Advanced Higher Maths

Complex Numbers

2001

(a) Given that $-1 = \cos\theta + i\sin\theta,\ -\pi < \theta < \pi,$ state the value of $\theta.$

(b) Use de Moivre's Theorem to find the non-real solutions, z_1 and z_2, of the equation $z^3 + 1 = 0.$

Hence show that $z_1^2 = -z_2$ and $z_2^2 = -z_1.$

(c) Plot all the solution of $z^3 + 1 = 0$ on an Argand diagram and state their geometrical significance.

(1,5,2,3 marks)

2002

Verify that i is a solution of $z^4 + 4z^3 + 3z^2 + 4z + 2 = 0.$
Hence find all the solutions **(5 marks)**

2003

① Identify the locus in the complex plane given by $|z + i| = 2$ **(3 marks)**

② Given that $w = \cos\theta + i\sin\theta,$ show that $\dfrac{1}{w} = \cos\theta - i\sin\theta.$

Use de Moivre's Theorem to prove $w^k + w^{-k} = 2\cos k\theta,$ where k is a natural number.

Expand $\left(w + w^{-1}\right)^4$ by the binomial theorem and hence show that

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}.$$

(1, 3, 5 marks)

2004

Given $z = 1 + 2i,$ express $z^2(z+3)$ in the form $a + ib.$
Hence, or otherwise, verify that $1 + 2i$ is a root of the equation

$$z^3 + 3z^2 - 5z + 25 = 0.$$

Obtain the other roots of this equation. **(2, 2, 2 marks)**

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2005

① Given the equation $z+2i\bar{z}=8+7i$, express z in the form $a+ib$. **(4 marks)**

② Let $z=\cos\theta+i\sin\theta$.

(a) Use the binomial expansion to express z^4 in the form $u+iv$, where u and v are expression involving $\sin\theta$ and $\cos\theta$.

(b) Use de Moivre's theorem to write down a second expression for z^4.

(c) Using the results of (a) and (b), show that

$$\frac{\cos 4\theta}{\cos^2\theta}=p\cos^2\theta+q\sec^2\theta+r, \text{ where } -\frac{\pi}{2}<\theta<\frac{\pi}{2},$$

stating the values of p, q and r.

(3, 1, 6 marks)

2006

Express the complex number $z=-i+\frac{1}{1-i}$ in the form $z=x+iy$, stating the values of x and y.

Find the modulus and argument of z and plot z and $\bar{z}$ on an Argand diagram.

(3, 4 marks)

2007

① Show that $z=3+3i$ is a root of the equation $z^3-18z+108=0$ and obtain the remaining roots of the equation.

(4 marks)

② Given that $|z-2|=|z+i|$, where $z=x+iy$, show that $ax+by+c=0$ for suitable values of a, b and c.

Indicate on an Argand diagram the locus of complex numbers z which satisfy $|z-2|=|z+i|$.

(3, 1 marks)

2008

Given $z=\cos\theta+i\sin\theta$, use de Moivre's theorem to write down an expression for z^k in terms of θ, where k is a positive integer.

Hence show that $\frac{1}{z^k}=\cos k\theta-i\sin k\theta$.

Deduce expression for $\cos k\theta$ and $\sin k\theta$ in terms of z.

Show that $\cos^2\theta\sin^2\theta=-\frac{1}{16}\left(z^2-\frac{1}{z^2}\right)^2$.

Hence show that $\cos^2\theta\sin^2\theta=a+b\cos 4\theta$, for suitable constants a and b.

(3, 2, 3, 2 marks)

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2009

Express $z = \frac{(1+2i)^2}{7-1}$ in the form $a+ib$ where a and b are real numbers.
Show z on an Argand diagram and evaluate $|z|$ and $\arg(z)$.

(6 marks)

2010

Given $z = r(\cos\theta + i\sin\theta)$, use de Moivre's theorem to express z^3 in polar form.
Hence obtain $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3$ in the form $a+ib$.
Hence, or otherwise, obtain the roots of the equation $z^3 = 8$ in Cartesian form.

Denoting the roots of $z^3 = 8$ by z_1, z_2, z_3:
(a) state the value $z_1 + z_2 + z_3$;
(b) obtain the value of $z_1^6 + z_2^6 + z_3^6$.

(1, 2, 4, 3 marks)

2011

Identify the locus in the complex plane given by

$$|z-1| = 3.$$

Show in a diagram the region given by $|z-1| \le 3$. **(5 marks)**

2012

① Given that $(-1+2i)$ is a root of the equation

$$z^3 + 5z^2 + 11z + 15 = 0,$$

obtain all roots.

Plot all the roots on an Argand diagram. **(4, 2 marks)**

② (a) Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all integers $n \ge 1$.

(b) Show that the real part of $\frac{\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right)^{11}}{\left(\cos\frac{\pi}{36} + i\sin\frac{\pi}{36}\right)^4}$ is zero.

(6, 4 marks)

2013

① Given that $z = 1 - \sqrt{3}i$, write down $\bar{z}$ and express $\bar{z}^2$ in polar form. **(4 marks)**

② Describe the loci in the complex plane given by:

(a) $|z + i| = 1$;

(b) $|z - 1| = |z + 5|$. **(2, 3 marks)**

2014

(a) Express -1 as a complex number in polar form and hence determine the solutions to the equation $z^4 + 1 = 0$.

(b) Write down the four solutions to the equation $z^4 - 1 = 0$.

(c) Plot the solutions of both equations on an Argand diagram.

(d) Show that the solutions of $z^4 + 1 = 0$ and the solutions of $z^4 - 1 = 0$ are also solutions of the equation $z^8 - 1 = 0$.

(e) Hence identify all the solutions to the equation

$$z^6 + z^4 + z^2 + 1 = 0.$$

(3, 2, 1, 2, 2 marks)

2015

By writing z in the form $x + iy$:

(a) solve the equation $z^2 = |z|^2 - 4$;

(b) find the solutions to the equation $z^2 = i\left(|z|^2 - 4\right)$. **(3, 4 marks)**