Page 3 Answers

a) $\underline{u} = 2\underline{i} - \underline{j} + 3\underline{k}, \underline{v} = \underline{i} + 2\underline{j}$

$$\underline{u} \times \underline{v} = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ 1 & 2 & 0 \end{vmatrix} = -6\underline{i} + 3\underline{j} + 5\underline{k}$$

b) $|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}| \sin(45°) = 4 \times 5 \times \sin(45°) = \frac{20}{\sqrt{2}} = 10\sqrt{2}$

c) $\underline{a} = 3\underline{i} - \underline{j} + 2\underline{k}, \underline{b} = 2\underline{i} + \underline{j} - \underline{k}, \underline{c} = \underline{i} - 2\underline{j} + 2\underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = -\underline{i} + 7\underline{j} + 5\underline{k}$$

$$(\underline{a} \times \underline{b}) \times c = \begin{vmatrix} i & j & k \\ -1 & 7 & 5 \\ 1 & -2 & 2 \end{vmatrix} = 24\underline{i} + 7\underline{j} - 5\underline{k}$$

$$\underline{b} \times \underline{c} = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & -2 & 1 \end{vmatrix} = \underline{i} - 5\underline{j} - 5\underline{k}$$

$$a \times (\underline{b} \times \underline{c}) = \begin{vmatrix} i & j & D \\ 3 & -1 & 2 \\ 0 & -5 & -5 \end{vmatrix} = 15\underline{i} + 15\underline{j} - 15\underline{k}$$

$$(\underline{a} \times \underline{b}) \times c \neq a \times (\underline{b} \times \underline{c})$$

d) $\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}, \underline{b} = 4\underline{i} + 3\underline{j} - \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 15\underline{i} - 10\underline{j} + 30\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(15)^2 + (-10)^2 + (30)^2} = 35$$

Unit vector $u_1 = \frac{3}{7}\underline{i} - \frac{2}{7}\underline{j} + \frac{6}{7}\underline{k}$

Unit vector $u_2 = -\frac{3}{7}\underline{i} + \frac{2}{7}\underline{j} - \frac{6}{7}\underline{k}$

e) i) $\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 3 & -1 & -1 \\ 1 & 4 & -2 \end{vmatrix} = 6\underline{i} + 5\underline{j} + 13\underline{k}$

ii) $\underline{b} \times \underline{a} = \begin{vmatrix} i & j & k \\ 1 & 4 & -2 \\ 3 & -1 & -1 \end{vmatrix} = -6\underline{i} - 5\underline{j} - 13\underline{k}$

iii) $\underline{a} + \underline{b} = 4\underline{i} + 3\underline{j} - 3\underline{k},\ \underline{a} - \underline{b} = 2\underline{i} - 5\underline{j} + \underline{k}$

$(\underline{a} + \underline{b}) \times (\underline{a} - \underline{b}) = \begin{vmatrix} i & j & k \\ 4 & 3 & -3 \\ 2 & -5 & 1 \end{vmatrix} = -12\underline{i} - 10\underline{j} - 26\underline{k}$

Page 4 Answers

a) $\underline{u} \times \underline{v} = \begin{vmatrix} i & j & k \\ -1 & 3 & 4 \\ 3 & 1 & -1 \end{vmatrix} = -7\underline{i} + 11\underline{j} - 10\underline{k}$

$|\underline{u} + \underline{v}| = \sqrt{(-7)^2 + (11)^2 + (-10)^2}$

Area $= 16.4$ units^2

b) i) $\underline{u} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} = -3\underline{i} + \underline{j} - 2\underline{k}$

$\underline{v} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} = 2\underline{i} + 5\underline{j} - 4\underline{k}$

$Area\ of\ triangle = \frac{1}{2}|(\underline{u} \times \underline{v})| = \frac{1}{2}\begin{vmatrix} i & j & k \\ -3 & 1 & -2 \\ 2 & 5 & -4 \end{vmatrix} = \frac{1}{2}|(6\underline{i} - 16\underline{j} - 17\underline{k})| =$
$\frac{1}{2}\sqrt{(6)^2 + (-16)^2 + (-17)^2} = \frac{1}{2}\sqrt{581} = 12.05$ units^2

ii) $\underline{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} = 4\underline{i} - 2\underline{j} - \underline{k}$

$\underline{v} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} = 5\underline{i} - \underline{j} + 2\underline{k}$

$Area\ of\ triangle = \frac{1}{2}\left|(\underline{u} \times \underline{v})\right| = \frac{1}{2}\begin{vmatrix} i & j & k \\ 4 & -2 & -1 \\ 5 & -1 & 2 \end{vmatrix} = \frac{1}{2}\left|(-5\underline{i} - 13\underline{j} + 6\underline{k})\right| =$
$\frac{1}{2}\sqrt{(-5)^2 + (-13)^2 + (6)^2} = \frac{1}{2}\sqrt{230} = 7.06 \text{ units}^2$

Page 5 Answers

a) i) $\underline{r} \cdot (\underline{s} \times \underline{t}) = \begin{vmatrix} 2 & 1 & 3 \\ -3 & 4 & -1 \\ -1 & 3 & -2 \end{vmatrix} = 2(-8+3) - (6-1) + 3(-9+4) = -10 - 5 - 15 =$
-30

ii) $\underline{s} \cdot (\underline{t} \times \underline{u}) = \begin{vmatrix} -3 & 4 & -1 \\ -1 & 3 & -2 \\ 5 & -2 & 1 \end{vmatrix} = -3(3-4) - 4(-1+10) - (2-15) = 3 - 36 + 13 =$
-20

iii) $\underline{u} \cdot (\underline{s} \times \underline{r}) = \begin{vmatrix} 5 & -2 & 1 \\ -3 & 4 & -1 \\ 2 & 1 & 3 \end{vmatrix} = 5(12+1) - 2(-9+2) - (-3-8) = 65 - 14 - 11 =$
40

b) $\underline{a} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$ $\underline{c} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$

$Volume = \underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \\ 4 & 2 & 2 \end{vmatrix} = (6-2) - 2(6-4) + 4(6-12) = 4 - 4 - 24 =$
$-24 = 24\ units^3$

Page 8 Answers

a) $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-1}{3} = \lambda$ $x = \lambda + 4, \quad y = \lambda + 2, \quad z = 3\lambda + 1$

b) $Direction\ vector\ \overrightarrow{AB} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$

$Choosing\ point\ (5, 5, 5)$

$\frac{x-5}{3} = \frac{y-5}{4} = \frac{z-5}{1} = \lambda$ $x = 5 + 3\lambda, \quad y = 5 + 4\lambda, \quad z = 5 + \lambda$

c) $\frac{x-3}{3} = \frac{y-2}{4} = \frac{z-7}{-1} = \lambda$ $x = 3 + 3\lambda, \quad y = 2 + 4\lambda, \quad z = 7 - \lambda$

d) $Direction\ vector\ \overrightarrow{AB} = \begin{pmatrix} a \\ 0 \\ 2a \end{pmatrix} - \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ -a \\ 2a \end{pmatrix}$

$Choosing\ point\ (0, a, 0)$

$\frac{x-0}{a} = \frac{y-a}{-a} = \frac{z-0}{2a} = \lambda \qquad x = a\lambda, \quad y = a - a\lambda, \quad z = 2a\lambda$

e) $Vector\ form: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -8 \end{pmatrix}$

$Symmetric\ form: \frac{x-2}{1} = \frac{y+1}{2} = \frac{z-6}{-8} = \lambda$

$Parametric\ form: x = 2 + \lambda, \quad y = -1 + 2\lambda, \quad z = 6 - 8\lambda$

Page 11 Answers

a) $Parametric\ form: x = 1 + 2\lambda, \quad y = 3 + 4\lambda, \quad z = 6 - 8\lambda$

$x = -1 + 2\mu, \quad y = 2 + 3\mu, \quad z = 7 - \mu$

$At\ intersection\ point:$

$x: 1 + 2\lambda = -1 + 2\mu$

$2 + 2\lambda = 2\mu$

$1 + \lambda = \mu \quad (1)$

$y: 3 + 4\lambda = 2 + 3\mu$

$1 + 3\lambda = 3\mu \quad (2)$

$z: 2 + \lambda = 7 - \mu$

$5 - \lambda = \mu \quad (3)$

$taking\ (1) - (3):$

$-4 + 2\lambda = 0$

$\lambda = 2$

$so\ \mu = 3\ (1)$

$1 + 4\lambda = 3\mu$
$1 + 4 \times 2 = 3\mu$
$\mu = 3 \quad (2)$

$5 - \lambda = \mu$
$5 - 2 = \mu$
$\mu = 3 \quad (3)$

$Hence\ consistent: \lambda = 2, \mu = 3 \quad Intersection\ point\ (5, 11, 4)$

$Angle\ directi⍰n\ vectors\ \underline{a} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \underline{b} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|a||b|} = \frac{(2 \times 2) + (4 \times 3) - (1 \times 1)}{\sqrt{21}\sqrt{14}} = \frac{15}{\sqrt{21}\sqrt{14}}$$

$$\theta = \cos^{-1}\left(\frac{15}{\sqrt{21}\sqrt{14}}\right) = 29°$$

b) $Parametric\ form: \ x = \lambda - 3, \quad y = 2 - \lambda, \quad z = 1 + 3\lambda$
$x = -4 = 2\mu, \quad y = 1 + \mu, \quad z = \mu$

$If\ intersection\ then:$

$x: \lambda - 3 = -4 - 2\mu$
$2\mu = -1 - \lambda \ (1)$

$y: 2 - \lambda = 1 + \mu$
$\mu = 1 - \lambda \ (2)$

$z: 1 + 3\lambda = \mu$
$\mu = 1 + 3\lambda \ (3)$

$taking\ (3) - (2):$

$0 = 3\lambda$
$\lambda = 0$

$Sub\ into\ (1)$
$2\mu = -1$
$\mu = \frac{-1}{2}$

$Sub\ into\ (2)$
$\mu = 1 - 0$
$\mu = 1$

$Hench\ inconsistent\ so\ lines\ do\ not\ intersect.$

c) $Parametric\ form:\ x = -2 + 2\lambda, \quad y = 1 - 3\lambda, \quad z = -1 + \lambda$
$x = -3 - \mu, \quad y = 4 + \mu, \quad z = -\mu$

$For\ intersection\ point:$

$x:\ -2 + 2\lambda = -3 - \mu$
$\mu = -1 - 2\lambda \quad (1)$

$y: 1 - 3\lambda = 4 + \mu$
$\mu = -3 - 3\lambda \quad (2)$

$z:\ -1 + \lambda = -\mu$
$\mu = 1 - \lambda \quad (3)$

$Using\ (1)\ and\ (3):$

$\mu = -1 - 2\lambda$
$\mu = 1 - \lambda$
$0 = -2 - \lambda$
$\lambda = -2$

$Check\ consistency:$

$\mu = -3 + 6 = 3$
$\mu = -1 + 4 = 3$
$\mu = 1 - (-2) = 3$

$Intersection\ point\ (-6, 7, -3)$

$\underline{Angle}$

$Direction\ vectors:$

$\underline{a} = (2, -3, 1)$
$\underline{b} = (-1, 1, -1)$

$$\cos\theta = \frac{\underline{a}\cdot\underline{b}}{|a||b|} = \frac{(-2)+(-3)+(-1)}{\sqrt{14}\sqrt{3}} = -0.926$$

$$\theta = \cos^{-1}\left(\frac{-6}{\sqrt{14}\sqrt{3}}\right) = 158° \qquad Acute\ angle = 180° - 158° = 22°$$

d) $Parametric\ form$: $x = 1 + 4\lambda, \quad y = 3, \quad z = -2 + \lambda$
$x = 5 - \mu, \quad y = 3 \quad z = 8 + 2\mu$

$For\ intersection\ point$:

x: $1 + 4\lambda = 5 - \mu$
$\mu = 5 - 4\lambda$ (1)

y: $3 = 3$ (2)

z: $-2 + \lambda = 8 + 2\mu$
$2\mu = -10 + \lambda$ (3)

$Using$ (1) and (3):

$\mu = 4 - 4\lambda$
$2\mu = -10 + \lambda$

$2\mu = 8 - 8\lambda$
$2\mu = -10 + \lambda$

$0 = -18 + 9\lambda$
$\lambda = 2$

$Subbing\ into$ (1):

$\mu = 4 - 4(2)$
$\mu = -4$

$Check\ consistency$:

$3 = 3$ (2)
$2(-4) = -10 + 2$ (3)

$Intersection\, point\ (9, 3, 0)$

$\underline{Angle}$

$Direction\, vectors:$

$$\underline{a} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \underline{b} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|a||b|} = \frac{(-4) + 0 + 2}{\sqrt{17}\sqrt{5}} = -0.217$$

$\theta = \cos^{-1}(-0.217) = 102.5°$ $\qquad Acute\ angle = 180° - 102.5° = 77.5°$

<u>Page 14 Answers</u>

a) $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\qquad \overrightarrow{OC} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

$Normal\ vector: \overrightarrow{OB} \times \overrightarrow{OC} = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ -2 & 1 & 2 \end{vmatrix} = 3\underline{i} - 4\underline{j} + 5\underline{k} = \underline{n}$

$\underline{p} \cdot \underline{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} = 3x - 4y + 5z$ $\qquad \underline{n}.p_0 = 3 - 8 + 5$

$3x - 4y + 5z = 0$

ii) $\overrightarrow{GH} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$ $\qquad \overrightarrow{GI} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

$\overrightarrow{GH} \times \overrightarrow{GI} = \begin{vmatrix} i & j & k \\ 4 & 2 & 3 \\ 3 & -2 & 2 \end{vmatrix} = 10\underline{i} + \underline{j} - 14\underline{k} = \underline{n}$

$\underline{p} \cdot \underline{n} = p \ \cdot n = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 1 \\ -14 \end{pmatrix} = 10x + y - 14z = 30 + 3 + (-42)$

$10x + y - 14z = -9$

b) $\overrightarrow{AB} = \begin{pmatrix}1\\-3\\-3\end{pmatrix} - \begin{pmatrix}-1\\3\\1\end{pmatrix} = \begin{pmatrix}2\\-6\\-4\end{pmatrix}$ $\qquad$ $\overrightarrow{GI} = \begin{pmatrix}3\\-1\\5\end{pmatrix} - \begin{pmatrix}-1\\3\\1\end{pmatrix} = \begin{pmatrix}4\\-4\\4\end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & -6 & -4 \\ 4 & -4 & 4 \end{vmatrix} = -40\underline{i} - 24\underline{j} + 16\underline{k} = \underline{n}$$

$$\underline{p} \cdot \underline{n} = p \cdot n = \begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}-40\\-24\\16\end{pmatrix} = -40x - 24y + 16z = 40 - 72 + 16$$

$-40x - 24y + 16z = -16$
$5x + 3y - 2z = 2$

$Verify\ (0, 2, 2) = 5(0) + 3(2) - 2(2) = 2 = RHS$

c) $\overrightarrow{SU} = \begin{pmatrix}1\\-4\\7\end{pmatrix} - \begin{pmatrix}5\\7\\-1\end{pmatrix} = \begin{pmatrix}4\\-11\\8\end{pmatrix}$ $\qquad$ $\overrightarrow{ST} = \begin{pmatrix}2\\-3\\6\end{pmatrix} - \begin{pmatrix}5\\7\\-1\end{pmatrix} = \begin{pmatrix}-3\\-10\\7\end{pmatrix}$

$$\overrightarrow{ST} \times \overrightarrow{SU} = \begin{vmatrix} i & j & k \\ -3 & -10 & 7 \\ -4 & -11 & 8 \end{vmatrix} = -3\underline{i} - 4\underline{j} - 7\underline{k} = \underline{n}$$

$$\underline{p} \cdot \underline{n} = p \cdot n = \begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}-3\\-4\\-7\end{pmatrix} = \begin{pmatrix}1\\-4\\7\end{pmatrix} \cdot \begin{pmatrix}-3\\-4\\-7\end{pmatrix}$$

$-3x - 4y - 7z = -3 + 16 - 49$
$-3x - 4y - 7z = -36$
$3x + 4y + 7z = 36$

$If\ coplanar\ the\ V(6, 1, 2) will\ satisfy\ the\ equation:$

$LHS\ of\ equation:$

$= 3(6) + 4(1) + 7(2)$
$= 18 + 4 + 14$
$= 36$
$= RHS$

$Hence\ coplanar$

Page 15 Answers

a) $\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 6 & 5 \end{vmatrix} = 6\underline{i} - 5\underline{j} + 6\underline{k} = \underline{n}$

$$\underline{p} \cdot \underline{n} = p \cdot n = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -5 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -5 \\ 6 \end{pmatrix}$$

$6x - 5y + 6z = -12 + 15 + 42$
$6x - 5y + 6z = 15$

b) $Parametric\ form$: $x = 1 + 3\lambda, \quad y = 4 - \lambda, \quad z = -7 + 2\lambda$
$x = -4 + 4\mu, \quad y = 3 - \mu \quad z = 3 + \mu$

$For\ intersection\ point$:

$x: 1 + 3\lambda = -4 + 4\mu$
$4\mu = 5 + 3\lambda\ (1)$

$y: 4 - \lambda = 3 - \mu$
$-\mu = 1 - \lambda\ (1)$

$z: -7 + 2\lambda = 3 + \mu$
$\mu = -10 + 2\lambda \quad (3)$

$Using\ (2)\ and\ (3)$:

$-\mu = 1 - \lambda$
$\mu = -10 + 2\lambda$

$0 = -9 + \lambda$
$\lambda = 9$

$Subbing\ into\ (2)$:

$-\mu = 1 - 9$
$\mu = 8$

$Check\ consistency$:

$5 + 27 = 4 \times 8 \ (1)$
$32 = 32 \ (1)$

$1 - 9 = -8\ (2)$
$-8 = -8\ (2)$

$-10 + 2(9) = 8\ (3)$
$8 = 8\ (3)$

$Intersection\,point\ (28, -5, 11)$

$$\underline{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \underline{b} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 4 & -1 & 1 \end{vmatrix} = \underline{i} + 5\underline{j} + \underline{k} = \underline{n}$$

$$\underline{p} \cdot \underline{n} = p\ \cdot n = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 28 \\ -5 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$$

$x + 5y + z = 28 - 25 + 11$
$x + 5y + z = 14$

<u>Page 15 Answers (Exercise 2)</u>

a) $$\underline{p} \cdot \underline{n} = p\ \cdot n = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$2x + 3y + z = 12$

b) $$\underline{p} \cdot \underline{n} = p\ \cdot n = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ -3 \end{pmatrix}$$

$5x + 4y - 3z = 17$

c) $$\underline{p} \cdot \underline{n} = p\ \cdot n = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$2x - 3y + z = -1$

d) $$\underline{p} \cdot \underline{n} = p\ \cdot n = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 6 \\ 7 \end{pmatrix}$$

$-4x + 6y + 7z = 12$

Page 17 Answers

a) $Direction\ vectors$:

$$\underline{a} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \underline{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|a||b|} = \frac{2 - 1 + 0}{\sqrt{5}\sqrt{3}} = \frac{1}{\sqrt{15}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{15}}\right) = 75°$$

b) $Direction\ vectors$:

$$\underline{a} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \underline{b} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|a||b|} = \frac{2 - 3 - 16}{\sqrt{18}\sqrt{29}} = \frac{-17}{\sqrt{18}\sqrt{29}} = -0.744$$

$$\theta = \cos^{-1}(-0.744) = 138° \ so\ acute\ angle = 42°$$

c) $Normal\ vectors$:

$$\underline{n_1} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{vmatrix} = 2\underline{i} - 4\underline{j} - 3\underline{k}$$

$$\underline{n_2} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 1 & 3 & -1 \\ 1 & 1 & -1 \end{vmatrix} = -2\underline{i} - 0\underline{j} - 2\underline{k}$$

$$\cos\theta = \frac{\underline{n_1\, n_2}}{|\underline{n_1}|\,|\underline{n_2}|} = \frac{-4 + 0 - 6}{\sqrt{29}\sqrt{8}} = \frac{-10}{\sqrt{29}\sqrt{8}} = -0.6565$$

$$\theta = \cos^{-1}(-0.6565) = 131° \ so\ acute\ angle = 49°$$

d) $\overrightarrow{OP} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \overrightarrow{OQ} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$\underline{n_1} = \overrightarrow{OP} \times \overrightarrow{OQ} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -\underline{i} - \underline{j} + \underline{k}$$

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

$$\underline{n_2} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & 2 & -3 \\ -1 & 3 & -2 \end{vmatrix} = 5\underline{i} + 7\underline{j} + 8\underline{k}$$

$$\cos\theta = \frac{\underline{n_1\, n_2}}{|\underline{n_1}|\,|\underline{n_2}|} = \frac{-5 - 7 + 8}{\sqrt{3}\sqrt{138}} = \frac{-4}{\sqrt{414}} = -0.1965$$

$$\theta = \cos^{-1}(-0.1965) = 101° \ so\ acute\ angle = 79°$$

Page 18 Answers

a) $\left(\begin{matrix}1 & -1 & -3 \\ 2 & 3 & -1\end{matrix}\ \middle|\begin{matrix}-7 \\ -4\end{matrix}\right) \rightarrow 2R_1 - 2R_2\ \left(\begin{matrix}1 & -1 & -3 \\ 0 & -5 & -5\end{matrix}\ \middle|\begin{matrix}-7 \\ -10\end{matrix}\right)$

$Let\ z = \lambda$

$-5y - 5\lambda = -10$
$y + \lambda = 2$
$y = 2 - \lambda$

$x - \text{y} - 3\lambda = -7$
$x = \text{y} + 3\lambda - 7$
$x = (2 - \lambda) + 3\lambda - 7$
$x = 2\lambda - 5$

b) $\left(\begin{matrix}2 & -1 & -2 \\ 1 & -2 & -2\end{matrix}\ \middle|\begin{matrix}1 \\ -8\end{matrix}\right) \rightarrow R_1 - 2R_2\ \left(\begin{matrix}2 & -1 & -2 \\ 0 & 3 & 2\end{matrix}\ \middle|\begin{matrix}1 \\ 17\end{matrix}\right)$

$Let\ z = \lambda$

$3y + 2\lambda = 17$
$y = \frac{-2}{3}\lambda + \frac{17}{3}$

$2x - \text{y} - 2\lambda = 1$
$2x = \text{y} + 2\lambda + 1$
$2x = \frac{-2}{3}\lambda + 2\lambda + \frac{17}{3} + 1$
$2x = \frac{4}{3}\lambda + \frac{20}{3}$
$x = \frac{2}{3}\lambda + \frac{10}{3}$

Page 22 Answers

a) $x = 12 + 5\lambda$
$y = -7 + 4\lambda$
$z = 5 + 3\lambda$

$Sub\ into\ plane\ equation$

$5(12 + 5\lambda) + 3(-7 + 4\lambda) - (5 + 3\lambda) = 0$
$60 + 25\lambda - 21 + 12\lambda - 5 - 3\lambda = 0$
$34 + 34\lambda = 0$
$\lambda = -1$

$Sub\ in\ line\ equation\ to\ find\ point:$

$x = 12 + 5(-1) = 7$
$y = -7 + 4(-1) = -11$
$z = 5 + 3(-1) = 2$

$P(7, -11, 2)$

b) $x = -7 + 3\lambda$
$y = 6 - \lambda$
$z = 17 - 5\lambda$

$Sub\ into\ plane\ equation$

$8(-7 + 3\lambda) + 5(6 - \lambda) - (17 - 5\lambda) = 0$
$-56 + 24\lambda + 30 - 5\lambda - 34 + 10\lambda + 2 = 0$
$-58 + 29\lambda = 0$
$\lambda = 2$

$Sub\ in\ line\ equation\ to\ find\ point:$

$x = -7 + 3(2) = -1$
$y = -6 - 2 = 4$
$z = 17 - 5(2) = 7$

$P(-1,4,7)$

<u>Page 23 Answers</u>

a) $Norm = l\ vector: \underline{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, Direction\ vector\ \underline{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|a||b|} = \frac{2 + 2 + 6}{\sqrt{14}\sqrt{9}} = \frac{-17}{\sqrt{126}} = 0.891$$

$\theta = \cos^{-1}(0.891) = 27°\ so\ true\ angle\ \emptyset = 90° - 27° = 63°$

$2x = -7 + \lambda$
$y = -5 + 2\lambda$
$3z = -9 + 2\lambda$

Sub into plane equation

$(-7 + \lambda) + (-5 + 2\lambda) - (-9 + 2\lambda) = 0$
$-21 + 5\lambda = 41$
$5\lambda = 21$
$\lambda = 12.4$

Sub in line equation to find point:

$2x = -7 + 12.4 \; so \; x = 2.7$
$y = -5 + 2(12.4) = 19.8$
$3z = -9 + 2 \times (12.4) so \; z = 5.26$

$P\left(\frac{27}{10}, \frac{99}{5}, \frac{79}{15}\right)$

b) $Normal\ vector: \underline{a} = \begin{pmatrix} 8 \\ 5 \\ -2 \end{pmatrix}, Direction\ vector\ \underline{b} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$

$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|a||b|} = \frac{16 + 25 + 2}{\sqrt{93}\sqrt{30}} = \frac{43}{\sqrt{93}\sqrt{30}} = 0.814$

$\theta = \cos^{-1}(0.814) = 35.5° \; so \; true \; angle \; \emptyset = 90° - 35.5° = 54.5°$

x=-9+2λ

y=-13+5λ

$\_ = 3 - \lambda$

Sub into plane equation

$8(-9 + 2\lambda) + 5(-13 + 5\lambda) - 2(3 - \lambda) = -14$
$-72 + 16\lambda + 65 + 25\lambda - 6 + 2\lambda = -14$
$43\lambda = 129$
$\lambda = 3$

$Sub\ in\ line\ equation\ to\ find\ point$:

$x = -9 + 2(3) = -3$
$y = -13 + 5(3) = 2$
$3z = 3 - (3) = 0$

$P(-3,2,0)$