example

The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ and its surface area by $A = 4\pi r^2$.

The volume is increasing at the steady rate of $5\ cm^3/s$.

a) Find the rate of change of the radius with respect to time, t seconds.

b) Calculate the value of $\frac{dA}{dt}$ when the radius = 4cm.

Solution

(a) $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ Since, $\frac{dV}{dt} = 5$, $\frac{dr}{dt} = 5 \div 4\pi r^2 = \frac{5}{4\pi r^2}$

(b) $\frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi r \frac{5}{4\pi r^2} = \frac{10}{r}$ When $r = 4$, $\frac{dA}{dt} = 2{\cdot}5\ cm^2/s$

Exercises

1.	Orlando Gough	Page 65	Ex. 1.3:6	Q. 12-19
2.	Sadler & Thorning	Page 324	Ex. 13C	Q. 1-10

FURTHER INTEGRATION

CONTENT
know the integrals of $\dfrac{1}{\sqrt{1-x^2}}, \dfrac{1}{1+x^2}$
use the substitution $x = at$ to integrate functions of the form $\dfrac{1}{\sqrt{a^2-x^2}}, \dfrac{1}{a^2+x^2}$

Comments

This content was contained in CSYS Paper 1.

Teaching notes

From the notes on differentiation we have:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

giving $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$ and $\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1}x + C$

$$\int \frac{1}{\sqrt{a^2-x^2}}$$

Using the substitution $x = at$ in $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}}$ we have:

$x = at \Rightarrow \quad x^2 = a^2t^2 \Rightarrow \quad a^2 - x^2 = a^2 - a^2t^2 = a^2(1 - t^2)$

$\dfrac{dx}{dt} = a \Rightarrow$ "$dx = adt$"

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{adt}{\sqrt{a^2(1-t^2)}} = \int \frac{adt}{a\sqrt{(1-t^2)}} = \int \frac{dt}{\sqrt{(1-t^2)}} = \sin^{-1}t + C = \sin^{-1}\frac{x}{a} + C$$

$\displaystyle\int \frac{dx}{a^2+x^2}$, using the same substitution we have:

$x = at \Rightarrow x^2 = a^2t^2 \Rightarrow a^2 + x^2 = a^2 + a^2t^2 = a^2(1 + t^2)$

$\dfrac{dx}{dt} = a \Rightarrow$ "$dx = adt$"

$$\int \frac{dx}{a^2+x^2} = \int \frac{adt}{a^2(1+t^2)} = \frac{1}{a}\int \frac{dt}{(1+t^2)} = \frac{1}{a}\tan^{-1}t + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Examples

Integrate the following with respect to x:

1. $\int \frac{dx}{\sqrt{9-x^2}}$

answer: $\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1} \frac{x}{3} + C$

2. $\int \frac{dx}{9+x^2}$

answer: $\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\frac{x}{3} + C$

3. $\int \frac{2dx}{\sqrt{1-4x^2}}$

answer: $\int \frac{2dx}{\sqrt{1-4x^2}} = \int \frac{2dx}{\sqrt{4(\frac{1}{4}-x^2)}} = \frac{2}{2}\int \frac{dx}{\sqrt{\frac{1}{4}-x^2}} = \sin^{-1} \frac{x}{\frac{1}{2}} + C = \sin^{-1} 2x + C$

4. $\int \frac{3dx}{4+9x^2}$

answer: $\int \frac{3dx}{4+9x^2} = \frac{3}{9}\int \frac{1}{(\frac{4}{9}+x^2)} = \frac{1}{3}\times\frac{1}{\frac{2}{3}}\tan^{-1}\frac{x}{\frac{2}{3}} + C = \frac{1}{2}\tan^{-1}\frac{3x}{2} + C$

Exercises

1.	Orlando Gough	Page 193	Ex. 4.2:3	Q. 1b,2b,5ii, 6ii, 8i, 12,16,17,19,20
2.	Sadler & Thorning	Page 376	Ex. 15H	Q. 13-24
		Page 501	Ex. 20A	Q. 17
3.	Hunter	Page 76	Ex. 7.1	Q. 1h,i,k,l

CONTENT
integrate rational functions, both proper and improper, by means of partial fractions; the degree of the denominator being $\leq$ 3, the denominator may include : (i) two separate or repeated linear factors **(ii) three linear factors [A/B]** **(iii) a linear factor and an irreducible quadratic factor of the form $x^2 + a^2$ [A/B]**

Comments

This content was contained in CSYS Paper 1

Teaching notes

Expressing a proper rational function in terms of partial fractions has been covered in Mathematics 1 (AH) Outcome 1. The procedure for integrating proper and improper rational functions is illustrated in the following examples.

Examples

1. Find $\displaystyle\int \frac{x+3}{(x+1)(x+2)}dx$

$$\frac{x+3}{(x+1)(x+2)} = \frac{2}{x+1} - \frac{1}{x+2} \qquad \text{(partial fractions)}$$

$$\int \frac{x+3}{(x+1)(x+2)}dx = \int \frac{2}{x+1}dx - \int \frac{1}{x+2}dx = 2\ln(x+1) - \ln(x+2) + c$$

$$= \ln\frac{C(x+1)^2}{(x+2)}, \text{ where } c = \ln C$$

2. Find $\displaystyle\int \frac{x-2}{(x+3)^2}dx$

$$\frac{x-2}{(x+3)^2} = \frac{1}{x+3} - \frac{5}{(x+3)^2}$$

$$\int \frac{x-2}{(x+3)^2}dx = \int \frac{1}{x+3}dx - \int \frac{5}{(x+3)^2}dx = \ln(x+3) + 5(x+3)^{-1} + C$$

3. Find $\displaystyle\int \frac{9x^2-6x+3}{(x+1)(2x-1)(x-2)}dx$

$$\frac{9x^2-6x+3}{(x+1)(2x-1)(x-2)} = \frac{2}{x+1} - \frac{1}{2x-1} + \frac{3}{x-2}$$

$$\int \frac{9x^2-6x+3}{(x+1)(2x-1)(x-2)}dx = \int \frac{2}{x+1}dx - \int \frac{1}{2x-1}dx + \int \frac{3}{x-2}dx$$

$$= 2\ln(x+1) - \frac{1}{2}\ln(2x-1) + 3\ln(x-2) + c$$

$$= \ln\frac{C(x+1)^2(x-2)^3}{(2x-1)^{1/2}}, \text{ where } c = \ln C$$

4. Find $\int \frac{3x^2 - 2x + 16}{(2-x)(x^2+4)}dx$

$$\frac{3x^2 - 2x + 16}{(2-x)(x^2+4)} = \frac{3}{2-x} + \frac{2}{x^2+4}$$

$$\int \frac{3x^2 - 2x + 16}{(2-x)(x^2+4)}dx = \int \frac{3}{2-x}dx + \int \frac{2}{x^2+4}dx = -3\ln|2-x| + \frac{2}{2}\tan^{-1}\frac{x}{2} + C$$

$$= -3\ln|2-x| + \tan^{-1}\frac{x}{2} + C$$

5. Find $\int \frac{x^2}{x^2-1}dx$

$$\frac{x^2}{x^2-1} = 1 + \frac{1}{x^2-1} = 1 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$ (by dividing out and then partial fractions)

$$\int \frac{x^2}{x^2-1}dx = \int 1dx + \int \frac{1}{2}\frac{1}{(x-1)}dx - \int \frac{1}{2}\frac{1}{(x+1)}dx = x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

6. Evaluate $\int_2^3 \frac{2}{(x+1)(x-1)}dx$

$$\frac{2}{(x+1)(x-1)} = \frac{1}{x-1} - \frac{1}{x+1}$$

$$\int_2^3 \frac{2}{(x+1)(x-1)}dx = \int_2^3 \frac{1}{x-1}dx - \int_2^3 \frac{1}{x+1}dx = \left[\ln|x-1| - \ln|x+1|\right]_2^3$$

$$= (\ln 2 - \ln 4) - (\ln 1 - \ln 3) = \ln 2 - \ln 4 + \ln 3 = \ln\frac{6}{4} = \ln 1{\cdot}5$$

Exercises

1.	Orlando Gough	Page 196	Ex. 4.2:5	Q. 1,2,3,4,6,7, 8,9,10
2.	Sadler & Thorning	Page 487	Ex .19B	Q. 35,36,39, 40,41,43f,46,47,48
		Page 508	Ex. 20C	Q. 27,38,56
3.	Hunter	Page 80	foot of page a - g	
		Page 85	Ex. 7.2	Q. 1c,h,v,w,x

CONTENT	
integrate by parts with one application	e.g. $\int x \sin x \, dx$
integrate by parts involving repeated applications [A/B]	e.g. $\int x^2 e^{3x} \, dx$

Comments

This content was contained in CSYS Paper 1.

Teaching notes

Consider the product rule for differentiation :

$$\frac{d}{dx} u.v = u\frac{dv}{dx} + v\frac{du}{dx}$$

integrating both sides we have :

$$u.v = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx \text{ and rearranging gives: } \int u\frac{dv}{dx}dx = u.v - \int v\frac{du}{dx}dx$$

To evaluate $\int f(x)g(x)dx$ we replace $f(x)$ or $g(x)$ by u and the other by $\frac{dv}{dx}$ and apply $\int u\frac{dv}{dx}dx = u.v - \int v\frac{du}{dx}dx$. For example, replacing $f(x)$ by u and applying the above gives $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\left(\int g(x)dx\right)dx$

Note : The choice of function is important since u will be differentiated and $\frac{dv}{dx}$ will be integrated. The choice will normally be dictated by the function which should be integrated to avoid introducing a more complicated integration.

Examples

1. $\int x \sin x \, dx$

   Let $u = x$ and $\frac{dv}{dx} = \sin x \Rightarrow \frac{du}{dx} = 1$ and $v = -\cos x$

   Now $\int u\frac{dv}{dx}dx = u.v - \int v\frac{du}{dx}dx$

   Thus $\int x\sin x dx = x(-\cos x) - \int(-\cos x).1\,dx = -x\cos x + \sin x + C$

2. $\int x^2 e^{3x}\,dx$

Let $u = x^2$ and $\frac{dv}{dx} = e^{3x} \Rightarrow \frac{du}{dx} = 2x$ and $v = \frac{1}{3}e^{3x}$

$$\int x^2 e^{3x} dx = x^2 \frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x} 2x dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int xe^{3x} dx$$

Note: for $\int xe^{3x} dx$ let $u = x$ and $\frac{dv}{dx} = e^{3x} \Rightarrow \frac{du}{dx} = 1$ and $v = \frac{1}{3}e^{3x}$

$$\int xe^{3x} dx = x\,\frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x} dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right) + C = \frac{1}{3}x^2 e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$$

This process is also useful in some situations where there is no obvious product.

3. $\int \ln x\,dx$

We write $\int \ln x\,dx$ as $\int \ln x.1\,dx$

Let $u = \ln x$ and $\frac{dv}{dx} = 1 \Rightarrow \frac{du}{dx} = \frac{1}{x}$ and $v = x$

$$\int \ln x\,dx = \int \ln x.1\,dx = x \ln x - \int x.\frac{1}{x} dx = x \ln x - \int 1\,dx = x \ln x - x + C$$

This procedure is useful when integrating inverse trigonometric functions.

Exercises

1.	Orlando Gough	Page 199	Ex. 4.2:6 Q. 1,2,4,5,6, 11,12,13
2.	Sadler & Thorning	Page 505	Ex. 20B Q. 1-25 (select)
		Page 508	Ex. 20C has mixed integration questions. Select from N[os] 1 – 56.
3.	Hunter	Page 77	Ex. 7.1 Q. 4

CONTENT
know the definition of a differential equation and the meaning of the terms linear, order, general solution, arbitrary constants, particular solution, initial condition solve first order differential equations (variables separable) i.e. can be written in the form $\frac{dy}{dx} = \frac{g(x)}{h(y)}$

Comments

The first part of the above content is background knowledge, the second part was contained in CSYS Paper 1.

Teaching notes

Any equation involving y as a function of x and any of its derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ .....
and possibly the independent variable x is called a **differential equation.**

The order of the highest derivative determines the **order** of the differential equation. Solving a differential equation of order n requires n integrations and we would have n **constants of integration.** At this stage we restrict ourselves to first order differential equations.

Consider : $y = x^2 - 3 \Rightarrow \frac{dy}{dx} = 2x$

also $\quad y = x^2 + 5 \Rightarrow \frac{dy}{dx} = 2x$

There are an infinite number of equations of the form $y = x^2 + C$ which have derivative $2x$.

We have, therefore, the first order differential equation : $\frac{dy}{dx} = 2x$ with **general solution** $y = x^2 + C$, where C is an arbitrary constant.

The solution is a family of functions. The solution is found by integration.

To find a particular member of the family, i.e. find a **particular solution** , we need to be told corresponding values of x and y. A condition such as ($y = 5$ when $x = 0$) where we are given the initial value of y is called an **initial condition.**

e.g.. $\frac{dy}{dx} = 2x \qquad y = 5$ when $x = 0$

$\Rightarrow \quad y = x^2 + C \Rightarrow 5 = 0^2 + C \Rightarrow C = 5 \Rightarrow y = x^2 + 5$

Example

$\frac{dy}{dx} = 4x - 3$ and $y = 9$ when $x = -1$

integrating both sides:

$y = 2x^2 - 3x + C$

$\Rightarrow \quad 9 = 2(-1)^2 - 3(-1) + C$
$9 = 2 + 3 + C$
$C = 4$

$y = 2x^2 - 3x + 4$ is the particular solution.

VARIABLES SEPARABLE

Here we consider differential equations of the form $\frac{dy}{dx} = \frac{g(x)}{h(y)}$

If $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ then $h(y)\frac{dy}{dx} = g(x)$

Now $\frac{d}{dx}\left(\int h(y)dy\right) = \frac{d}{dy}\left(\int h(y)dy\right)\frac{dy}{dx} = h(y)\frac{dy}{dx} = g(x)$
so $\int h(y)dy = \int g(x)dx$

Thus $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ can be solved by writing it in the form $\int h(y)dy = \int g(x)dx$
(Hence the description 'variables separable')

The procedure for solving such equations is illustrated in the following examples.

Examples

1. Solve the equation $y^2\frac{dy}{dx} = 4x$

$$\int y^2\,dy = \int 4x\,dx$$

$$\Rightarrow \quad \frac{1}{3}y^3 = 2x^2 + C$$

$$\Rightarrow \quad y^3 = 6x^2 + k \quad \text{is a general solution}$$

Note that C is an arbitrary constant and when the solution is multiplied by 3 the arbitrary constant $3C$ is written as k.

2. Solve the equation $(x-1)\frac{dy}{dx}=y$ for $y>0$ and $x>1$

$$\int\frac{1}{y}dy=\int\frac{1}{x-1}dx$$

$$\ln y=\ln(x-1)+C$$

Since C is an arbitrary constant it can be replaced by $\ln k$ giving

$$\ln y=\ln(x-1)+\ln k=\ln k(x-1)$$

$\ln y = \ln k(x-1) \quad \Rightarrow \quad y=k(x-1)$ is a general solution

3. Solve $e^x\frac{dy}{dx}=xy^2$ given $y=1$ when $x=0$

$$\int y^{-2}\,dy = \int x\,e^{-x}\,dx$$

To evaluate the right hand side let $u=x$ and $\frac{dv}{dx}=e^{-x}$ giving $\frac{du}{dx}=1$ and $v=-e^{-x}$

$$\int x\,e^{-x}\,dx = -x.\,e^{-x} - \int -e^{-x}.\,1\,dx = -x\;e^{-x} - e^{-x}$$

i.e. $-y^{-1} = -x\;e^{-x} - e^{-x}+C$

substitute x and y values $\qquad -1=0\,e^0-e^0+C \;\Rightarrow\; C=0$

i.e. $-\frac{1}{y}=-\frac{x}{e^x}-\frac{1}{e^x} \;\Rightarrow\; \frac{1}{y}=\frac{x+1}{e^x} \;\Rightarrow\; y=\frac{e^x}{x+1}$

Note

To solve $\frac{dy}{dx}=\frac{h(y)}{g(x)}$ the equivalent expression is $\int\frac{dy}{h(y)}=\int\frac{dx}{g(x)}$.
The solution of examples of this type are illustrated in the next section.

Exercises

1.	Orlando Gough	Page 206	Ex. 4.4:2	Q.1-4 (select)
2.	Sadler & Thorning	Page 515	Ex. 20D	Q. 1-32 (select)
3.	Hunter	Page 78	Ex. 7.1	Q. 6

CONTENT
formulate a simple statement involving rate of change as a simple separable first order differential equation, including the finding of a curve in the plane, given the equation of the tangent at (x, y), which passes through a given point know the laws of growth and decay : applications in practical contexts

Teaching notes

In Mathematics 1 (AH) the relationship between displacement, velocity and acceleration was covered. The concept of 'solving for *C*' given the gradient of the tangent to a curve and a given point on that curve is contained within the Higher course.

Examples

1. The rate of change of the potential ,V, of a charged condenser is given by the differential equation $\frac{dV}{dt} = -\frac{V}{CR}$ where t is the time, C is the capacity of the condenser and R is the resistance to earth. Given that $C = 6 \times 10^{-6}$ units and V falls from 200 to 160 units in 50 units of time calculate the value of R.

$$\int \frac{dV}{V} = -\int \frac{dt}{CR} \quad \Rightarrow \quad \ln V = -\frac{t}{CR} + k$$

When $t = 0 \quad V = 200 \Rightarrow \quad \ln 200 = 0 + k \Rightarrow \quad k = \ln 200$

Hence $\ln V = -\frac{t}{6 \times 10^{-6} R} + \ln 200$

$$V = 160,\ t = 50 \Rightarrow \ln \frac{160}{200} = -\frac{50}{6 \times 10^{-6} R} \quad \Rightarrow R = 3.73 \times 10^{7}$$

2. The gradient of the tangent to a curve is given by $\frac{dy}{dx} = \frac{1+y^2}{\tan x}$

   Find the equation of the curve if it passes through the point ($\frac{\pi}{2}$, 1)

$$\int \frac{1}{1+y^2} dy = \int \frac{1}{\tan x} dx \Rightarrow \tan^{-1} y = \int \frac{\cos x}{\sin x} dx + C \Rightarrow \tan^{-1} y = \ln|\sin x| + C$$

Given $x = \frac{\pi}{2}$, $y = 1$: $\tan^{-1} 1 = \ln|\sin \frac{\pi}{2}| + C \Rightarrow \quad \frac{\pi}{4} = \ln 1 + C \Rightarrow \quad C = \frac{\pi}{4}$

Hence $\tan^{-1} y = \ln|\sin x| + \frac{\pi}{4}$

3. If I is the intensity of light at a depth x metres below the surface of the sea then $\frac{dI}{dx} = -kI$ where k is a positive constant.

If I_0 is the light intensity at the surface express I in terms of $k, x,$ and I_0.

$$\int \frac{1}{I} dI = -\int k dx \qquad \Rightarrow \qquad \ln I = -kx + C$$

When $x = 0$, $I = I_0 \Rightarrow \ln I_0 = -k.0 + C$
$C = \ln I_0$

$$\ln I = -kx + \ln I_0$$
$$\ln I - \ln I_0 = -kx$$
$$\ln \frac{I}{I_0} = -kx \quad \Rightarrow \quad \frac{I}{I_0} = e^{-kx} \quad \Rightarrow \quad I = I_0 e^{-kx}$$

Note
The above is a particular example of a relationship displaying exponential decay. When the coefficient of x is positive we have exponential growth.

Exercises

1.	Orlando Gough	Page 208	Ex. 4.4:3 Q. 1-6
2.	Sadler & Thorning	Page 516	Ex. 20D Q. 41,42
3.	Hunter	Page 78	Ex. 7.1 Q. 7,8,9

COMPLEX NUMBERS

CONTENT
know the definition of i as a solution of $z^2 + 1 = 0$, so that $i = \sqrt{-1}$
know the definition of the set of complex numbers as $\mathbf{C} = \{a + ib : a, b \in \mathbf{R}\}$
know the definition of real and imaginary parts
know the terms complex plane, Argand diagram
plot complex numbers as points in the complex plane

Comments

This content was contained in CSYS Paper 1. All except the last statement could be described as background knowledge.

Teaching notes

When solving quadratic equations the value of the discriminant had to be considered to determine the nature of the roots. There were three possibilities:

$b^2 - 4ac = 0 \Rightarrow$ real and equal roots
$b^2 - 4ac > 0 \Rightarrow$ real and unequal roots
$b^2 - 4ac < 0 \Rightarrow$ no real roots

It is the third possibility in which we are interested in this section.

Complex Numbers

Firstly, consider the equation $z^2 + 1 = 0$ for which $z^2 = -1$.

This equation has no real roots. The solution of this equation is defined to be the **imaginary number i** which stands for $\sqrt{-1}$.

Secondly, consider the equation $x^2 - 2x + 10 = 0$, say

The roots are : $x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4\times1\times10}}{2\times1} = \dfrac{2 \pm \sqrt{-36}}{2}$

Assuming that $\sqrt{-36} = \sqrt{36}\sqrt{-1} = 6i$ we have $x = \dfrac{2 \pm 6i}{2} = 1 \pm 3i$

The **set of complex numbers** is defined to be $\mathbf{C} = \{a + ib : a, b \in \mathbf{R}\}$.

We often use $z = a + ib$ as a member of this set.

The **real part** of z is 'a' ($\text{Re}(z) = a$) and the **imaginary part** is 'b' ($\text{Im}(z) = b$).

Examples

1. Solve $z^2 = -9$

$z^2 = -9 \Rightarrow z^2 = 9i^2 \Rightarrow z^2 = \pm 3i$

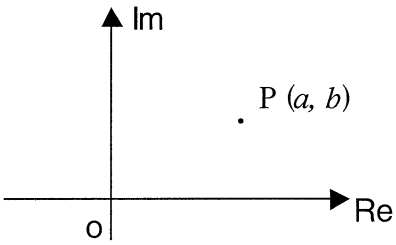
2. Solve $z^2 + 2z + 4 = 0$

$$z = \frac{-2 \pm \sqrt{2^2 - 4.1.4}}{2} = \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm \sqrt{12i^2}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2}$$

$z = -1 \pm \sqrt{3}i$

Complex Plane and Argand Diagrams

Complex numbers can be represented geometrically using the x-axis as the real axis and the y-axis as the imaginary axis. This is known as the **complex plane** and a diagram showing complex numbers is called an **Argand diagram.**

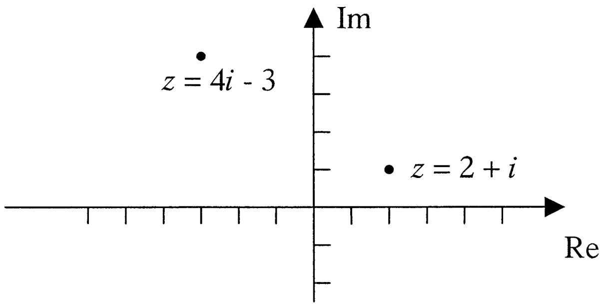


The complex number $z = a + ib$ is represented by the point P(a, b)

Example Show the following complex numbers on an Argand diagram :

a) $z = 2 + i$

b) $z = 4i - 3$



Exercises

1.	Orlando Gough	Page 344	Ex. 6.5:1 Q. 1 – 5
		Page 353	Ex. 6.5:4 Q. 1 – 7
2.	Sadler & Thorning	Page 463	Ex. 18D Q. 6
		Page 467 _	Ex. 18E Q. 1,2
3.	Hunter	Page 39	Ex. 4.2 Q. 1,2,3

Note that Orlando Gough does not mention the 'complex plane'.

CONTENT
perform algebraic operations on complex numbers : equality (equating real and imaginary part), addition, subtraction, multiplication and division evaluate the modulus, argument and conjugate of complex numbers convert between Cartesian and polar form

Comments

This content was contained in CSYS Paper 1.

Teaching notes

As with other sets of numbers, we can perform algebraic operations on complex numbers:

Equality: Given $a + ib = c + id$ then $a = c$ (equating real parts) and $b = d$ (equating imaginary parts).

Addition/subtraction: Given $z_1 = a + ib$ and $z_2 = c + id$ then

$z_1 + z_2 = a + ib + c + id = (a + c) + i(b + d)$

$z_1 - z_2 = a + ib - (c + id) = (a - c) + i(b - d)$

Multiplication/division: Given $z_1 = a + ib$ and $z_2 = c + id$ then

$$z_1 . z_2 = (a + ib) . (c + id) = ac + iad + ibc + i^2 bd = (ac - bd) + i(ad + bc)$$

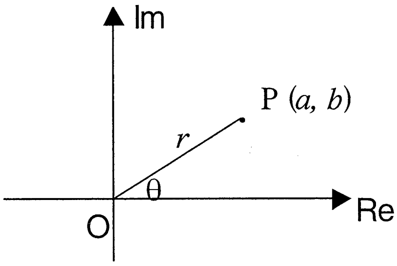
It is preferable to introduce the notion of the conjugate of a complex number before tackling division.

If $z = a + ib$ then the **conjugate** of z is $a - ib$ and is denoted by $\bar{z}$

Note that $z\bar{z} = (a + ib)(a - ib) = a^2 -iab + iab - i^2b^2 = a^2 + b^2$, i.e. $z\bar{z}$ is real.

$z_1 \div z_2 = \frac{a+ib}{c+id}$ is simplified by multiplying both numerator and denominator by the complex conjugate of the denominator: $\frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$

The modulus and argument of a complex number



P(a, b) represents $z = a + ib$.

r, the length of OP is called the **modulus** of z. We denote the modulus of z by $|z|$.

Thus $|z| = \sqrt{a^2 + b^2}$

The angle, θ, that OP makes with the positive direction of the Real axis is called the **argument** of z, commonly written as arg(z)

Thus **arg**$(z) = \theta = \tan^{-1}\frac{b}{a}$. Commonly $-\pi < \theta \leq \pi$

Cartesian/polar form

$z = a + ib$ is termed the **Cartesian** form of the complex number z.

Now $a = r\cos\theta$ and $b = r\sin\theta$ so z can be expressed in the form $z = r(\cos\theta + i\sin\theta)$. This is the **polar** form of z.

The modulus and argument of a product of complex numbers

Suppose that $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\begin{aligned} z_1 \times z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2([\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2] + i[\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2]) \\ &= r_1 r_2(\cos[\theta_1 + \theta_2] + i\sin[\theta_1 + \theta_2]) \end{aligned}$$

Thus $|z_1 \times z_2| = r_1 r_2 = |z_1| \times |z_2|$ and $\arg(z_1 \times z_2) = \arg(z_1) + \arg(z_2)$.

It is easily shown that:

$|z_1 \times z_2 \times z_3 \times \ldots \times z_n| = |z_1| \times |z_2| \times |z_3| \times \ldots \times |z_n|$, and

$\arg(z_1 \times z_2 \times z_3 \times \ldots \times z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \ldots + \arg(z_n)$.

For the special case where $z_1 = z_2 = z_3 = \ldots = z_n = z$

$|z^n| = |z|^n$ and $\arg(z^n) = n \times \arg(z)$ ie $z^n = r^n(\cos n\theta + i\sin n\theta)$

The modulus and argument of iz

$|iz| = |i| \times |z| = 1 \times |z| = |z|$ and $\arg(iz) = \arg(i) + \arg(z) = \frac{\pi}{2} + \arg(z)$

Thus multiplying by i is equivalent to a rotation of $\frac{\pi}{2}$ about the origin.

The modulus and argument of $\bar{z}$

If $z = a + ib$ then $\bar{z} = a - ib$ and hence,

$$|\bar{z}| = a^2 + (-b)^2 = a^2 + b^2 = |z| \text{ and } \arg(\bar{z}) = \tan^{-1}\frac{(-b)}{a} = -\tan^{-1}\frac{b}{a} = -\arg(z)$$

Thus "$\bar{z}$" is the image of "z" under reflection in the Real axis.

The modulus and argument of $z\bar{z}$

$z\bar{z} = a^2 + b^2$ thus $z\bar{z}$ is real, $|z\bar{z}| = |z| \times |\bar{z}| = |z|^2$ and $\arg(z\bar{z}) = 0$

Examples

1. Simplify :

   a) $(2 + 3i)(i - 2)$ answer: $(2 + 3i)(i - 2) = 2i - 4 + 3i^2 - 6i = -7 - 4i$

   b) $\dfrac{4}{3+i}$ answer: $\dfrac{4}{3+i} = \dfrac{4(3-i)}{(3+i)(3-i)} = \dfrac{12-4i}{9+1} = \dfrac{6}{5} - \dfrac{2}{5}i$

2. Express $z = 3 + 4i$ in polar form

   answer: $a = 3$ and $b = 4$, $\therefore |z| = \sqrt{3^2 + 4^2} = 5$, argument $= \tan^{-1}\dfrac{4}{3} = 53{\cdot}1°$

   $\therefore z = 5(\cos 53{\cdot}1° + i\sin 53{\cdot}1°)$

3. Express $z = \sqrt{2}(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4})$ in Cartesian form

   answer: $z = \sqrt{2}\left(\dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}\right) = 1 + i$

4. Express $\dfrac{-8}{i - \sqrt{3}}$ in polar form (hint: change to Cartesian form first)

$$\frac{-8}{i-\sqrt{3}} = \frac{-8(i+\sqrt{3})}{(i-\sqrt{3})(i+\sqrt{3})} = \frac{-8(i+\sqrt{3})}{-4} = 2(i+\sqrt{3})$$

Modulus = $\sqrt{16} = 4$, argument $= \tan^{-1}\dfrac{2}{2\sqrt{3}} = \dfrac{\pi}{6}$ $\therefore z = 4(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6})$

Exercises

1.	Orlando Gough	Page 347	Ex.6.5:2 Q. 1–33 (Select)
		Page 356	Ex.6.5:6 Q. 1– 8 (Select)
		Page 357	Ex.6.5:7 Q. 1– 8 (Select)
2.	Sadler & Thorning	Page 463	Ex. 18D Q. 1 – 5
		Page 467	Ex. 18E Q. 3 – 7 (select)
3.	Hunter	Page 37	Ex. 4.1 Q. 1 – 4
		Page 40	Ex. 4.2 Q. 5,6,9

CONTENT
know the fundamental theorem of algebra and the conjugate roots property
factorise polynomials with real coefficients *e.g. Find the roots of a quartic when one complex root is given*
solve simple equations involving a complex variable by equating real and imaginary parts

Comments

This content was contained in CSYS Paper 1.

The Fundamental Theorem of Algebra

The fundamental theorem of algebra states that every **polynomial** equation of the form $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + .. + a_2x^2 + a_1x + a_0 = 0$ (where $a_n, a_{n-1}, \ldots.\ a_2, a_1, a_0$ are complex numbers) has at least one complex root. Using the Remainder Theorem, it can be proved by induction that the equation has n complex roots.

If the roots are $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots.\ \alpha_{n-1}, \alpha_n$ (Some or all of these may be real and not all need be distinct) then it follows that

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 = a_n(x - \alpha_1)(x - \alpha_2)\ldots..(x - \alpha_{n-2})(x - \alpha_{n-1})(x - \alpha_n)$$

Since the real numbers form a subset of the complex numbers the above results are true when the coefficients are all real.

Conjugate Root Property

When the coefficients are real it further follows that if $z = a + ib$ is a root of the polynomial equation then $\bar{z} = a - ib$ will also be a root of the equation.

Proof

Suppose that z is a root of $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 = 0$ **where the coefficients are real** then $a_nz^n + a_{n-1}z^{n-1} + \ldots + a_2z^2 + a_1z + a_0 = 0$

Taking the complex conjugates of each side we have

$$\bar{a}_n\overline{z^n} + \bar{a}_{n-1}\overline{z^{n-1}} + \ldots\ldots. + \bar{a}_2\overline{z^2} + \bar{a}_1\bar{z} + \bar{a}_0 = \bar{0}$$

$$\Rightarrow \quad a_n(\bar{z})^n + a_{n-1}(\bar{z})^{n-1} + \ldots..a_2(\bar{z})^2 + a_1\bar{z} + a_0 = 0$$

(since a_r *is* real for $r = 0$ to n, then $\bar{a}_r = a_r$ for $r = 0$ to n, and $\overline{z^r} = \bar{z}^r$)

Hence $\bar{z}$ is also a root of the equation. z and $\bar{z}$ are known as **conjugate roots.**

Factorising Polynomials with Real Coefficients

Additionally, if $z = a + ib$ and $\bar{z} = a - ib$ are roots of the equation, then $(x - z)$ and $(x - \bar{z})$ are factors of the polynomial and $(x - z)(x - \bar{z})$ $= x^2 - (z + \bar{z})x + z\bar{z} = x^2 - 2ax + (a^2 + b^2)$ is a quadratic factor with real coefficients. This result is used to factorise polynomials with real coefficients.

Examples

1. Given that $z = 1 - i$ is a root of the polynomial equation $z^4 + 4z^3 - 8z + 20 = 0$, find the other roots.

If $z = 1 - i$ is a root then so is $z = 1 + i$ and thus $(z - 1 + i)$ and $(z - 1 - i)$ are factors of $z^4 + 4z^3 - 8z + 20$.

Hence, $(z - 1 + i)(z - 1 - i) = z^2 - 2z + 2$ is a factor of $z^4 + 4z^3 - 8z + 20$

Dividing $z^4 + 4z^3 - 8z + 20$ by $z^2 - 2z + 2$

$$
\begin{array}{r|rrrrr}
 & z^2 + & 6z + & 10 & & \\
\hline
z^2 - 2z + 2 & z^4 + & 4z^3 + & 0z^2 & - 8z & + 20 \\
 & z^4 - & 2z^3 + & 2z^2 & & \\
\cline{2-4}
 & & 6z^3 - & 2z^2 & - 8z & \\
 & & 6z^3 - & 12z^2 & + 12z & \\
\cline{3-5}
 & & & 10z^2 & - 20z & + 20 \\
 & & & 10z^2 & - 20z & + 20 \\
\cline{4-6}
 & & & & & 0
\end{array}
$$

Thus the other quadratic factor is $z^2 + 6z + 10$

For $z^2 + 6z + 10 = 0$, $z = \dfrac{-6 \pm \sqrt{36 - 40}}{2} = \dfrac{-6 \pm 2i}{2} = -3 \pm i$.

Hence, roots of $z^4 + 4z^3 - 8z + 20 = 0$ are $1 \pm i$ and $-3 \pm i$

2. Find the 'square roots' of $5 + 12i$

Let $z = a + ib$ be a 'square root' of $5 + 12i$

$(a + ib)^2 = 5 + 12i \Rightarrow a^2 + 2iab - b^2 = 5 + 12i$

Equating real and imaginary parts we have:

$a^2 - b^2 = 5$ and $2ab = 12 \Rightarrow ab = 6 \Rightarrow a = \dfrac{6}{b}$

$$\left(\frac{6}{b}\right)^2 - b^2 = 5 \quad \Rightarrow 36 - b^4 = 5b^2$$

$$\Rightarrow b^4 + 5b^2 - 36 = 0$$
$$\Rightarrow (b^2 + 9)(b^2 - 4) = 0$$
$$\Rightarrow (b^2 + 9)(b - 2)(b + 2) = 0$$

since $b^2 + 9 \neq 0$ $\quad b = -2$ (and so $a = -3$) or $b = 2$ (and so $a = 3$)

Thus the 'square roots' are $3 + 2i$ and $-3 - 2i$

Exercises

1.	Orlando Gough	Page 351	Ex.6.5:3 Q. 1–12,15-20, 23-27 (select)
		Page 354	Ex.6.5:5 Q. 1– 26 (select)
2.	Sadler & Thorning	Page 463	Ex. 18D Q. 6 – 13 (Select)
3.	Hunter	Page 37	Ex. 4.1 Q. 5; Page 40 Ex. 4.2 Q. 7,8

Note: Sadler & Thorning does not cover the 'fundamental theorem of algebra'.

CONTENT
interpret geometrically certain equations or inequations in the complex plane *e.g.* $\mid z \mid = 1$, $\mid z - a \mid = b$, $\mid z - 1 \mid = \mid z - i \mid$, $\mid z - a \mid > b$

Teaching Notes

Geometrical interpretation of equations and inequalities

The solution sets of complex equations and inequalities can be represented by sets of points in an Argand diagram. For example:

(a) $\mid z \mid = 1$

If $z = x + iy$ and $\mid z \mid = 1$ then $\sqrt{x^2 + y^2} = 1$

Thus all the points representing complex numbers with modulus = 1 lie on the circumference of a circle, centre the origin and radius 1.

$\mid z \mid = 1$ is called the **complex equation** of this circle

(b) $\mid z - a \mid = b$

If $z = x + iy$ then $z - a = (x - a) + iy$

so $\mid z - a \mid = b \Rightarrow (x - a)^2 + y^2 = b^2$

Thus all the points representing complex numbers z, for which $\mid z - a \mid = b$ lie on the circumference of a circle, centre $(a, 0)$ and radius b.

$\mid z - a \mid = b$ is the complex equation of this circle

(c) $\mid z - 1 \mid = \mid z - i \mid$

If $z = x + iy$ then $z - 1 = (x - 1) + iy$ and $z - i = x + (y - 1)i$.

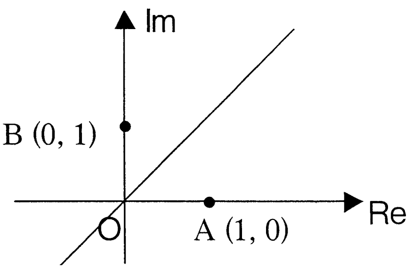
so $\mid z - 1 \mid = \mid z - i \mid \Rightarrow (x - 1)^2 + y^2 = x^2 + (y - 1)^2$

$\Rightarrow x^2 - 2x + 1 + y^2 = x^2 + y^2 - 2y + 1$

$\Rightarrow x = y$

Alternatively, using an Argand diagram

1 is represented by A (1, 0)
i is represented by B (0, 1)
$\mid z - 1 \mid = \mid z - i \mid \Rightarrow$ point representing z is equidistant from A and B ie lies on the perpendicular bisector of AB, ie $y = x$



Further examples are to be found in the recommended textbooks with the exception of Orlando Gough which does not cover the geometric interpretation of 'certain inequalities or inequalities in the complex plane'.

CONTENT
know and use de Moivre's Theorem with positive integer indices and **fractional indices [A/B]** *e.g.* $(\cos\theta + i\sin\theta)^3$ **$(\cos\theta + i\sin\theta)^{1/2}$ [A/B]** **apply de Moivre's theorem to multiple angle trigonometric formulae [A/B]** *e.g.* ***Express $\sin 5\theta$ in terms of $\sin\theta$ only. [A/B]*** ***Express $\cos^3\theta$ in terms of $\cos\theta$ and $\cos 3\theta$. [A/B]*** **apply de Moivre's theorem to find nth roots of unity [A/B]**

Comments

This content, apart from the last statement, which was in CSYS Paper 2 (Unrevised), was contained in CSYS Paper 1.

Teaching notes

De Moivre's theorem

Given $z = \cos\theta + i\sin\theta$ (The locus of such points is $x^2 + y^2 = 1$)

$$z^2 = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) = \cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta$$
$$= \cos^2\theta - \sin^2\theta + i\,2\sin\theta\cos\theta$$
$$= \cos 2\theta + i\sin 2\theta$$

$$z^3 = (\cos 2\theta + i\sin 2\theta)(\cos\theta + i\sin\theta)$$
$$= \cos 2\theta\cos\theta + i^2\sin 2\theta\sin\theta + i\cos 2\theta\sin\theta + i\sin 2\theta\cos\theta$$
$$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta + i(\cos 2\theta\sin\theta + \sin 2\theta\cos\theta)$$
$$= \cos 3\theta + i\sin 3\theta$$

This leads to : $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where n is an integer

and $r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$, where n is an integer

These results are proved, later, by induction for integer powers. They are in fact true for rational powers.

De Moivre's theorem also gives a method for finding the n^{th} roots of a complex number.

Examples

1. Evaluate $(1-i)^7$

Let $z = 1 - i$ (Cartesian form) $= \sqrt{2}\,(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right))$

$$z^7 = (\sqrt{2}\,(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)))^7$$

$$= (\sqrt{2})^7 \cos\left(-\frac{7\pi}{4}\right) + i\sin\left(-\frac{7\pi}{4}\right))$$

$$= 8\sqrt{2}\,(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)) = 8\sqrt{2}\,(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) = 8 + 8i$$

2. Evaluate $z = \left(4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})\right)^{\frac{1}{2}}$

$$z = 4^{\frac{1}{2}}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \pm 2(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = \sqrt{3} + i \text{ or } -\sqrt{3} - i$$

3. Express $\sin 5\theta$ in terms of $\sin\theta$.

Applying de Moivre's theorem to $(\cos\theta + i\sin\theta)^5$

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$$

Applying the Binomial Theorem to $(\cos\theta + i\sin\theta)^5$

$$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta$$
$$+10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$$

Equating imaginary parts:

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$= 5(\cos^2\theta)^2\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$= 5(1 - \sin^2\theta)^2\sin\theta - 10(1 - \sin^2\theta)\sin^3\theta + \sin^5\theta$$
$$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 11\sin^5\theta$$

i.e. $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$

4. Express $\cos^3\theta$ in terms of $\cos\theta$ and $\cos 3\theta$

Let $z = \cos\theta + i\sin\theta$, then

$$\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} = \frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta$$

Thus $z + \frac{1}{z} = 2\cos\theta$ and so $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$ and

$$z^k + (\frac{1}{z})^k = (\cos\theta + i\sin\theta)^k + (\cos\theta - i\sin\theta)^k = (\cos\theta + i\sin\theta)^k + (\cos(-\theta) - i\sin(-\theta))^k$$

$$= (\cos k\theta + i\sin k\theta) + (\cos(-k\theta) - i\sin(-k\theta)) = 2\cos k\theta$$

Hence $\cos^3\theta = [\tfrac{1}{2}(z + \frac{1}{z})]^3 = \frac{1}{8}[z^3 + 3z^2(\frac{1}{z}) + 3z(\frac{1}{z})^2 + (\frac{1}{z})^3]$

$$= \frac{1}{8}[z^3 + 3z + 3(\frac{1}{z}) + (\frac{1}{z})^3]$$

$$= \frac{1}{8}[z^3 + (\frac{1}{z})^3 + 3z + 3(\frac{1}{z})]$$

$$= \frac{1}{8}(2\cos 3\theta + 3\times 2\cos\theta) = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$$

5. If $z = -8i$, find the cube roots of z.

We firstly express z in the form $\cos\theta + i\sin\theta$.

Taking $z = 8(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right))$

we get $z^{\frac{1}{3}} = 8^{\frac{1}{3}}(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right))^{\frac{1}{3}}$

This gives us only one solution and the fundamental theorem of algebra tells us that there are three roots.

Consider, instead, $z = 8(\cos(2n\pi - \frac{\pi}{2}) + i\sin(2n\pi - \frac{\pi}{2}))$ which takes the value $-8i$ for all integral values of n.

In this case $z^{\frac{1}{3}} = 2\left(\cos(2n\pi - \frac{\pi}{2}) + i\sin(2n\pi - \frac{\pi}{2})\right)^{\frac{1}{3}}$

We now take $n = 0$, $n = 1$ and $n = 2$ giving

$n = 0$: $\quad 2(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)) = \sqrt{3} - i$

$n = 1$: $\quad 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 2i$

$n = 2$: $\quad 2(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}) = -\sqrt{3} - i$

Thus there are three distinct roots $\sqrt{3} - i$, $2i$, and $-\sqrt{3} - i$

Had we taken $n = 3$, $n = 4$ and $n = 5$ these would have yielded $\sqrt{3} - i$, $2i$, and $-\sqrt{3} - i$ respectively. As would have been the case had we taken $n = 6$, $n = 7$ and $n = 8$ etc.

6. Find the n^{th} roots of unity ie the solutions to the equation $z^n = 1$

$z^n = 1 \Rightarrow |z^n| = 1 \Rightarrow |z|^n = 1 \Rightarrow |z| = 1$

Then z can be expressed in the form $\cos\theta + i\sin\theta$ for some θ

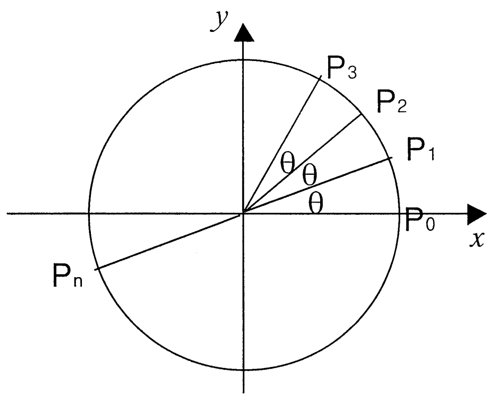
Now $\quad z^2 = (\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

$z^3 = (\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

……………………..

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Thus if the points P_r for $r = 1$ to n represent z^r for $r = 1$ to n respectively, then they lie, equally spaced, on the circle, centre the origin, radius 1 and $\angle P_0OP_r = r\theta$ as shown in the Argand diagram on the right.



Now $z^n = 1 \Leftrightarrow P_n$ coincides with $P_0 \Rightarrow n\theta = 2\pi \Rightarrow \theta = \frac{2\pi}{n}$.

Thus $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ is an n^{th} root of unity.

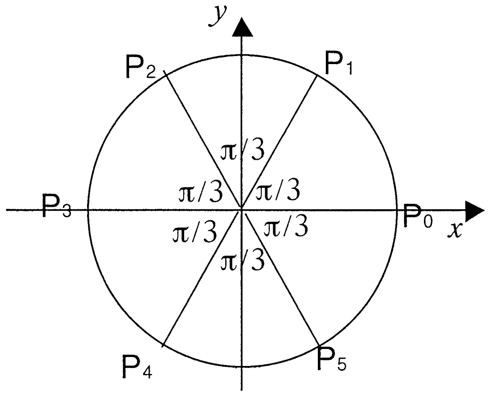
There should of course be n n^{th} roots of unity.

Clearly, since $1^n = 1$, 1 is an n^{th} root of unity.

Now $(z^r)^n = (z^n)^r$ hence:

z is a root of unity $\Rightarrow z^n = 1 \Rightarrow (z^n)^r = 1 \Rightarrow (z^r)^n = 1 \Rightarrow z^r$ is a root of unity.

Thus the n n^{th} roots of unity are represented by n equally spaced points on the unit circle, including the point (1,0).



For example the 6^{th} roots of unity are represented by the points P_0, P_1, P_2, P_3, P_4, P_5 in the Argand diagram on the left.

Exercises

1.	Orlando Gough	Page 359	Ex.6.5:8 Q. 1–13 (select)
2.	Sadler & Thorning	Page 476	Ex. 18G Q. 19 -22
3.	Hunter	Page45	Ex. 4.3 Q. 1 – 10 (select)

Note that Orlando Gough does not cover the application of de Moivre's Theorem to multiple angle trig formulae. Neither Orlando Gough nor Sadler & Thorning mention n^{th} roots of unity, although both contain some examples.

Note

1. The n^{th} roots of unity can be found using de Moivre's Theorem as illustrated in example 5 above.
   In this case we would take $z = \cos 2n\pi + i\sin 2n\pi$

2. The exponential form $z = re^{i\theta}$, which provides a link with power series, is addressed in the Teachers' Notes for Mathematics 3(AH).

SEQUENCES AND SERIES

CONTENT
know the meaning of the terms infinite sequence, infinite series, n^{th} term, sum to n terms (partial sum), limit, sum to infinity (limit to infinity of the sequence of partial sums), common difference, arithmetic sequence, common ratio, geometric sequence, recurrence relation
know and use the formulae $u_n = a + (n-1)d$ and $S_n = \frac{n}{2}(2a + (n-1)d)$ for the n^{th} term and the sum to n terms of an arithmetic sequence, respectively.
know and use the formulae $u_n = ar^{n-1}$ and $S_n = \frac{a(1-r^n)}{1-r}, r \neq 1,$ for the n^{th} term and the sum to n terms of a geometric series, respectively.
know and use the condition on r for the sum to infinity to exist and the formula $S_\infty = \frac{a}{1-r}$ for the sum to infinity of a geometric series where $\lvert r \rvert < 1$

Comments

This content was contained in CSYS Paper 1.

Teaching notes

Higher Mathematics involved the study of recurrence relations of the form $u_{n+1} = mu_n + c$ and the sequences they produced.

In this section we consider two particular types of sequence arising from this form of recurrence relation.

First Type

$u_{n+1} = mu_n + c$ with $u_1 = a$, $m = 1$ and c is any number d

The recurrence relation is then defined by $u_{n+1} = u_n + d$, $u_1 = a$ which gives rise to the ***infinite sequence*** $a, a + d, a + 2d, a + 3d, \ldots\ldots$ where $\boldsymbol{u_n = a + (n - 1)d}$

e.g. if we take $a = 5$ and $d = 3$ then we obtain the sequence 5, 8, 11, 14, ...
if we take $a = 20$ and $d = -5$ then we obtain the sequence 20, 15, 10, 5, ...

Since $u_{n+1} - u_n = d$ there is a ***common difference*** between terms (hence the reason for using d). Such sequences are called ***arithmetic sequences.***

$a, a + d, a + 2d, a + 3d, \ldots\ldots$ is called the general arithmetic sequence

The sum of the terms of an arithmetic sequence is called an ***arithmetic series.***
$a + (a + d) + (a + 2d) + (a + 3d), \ldots\ldots$ is called the general arithmetic series.
and the ***n^{th} term*** of the series is $\boldsymbol{u_n = a + (n - 1)d}$.

Particular examples are $5 + 8 + 11 + 14 + \ldots.$ and $20 + 15 + 10 + 5 + \ldots.$

The sum to n terms (partial sum) of the series is denoted by $\boldsymbol{S_n}$.

i.e. $S_n = a + (a + d) + (a + 2d) + (a + 3d), \ldots\ldots + (a + (n - 1)d)$

Now	$\boldsymbol{S_n}$ =	a	+	$(a+d)$	+	.....	+	$(a+[n-2]d$	+	$a+[n-1]d$
also	$\boldsymbol{S_n}$ =	$a+[n-1]d$	+	$(a+[n-2]d$	+	......	+	$(a+d)$	+	a
adding	$\boldsymbol{2S_n}$ =	$2a+[n-1]d$	+	$2a+[n-1]d$	+	......	+	$2a+[n-1]d$	+	$2a+[n-1]d$

Thus $S_n = \frac{n}{2}(2a + (n-1)d)$

e.g. for $5 + 8 + 11 + 14 + \ldots.\ a = 5, \quad d = 3$ so $S_n = \frac{n}{2}(2a + (n-1)d) = \frac{1}{2}n(3n + 7)$

Example: Find the sum of 4 + 9 + ... 84

First we need to find the value of n.

$u_n = a + (n - 1)d \Rightarrow 84 = 4 + (n - 1)5 \Rightarrow 85 = 5n \Rightarrow n = 17$

$S_{17} = \frac{17}{2}(2\times 4 + 16\times 5) = 748$

Exercises

1.	Orlando Gough	Page 218	Ex. 5.1:3 Q. 1 - 7, 12 -17, 23 – 26, 30
2.	Sadler & Thorning	Page 210	Ex. 8A Q. 1 – 30 (select)
3.	Hunter	Page 25	Ex. 3.1 Q. 4 – 9

Note that Sadler & Thorning does not cover recurrence relations.

Second Type

$u_{n+1} = mu_n + c$ with $u_1 = a$, $m = r$ and $c = 0$

The recurrence relation is then defined by $u_{n+1} = ru_n$, $u_1 = a$ which gives rise to the infinite sequence $a, ar, ar^2, ar^3, \ldots\ldots$ where the ***n^th^ term*** is given by $\boldsymbol{u_n = ar^{n-1}}$

e.g. if we take $a = 3$ and $r = 2$ then we obtain the sequence 3, 6, 12, 24

For this type of sequence $u_{n+1} / u_n = r$, i.e. there is a ***common ratio*** between terms (hence the reason for using r). Such sequences are called ***geometric sequences.***

$a, ar, ar^2, ar^3, \ldots\ ar^{n-1}, \ldots..$ is called the general geometric sequence.

The sum of the terms of a geometric sequence is called a ***geometric series.***

and $a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$, ...is called the general geometric series.

The sum to n terms (partial sum) of the series is denoted by S_n.

i.e. $S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

Now	S_n	=	a	+	ar	+ ..... +	ar^{n-2}	+	ar^{n-1}		
also	rS_n	=			ar	+ ...... +	ar^{n-2}	+	ar^{n-1}	+	ar^n
subtracting	$(1-r)S_n$	=	a		0	+ ...... +	0	+	0		$-ar^n$

Thus $S_n = \dfrac{a(1-r^n)}{1-r}, r \neq 1,$

e.g. for 3, 6, 12, 24, ... $a = 3$, $r = 2$ $u_n = 3.2^{n-1}$ and $S_n = \dfrac{a(1-r^n)}{1-r} = 3(2^n - 1)$

Example: Find the first terms and common ratios of two possible sequences whose fourth terms are 4 and whose eighth terms are 0·25.

$ar^7 = 0.25$ (i) and $ar^3 = 4$ (ii)

Dividing (i) by (ii) we get $r^4 = \dfrac{1}{16} \Rightarrow r = \dfrac{1}{2}$ or $-\dfrac{1}{2}$,

giving $a = \dfrac{4}{0.125} = 32$ when $r = \dfrac{1}{2}$ and $a = -32$ when $r = -\dfrac{1}{2}$

Sum to infinity of a geometric series

We have found that $S_n = \dfrac{a(1-r^n)}{1-r}, r \neq 1,$

Now, if $-1 < r < 1$ then $r^n \to 0$ as $n \to \infty$ and so $S_n \to \dfrac{a}{1-r}$ as $n \to \infty$

We write $S_\infty = \dfrac{a}{1-r}$ and call S_∞ the ***sum to infinity of the geometric series*** or alternatively the ***limit to infinity of the sequence of partial sums***.

Example: Find the sum to infinity of the series $2 + 1 + \dfrac{1}{2} + \dfrac{1}{4} + \ldots$.

$a = 2$ and $r = \dfrac{1}{2}$ $\Rightarrow$ $S_\infty = \dfrac{a}{1-r} = \dfrac{2}{1-\frac{1}{2}} = 4$

Exercises

1.	Orlando Gough	Page 221	Ex. 5.1:4 Q. 1-7, 25-27, 34,35
		Page 227	Ex. 5.1:6 Q. 2-6, 9-11
2.	Sadler & Thorning	Page 216	Ex. 8B Q. 1 – 28 (select)
3.	Hunter	Page 25	Ex. 3.1 Q. 10-16, 20-22

CONTENT
Expand $\frac{1}{1-r}$ as a geometric series and **extend to** $\frac{1}{\boldsymbol{a+b}}$ **[A/B]**
know the sequence $\left(1+\frac{1}{n}\right)^n$ and its limit
know and use the $\sum$ notation
know the formula $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ and apply it to simple sums
e.g $\sum_{r=1}^{n}(ar+b) = a\sum_{r=1}^{n} r + \sum_{r=1}^{n} b$

Comments

The last statement of content was contained in CSYS Paper 1. The sequence $\left(1+\frac{1}{n}\right)^n$ and its limit should be explored, although it is unlikely that it will be assessed directly.

Expansion of $\frac{1}{1-r}$ as a geometric series

Consider the series $1 + r + r^2 + r^3 + r^4 + \ldots\ldots\ldots$ where $-1 < r < 1$ i.e. the infinite geometric series with $a = 1$.

This geometric series has a sum to infinity $\frac{1}{1-r}$ i.e. $1 + r + r^2 + r^3 + r^4 + \ldots. = \frac{1}{1-r}$

Writing this 'the other way round' we have $\frac{1}{1-r} = 1 + r + r^2 + r^3 + r^4 + \ldots$

ie provided $|r| < 1$ we can expand $\frac{1}{1-r}$ as the geometric series $1 + r + r^2 + r^3 + r^4 + ..$

Example 1

Find the value of $\frac{1}{0\cdot 99}$ correct to 5 decimal places.

$$\frac{1}{0\cdot 99} = \frac{1}{1-0\cdot 01} = 1 + (0\cdot 01) + (0\cdot 01)^2 + (0\cdot 01)^3 + (0\cdot 01)^4 + .. \quad (\text{since } -1 < 0\cdot 01 < 1)$$

$$= 1 + 0\cdot 01 + 0\cdot 0001 + 0\cdot 000001 + 0\cdot 00000001 + \ldots$$

$$= 1\cdot 01010 \text{ (correct to 5 decimal places)}$$

Example 2

Expand $\frac{1}{2x+1} = \frac{1}{1-(-2x)}$, where $|x| < \frac{1}{2}$

$$\frac{1}{2x+1} = \frac{1}{1-(-2x)} = 1 + (-2x) + (-2x)^2 + (-2x)^3 + \ldots = 1 - 2x + 4x^2 - 8x^3 + \ldots$$

The expansion of $\frac{1}{a+b}$ as a geometric series

(i) If $|b| < |a|$ we express $\frac{1}{a+b}$ as $\frac{1}{a\left(1+\frac{b}{a}\right)}$

Then if we put $r = -\frac{b}{a}$ we have $\frac{1}{a+b} = \frac{1}{a}\frac{1}{1-r}$ and $|r| < 1$

Since $\frac{1}{1-r} = 1 + r + r^2 + r^3 + r^4 + \ldots$ we have

$$\frac{1}{a+b} = \frac{1}{a}\left(1+\left(-\frac{b}{a}\right)+\left(-\frac{b}{a}\right)^2+\left(-\frac{b}{a}\right)^3+\ldots\ldots\ldots\right)$$

$$= \frac{1}{a}\left(1-\left(\frac{b}{a}\right)+\left(\frac{b}{a}\right)^2-\left(\frac{b}{a}\right)^3+\ldots\ldots\right).$$

(ii) If $|a| < |b|$ then we express $\frac{1}{a+b}$ as $\frac{1}{b\left(\frac{a}{b}+1\right)} = \frac{1}{b\left(1+\frac{a}{b}\right)}$

Then if we put $r = -\frac{a}{b}$ we have $\frac{1}{a+b} = \frac{1}{b}\frac{1}{1-r}$ and $|r| < 1$

Hence $\frac{1}{a+b} = \frac{1}{b}\left(1-\left(\frac{a}{b}\right)+\left(\frac{a}{b}\right)^2-\left(\frac{a}{b}\right)^3+\ldots\ldots\right)$

A Further Look at $(x+y)^n$

In Mathematics 1(AH), we encountered the binomial expansion of $(x+y)^n$, viz.

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \ldots \text{where } n \in \mathbf{N}.$$

(1) putting $x = 1, y = -r$ and $n = -1$ in this expression we obtain

$$(1-r)^{-1} = 1 + (-1)(-r) + \frac{(-1)\left((-1)-1\right)}{2!}(-r)^2 + \frac{(-1)\left((-1)-1\right)\left((-1)-2\right)}{3!}(-r)^3 + \ldots$$

$$= 1 + r + r^2 + r^3 + \ldots\ldots$$

Comparing this with the expansion of $\frac{1}{1-r}$ it would appear that the 'binomial expansion' holds when $n = -1$

(2) Putting $x = 1$ and $y = \frac{1}{n}$ in the expression we obtain

$$\left(1+\frac{1}{n}\right)^n = 1^n + n1^{n-1}(n^{-1}) + \frac{n(n-1)}{2!}1^{n-2}(n^{-1})^2 + \frac{n(n-1)(n-2)}{3!}1^{n-3}(n^{-1})^3 + \ldots$$

$$= 1 + 1 + \frac{(n-1)}{2!n} + \frac{(n-1)(n-2)}{3!n^2} + \ldots.$$

$$= 2 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \frac{1}{4!}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n}) + \ldots\ldots$$

So, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots\ldots$

$$= 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \ldots. = 2{\cdot}718 \text{ (correct to 3dp)}$$

The sum is in fact the non-recurring decimal number which we denote by *e*.

$$\text{i.e. } e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Exercises

1. Sadler & Thorning Page 231 Ex. 8F Q. 1 – 5

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ (A proof of this result will be met in the next section.)

Examples

1. $\sum_{r=1}^{10} r = \frac{1}{2}\times 10\times(10+1) = \frac{1}{2}\times 10\times 11 = 55$
2. $\sum_{r=1}^{8} 2r = 2\sum_{r=1}^{8} r = 2\times\frac{1}{2}\times 8\times 9 = 8\times 9 = 72$
3. $\sum_{r=1}^{6}(r+3) = \sum_{r=1}^{6} r + \sum_{r=1}^{6} 3 = \frac{1}{2}\times 6\times 7 + 3\times 6 = 21+18 = 39$ $[\sum_{r=1}^{6} 3 = 3+3+3+3+3+3]$
4. $\sum_{r=1}^{5}(2r-3) = 2\sum_{r=1}^{5} r - \sum_{r=1}^{5} 3 = 2\times\frac{1}{2}\times 5\times 6 - 3\times 5 = 30 - 15 = 15$
5. $\sum_{r=4}^{9} r = \sum_{r=1}^{9} r - \sum_{r=1}^{3} r = \frac{1}{2}\times 9\times 10 - \frac{1}{2}\times 3\times 4 = 45 - 6 = 39$
6. $\sum_{r=1}^{n}(2r-1) = 2\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 = n(n+1) - n = n^2$

   [ie the sum of the first n odd numbers is a perfect square]

Exercises

1. Sadler & Thorning Page 220 Ex. 8C Q. 1a,c,2a,c,d,4a,4d,10a,d

ELEMENTARY NUMBER THEORY AND METHODS OF PROOF

CONTENT
understand the nature of mathematical proof *simple examples involving the real number line, inequalities and modulus.* *e.g.* $\lvert x+y \rvert \leq \lvert x \rvert + \lvert y \rvert$
understand and make use of the notations $\Rightarrow$, $\Leftarrow$ and $\Leftrightarrow$
know the corresponding terminology implies, implied by, equivalence
know the terms natural number, prime number, rational number, irrational number
know and use the fundamental theorem of arithmetic
disprove a conjecture by providing a counter-example
use proof by contradiction in simple examples *e.g. the irrationality of* $\sqrt{2}$*, the infinity of primes.*

Comments
Knowledge of notation and terminology will be assumed in the wording of questions.

Notation and Terminology
A statement of the form 'if........then ....' is called an **implication**
e.g. if $x = 3$ then $2x + 1 = 7$.

The shorthand notation for 'if p then q' is $p \Rightarrow q$
Similarly $p \Rightarrow q$ can be read as $q \Leftarrow p$ which means that q **is implied by** p. Where statements $p \Rightarrow q$ and $q \Rightarrow p$ are both true then we write $p \Leftrightarrow q$
i.e. p is **equivalent** to q.

E.g. $x = 3 \Leftrightarrow 2x + 1 = 7$

Sets of numbers

N = the set of natural numbers = {1, 2, 3, 4,....}
Z = the set of integers = {....-3, -2, -1, 0, 1, 2, 3, ...}

Q = the set of rational numbers $= \{\frac{m}{n} : m \in \mathbf{Z}, n \in \mathbf{Z}, n \neq 0\}$

Irrational numbers are real numbers that cannot be expressed in the form $\frac{m}{n} : m \in \mathbf{Z}$
One such number is $\sqrt{2}$. We prove that $\sqrt{2}$ is irrational later.

Any positive integer $p > 1$ whose only positive divisors are 1 and p is called a ***prime number.***

The *fundamental theorem of arithmetic* states that every positive integer can be expressed uniquely as the product of one or more primes.

e.g. $360 = 2.2.2.3.3.5 = 2^3.3^2.5$

$416 = 2.2.2.2.2.13 = 2^5.13$

$37 = 37$

MATHEMATICAL PROOFS

In the course of carrying out investigations we make conjectures. For example,in the course of investigating the relationship between the modulii of a pair of numbers, x and y and the modulus of their sum we might make the conjecture:

$$|x+y| \leq |x|+|y|$$

Such conjectures have then to be proved (or disproved as the case may be).

One form of proof is to work from the truth of one or more already known statements to the statement we wish to prove true. Linking statements in this way provides a direct proof.

For example, prove that $|x+y| \leq |x|+|y|$

We know that

(a) $|x| = +\sqrt{x^2}$

(b) $|xy| = |x||y|$

and, since one the following must be true: $xy > 0$ or $xy < 0$ or $xy = 0$, that

(c) $xy \leq |xy|$

(b) and (c) are true $\Rightarrow xy \leq |x||y|$

$\Rightarrow x^2 + 2xy + y^2 \leq x^2 + 2|x||y| + y^2$

$\Rightarrow x^2 + 2xy + y^2 \leq |x|^2 + 2|x||y| + |y|^2$ [using (a)]

$\Rightarrow (x+y)^2 \leq (|x|+|y|)^2$

$\Rightarrow \sqrt{(x+y)^2} \leq \sqrt{(|x|+|y|)^2}$

$\Rightarrow |x+y| \leq |x|+|y|$

Sometimes it may be difficult/impossible to prove a conjecture by direct reasoning as above and we use an **indirect method**. One such method is **proof by contradiction**. In this method we make the assumption that the statement we are trying to prove true is in fact false. We then proceed, by means of a direct argument to obtain a contradiction. Since this is wrong, our earlier assumption must be false and so the statement must be true.

Prove that $\sqrt{2}$ is irrational

The statement we wish to prove true is "$\sqrt{2}$ is irrational".

We assume that this statement is false i.e. that "$\sqrt{2}$ is rational".

$\sqrt{2}$ is rational $\Leftrightarrow \sqrt{2} = \dfrac{m}{n}$ $\quad m, n \in \mathbf{Z}$ and m and n have no common factor

$\Rightarrow 2 = \dfrac{m^2}{n^2}$

$\Rightarrow m^2 = 2n^2$

$\Rightarrow m^2$ is a multiple of 2

$\Rightarrow m$ is a multiple of 2 (since 2 is prime)

$\Rightarrow m = 2p$ for some integer p

$\Rightarrow m^2 = 4p^2$

but $m^2 = 2n^2 \Rightarrow 2n^2 = 4p^2$

$\Rightarrow n^2 = 2p^2$

$\Rightarrow n^2$ is a multiple of 2

$\Rightarrow n$ is a multiple of 2

$\Rightarrow n = 2q$

We assumed that $\sqrt{2} = \dfrac{m}{n}$, m and n have no common factor.

We have reasoned that m and n have a common factor 2.

} hence, contradiction!

Because of this contradiction our initial assumption must be false.

$\therefore \sqrt{2}$ is irrational.

The infinity of Primes

There is an infinite number of primes.

Proof

Suppose that there is a finite number of primes $p_1, p_2, p_3, ..., p_n$ for some n.

Consider the number $N = (p_1 \times p_2 \times p_3 \times .... \times p_n) + 1$.

Now N is not divisible by $p_1, p_2, p_3, ...$ or p_n since on dividing N by any one of them leaves a remainder 1.

But N must have at least one prime factor so there is a prime number other than $p_1, p_2, p_3, ...$ or p_n.

This contradicts our initial assumption which must then be false. Hence there is an infinity of primes.

DISPROVING CONJECTURES

Not all of the conjectures made during investigations are true and sometimes we need to prove that a conjecture is false. For example, mathematicians have been fascinated by prime numbers and many attempts have been made to find an expression which would yield primes. One such attempt was:

$$n^2 + n + 41 \text{ is a prime number for all } n \in \mathrm{N}$$

For $n = 1, 2, 3, 4, 5,....$ the expression yields 43, 47, 53, 61, 71,..... and one might be tempted to accept that it will always yield primes. However this is not the case.

We can easily disprove the statement by considering $n = 41$.
When $n = 41$ the expression yields $41^2 + 41 + 41 = 41 \times 43$.

The initial statement that "$n^2 + n + 41$ is a prime number for <u>all</u> $n \in \mathbf{N}$" can be disproved by giving a ***counter-example***, viz "$n^2 + n + 41 = 41 \times 43$ when $n = 41$".

Exercises

1.	Orlando Gough	Page 2	Ex. 1.1:1 Q. 1,5
2.	Hunter	Page 4	Ex. 1.1 Q. 4b,7
		Page 13	Ex. 2.2 Q. 2a,b,c,d

Note the fundamental theorem of arithmetic is not contained in Sadler & Thorning or Orlando Gough.

CONTENT
use proof by mathematical induction in simple examples prove the following results : $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ - the Binomial theorem for positive integers - de Moivre's theorem for positive integers

Comments

Straightforward proofs could be asked for in assessments without guidance,
e.g. $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all $n \in \mathbf{N}$
e.g. 8^n is a factor of $(4n)!$ for all $n \in \mathbf{N}$
e.g. $n < 2^n$ for all $n \in \mathbf{N}$

Teaching notes

Mathematical induction is often used to prove results containing universal statements involving integers – statements such as " p(n) is true for $n \geq 2$, $n \in \mathbf{Z}$."

The method is as follows:

(a) prove p(l) is true, where l is the lowest value of n
(b) assume p(k) is true, where k is an arbitrary integer $\geq l$
(c) prove p(k+ 1) is true

The inductive reasoning then is as follows:

p(l) is true, so it follows from (c) that p(l+1) is true.
p(l+1) is true, so it follows from (c) that p(l+2) is true.
p(l+2) is true, so it follows from (c) that p(l+3) is true.
and so on ad infinitum.

Examples

1. Show that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ for all $n \in \mathbf{N}$.

- Prove true for n =1.

  When n = 1, LHS = 1 and RHS = $\frac{1}{2} \times 1 \times 2 = 1$ $\quad \therefore$ LHS = RHS

  Hence statement is true for n =1.

- Assume true for $n = k$ i.e. assume $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$.

- Prove true for $n = k + 1$.

  i.e. prove that $\sum_{r=1}^{k+1} r = \frac{1}{2}\overline{k+1}(\overline{k+1}+1) = \frac{1}{2}(k+1)(k+2)$

$$\sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1)$$

$$= \frac{1}{2}k(k+1) + (k+1)$$

$$= \frac{1}{2}(k+1)(k+2)$$

Hence statement is true for $n = k + 1$.

Hence, by induction $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ for all $n \in \mathbf{N}$.

2. Show that for $n \in \mathbf{N}$,

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \ldots.. + \frac{n(n-1)}{2}x^2y^{n-2} + nxy^{n-1} + y^n$$

- Prove true for $n = 1$.

  When, $n = 1$, LHS = $(x+y)^1$ and RHS = $1.x^1 + 1.y^1 = x + y$ $\therefore$ LHS = RHS

  Hence statement is true for $n = 1$.

- Assume true for $n = k$.

i.e. $(x+y)^k = x^k + kx^{k-1}y + \frac{k(k-1)}{2}x^{k-2}y^2 + \ldots.. + \frac{k(k-1)}{2}x^2y^{k-2} + kxy^{k-1} + y^k$

- Prove true for $n = k + 1$.

i.e. prove that $(x+y)^{k+1} = x^{k+1} + (k+1)x^ky + \frac{k(k+1)}{2}x^{k-1}y^2 + \ldots..$

$$+ \frac{k(k+1)}{2}x^2y^{k-1} + (k+1)xy^k + y^{k+1}$$

$(x+y)^{k+1} = (x+y)^k(x+y)$

$= (x^k + kx^{k-1}y + \frac{k(k-1)}{2}x^{k-2}y^2 + \ldots.. + \frac{k(k-1)}{2}x^2y^{k-2} + kxy^{k-1} + y^k)(x+y)$

(this can be simplified to:)

$x^{k+1} + (k+1)x^ky + \frac{k(k+1)}{2}x^{k-1}y^2 + \ldots.. + \frac{k(k+1)}{2}x^2y^{k-1} + (k+1)xy^k + y^{k+1}$

Hence statement is true for $n = k + 1$.

Hence, by induction,

$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + .. + \frac{n(n-1)}{2}x^2y^{n-2} + nxy^{n-1} + y^n$ for $n \in \mathbf{N}$.

3. Show that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for $n \in \mathbf{N}$.

- Prove true for $n = 1$.

  When $n = 1$, LHS = $(\cos\theta + i\sin\theta)$ and RHS = $\cos\theta + i\sin\theta$ $\therefore$ LHS = RHS

  Hence statement is true or $n = 1$.

- Assume true for $n = k$.

  i.e. assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

- Prove true for $n = k + 1$.

  i.e. prove that $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)\\ &= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)\\ &= (\cos k\theta \cos\theta + i\cos k\theta \sin\theta + i\sin k\theta \cos\theta + i^2\sin k\theta \sin\theta)\\ &= \cos k\theta \cos\theta - \sin k\theta \sin\theta + i(\sin k\theta \cos\theta + \cos k\theta \sin\theta)\\ &= \cos(k\theta + \theta) + i\sin(k\theta + \theta)\\ &= \cos(k+1)\theta + i\sin(k+1)\theta\end{aligned}$$

Hence statement is true for $n = k + 1$.

Hence, by induction, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for $n \in \mathbf{N}$.

4. Show that $\sum_{r=3}^{n}(2r-1) = (n-2)(n+2)$ for $n \in \mathbf{N}$, $n \geq 3$.

- Prove true for $n = 3$.

  When $n = 3$, LHS = $2\times3 - 1 = 5$, RHS = $(3-2)(3+2) = 1\times5 = 5$

  $\therefore$ LHS = RHS

  Hence statement is true for $n = 3$.

- Assume true for $n = k$.

  $\sum_{r=3}^{k}(2r-1) = (k-2)(k+2)$

- Prove true for $n = k + 1$.

  i.e prove that $\sum_{r=3}^{k+1}(2r-1) = (k-1)(k+3)$

$$\sum_{r=3}^{k+1}(2r-1) = \sum_{r=3}^{k}(2r-1) + 2(k+1) - 1$$

$= (k-2)(k+2) + 2k + 2 - 1$

$= k^2 - 4 + 2k + 1$

$= k^2 + 2k - 3$

$= (k-1)(k+3)$

Hence statement is true for $n = k + 1$.

Hence, by induction, $\sum_{r=3}^{n}(2r-1) = (n-2)(n+2)$ for $n \in \mathbf{N}, n \geq 3$.

5. Prove that $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all $n \in \mathbf{N}$.

- Prove true for $n = 1$.

  When $n = 1$, LHS $= 1 + 2 = 3$ and RHS $= 2^2 - 1 = 3$ $\therefore$ LHS = RHS

  Hence statement is true for $n = 1$.

- Assume true for $n = k$.

  i.e. assume that $1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$

- Prove true for $n = k + 1$.

  i.e. prove that $1 + 2 + 2^2 + \ldots + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$

  $1 + 2 + 2^2 + \ldots + 2^{k+1} = (1 + 2 + 2^2 + \ldots + 2^k) + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$

  $= 2.\,2^{k+1} - 1 = 2^{k+2} - 1$

  Hence statement is true for $n = k + 1$.

  Hence, by induction, $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all $n \in \mathbf{N}$.

6. Prove that 8^n is a factor of $(4n)!$ for all $n \in \mathbf{N}$.

- Prove true for $n = 1$.

  When $n = 1$, $8^n = 8$ and $(4n)! = 4.3.2.1 = 24$

  8 is a factor of 24, hence statement is true for $n = 1$.

- Assume true for $n = k$.

  i.e. assume that 8^k is a factor of $(4k)!$

- Prove true for $n = k + 1$.

  i.e prove that 8^{k+1} is a factor of $(4(k+1))!$ i.e. of $(4k+4)!$

  $(4k+4)! = (4k+4)(4k+3)(4k+2)(4k+1)(4k)!$

$= 4(k + 1)(4k + 3)2(2k + 1)(4k + 1)(4k)!$

$= 8(4k)!(k + 1)(4k + 3)(2k + 1)(4k + 1)$

Thus $8(4k)!$ is a factor of $(4k + 4)!$ But 8.8^k is a factor of $8(4k)!$

Hence 8^{k+1} is a factor of $(4k + 4)!$

Hence, by induction, 8^n is a factor of $(4n)!$ for all $n \in \mathbf{N}$.

7. Prove that $n < 2^n$ for all $n \in \mathbf{N}$.

- Prove true for $n = 1$.

  When $n = 1$, $2^n = 2$

  Hence statement is true for $n = 1$.

- Assume true for $n = k$.

  i.e. assume that $k < 2^k$

- Prove true for $n = k + 1$.

  i.e. prove that $k + 1 < 2^{k+1}$

  $2^{k+1} = 2.2^k > 2k = k + k \geq k + 1$

  Hence statement is true for $n = k + 1$.

Hence, by induction, $n < 2^n$ for all $n \in \mathbf{N}$.

Exercises

1.	Orlando Gough	Page 235	Ex. 5.2 Q. 1- 18 (select)
2.	Sadler & Thorning	Page 220	Ex. 8C Q. 3,12
3.	Hunter	Page 13	Ex. 2.2 Q. 3a,sc,d,e,f,g,i