Advanced Higher Maths

Proof

2001

Prove by induction , that for all integers $n \geq 1$,

$$2+5+8+\ldots+(3n-1)=\frac{1}{2}n(3n+1).$$

(5 marks)

2002

Prove by induction that $4^n - 1$ is divisible by 3 for all positive integers n.

(5 marks)

A matrix $A=\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$. Prove by induction that

$$A^n=\begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix},$$

where n is any positive integer. **(6 marks)**

2003

① Given that $p(n)=n^2+n$, where n is a positive integer, consider the statements:

A $\quad p(n)$ is always even

B $\quad p(n)$ is always a multiple of 3.

For each statement, prove if it is true or, otherwise, disprove it.

(4 marks)

② (a) Prove by induction that for all natural numbers $n \geq 1$

$$\sum_{r=1}^{n} 3(r^2-r)=(n-1)n(n+1).$$

(b) Hence evaluate $\sum_{r=11}^{40} 3(r^2-r)$. **(4, 2 marks)**

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2004

Prove by induction that $\frac{d^n}{dx^n}(xe^x) = (x+n)e^x$ for all integers $n \geq 1$. **(5 marks)**

2005

Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

State the value of $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$. **(5, 1 marks)**

2006

① For all natural numbers n, prove whether the following results are true or false.

(a) $n^3 - n$ is always divisible by 6.

(b) $n^3 + n + 5$ is always prime. **(5 marks)**

② The square matrices A and B are such that $AB = BA$. Prove by induction that $A^n B = BA^n$ for all integers $n \geq 1$.

5 marks)

2007

Prove by induction that for $a > 0$,

$$(1+a)^n \geq 1 + na$$

For all positive integers n. **(5 marks)**

2008

For each of the following statements, decide whether it is true or false and prove your conclusion.

A For all natural numbers m, if m^2 is divisible by 4 then m is divisible by 4.

B The cube of any odd integer p plus the square of any even integer q is always odd.

(5 marks)

Advanced Higher Maths

2009

Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}.$$

(5 marks)

2010

① (a) Prove that the product of two odd integers is odd.

(b) Let p be an odd integer. Use the result of (a) to prove by induction that p^n is odd for all positive integers n.

(2, 4 marks)

② Prove by contradiction that if x is an irrational number, then $2+x$ is irrational.

(4 marks)

2011

Prove by induction that $8^n + 3^{n-2}$ is divisible by 5 **for all integers** $n \geq 2$. **(5 marks)**

2012

Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all integers $n \geq 1$. **(6 marks)**

2013

① Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} (4r^3 + 3r^2 + r) = n(n+1)^3$$

(6 marks)

② Let n be a natural number.
For each of the following statements, decide whether it is true or false.
If true, give a proof; if false, give a counterexample.

A If n is a multiple of 9, then so is n^2.

B If n^2 is a multiple of 9, then so is n.

(4 marks)

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2014

Given A is the matrix $\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$,

Prove by induction that

$$A^n = \begin{pmatrix} 2^n & a(2^n - 1) \\ 0 & 1 \end{pmatrix}, \ n \geq 1.$$

(4 marks)

2015

Prove that the difference between the squares of any two consecutive odd numbers is divisible by 8.

(3 marks)