Section A (Mathematics 1 and 2)

All candidates should attempt this Section.

Marks

Answer all the questions.

A1. Use Gaussian elimination to solve the following system of equations

$$\begin{aligned} x + y + z &= 10 \\ 2x - y + 3z &= 4 \\ x \quad\quad + 2z &= 20. \end{aligned}$$ 5

A2. Differentiate with respect to x

(*a*) $f(x) = (2+x)\tan^{-1}\sqrt{x-1},\ x > 1,$ 4

(*b*) $g(x) = e^{\cot 2x},\ 0 < x < \frac{\pi}{2}.$ 2

A3. Find the value of

$$\int_0^{\pi/4} 2x \sin 4x \, dx.$$ 5

A4. Prove by induction that, for all integers $n \geq 1$,

$$2+5+8+ \ \ldots \ +(3n-1) = \tfrac{1}{2}n(3n+1).$$ 5

A5. (*a*) Obtain partial fractions for

$$\frac{x}{x^2-1}, \qquad x > 1.$$ 2

(*b*) Use the result of (*a*) to find

$$\int \frac{x^3}{x^2-1}\,dx, \qquad x > 1.$$ 4

A6. Expand

$$\left(x^2 - \frac{2}{x}\right)^4, \qquad x \neq 0$$

and simplify as far as possible. 5

Marks

A7. A curve has equation $xy + y^2 = 2$.

(*a*) Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y. **3**

(*b*) Hence find an equation of the tangent to the curve at the point (1, 1). **2**

A8. A function f is defined by $f(x) = \frac{x^2 + 6x + 12}{x + 2}$, $x \neq -2$.

(*a*) Express $f(x)$ in the form $ax + b + \frac{b}{x+2}$ stating the values of a and b. **2**

(*b*) Write down an equation for each of the two asymptotes. **2**

(*c*) Show that $f(x)$ has two stationary points.
Determine the coordinates and the nature of the stationary points. **4**

(*d*) Sketch the graph of f. **1**

(*e*) State the range of values of k such that the equation $f(x) = k$ has no solution. **1**

A9. (*a*) Given that $-1 = \cos\theta + i\sin\theta$, $-\pi < \theta \leq \pi$, state the value of θ. **1**

(*b*) Use de Moivre's Theorem to find the non-real solutions, z_1 and z_2, of the equation $z^3 + 1 = 0$. **5**

Hence show that $z_1^2 = -z_2$ and $z_2^2 = -z_1$. **2**

(*c*) Plot all the solutions of $z^3 + 1 = 0$ on an Argand diagram and state their geometrical significance. **3**

[Turn over

Marks

A10. A chemical plant food loses effectiveness at a rate proportional to the amount present in the soil. The amount M grams of plant food effective after t days satisfies the differential equation

$$\frac{dM}{dt} = kM, \text{ where } k \text{ is a constant.}$$

(*a*) Find the general solution for M in terms of t where the initial amount of plant food is M_0 grams. **3**

(*b*) Find the value of k if, after 30 days, only half the initial amount of plant food is effective. **3**

(*c*) What percentage of the original amount of plant food is effective after 35 days? **2**

(*d*) The plant food has to be renewed when its effectiveness falls below 25%. Is the manufacturer of the plant food justified in calling its product "sixty day super food"? **2**

[*END OF SECTION A*]

Candidates should now attempt ONE of the following

Section B (Mathematics 3) on *Page five*
Section C (Statistics 1) on *Page six*
Section D (Numerical Analysis 1) on *Page eight*
Section E (Mechanics 1) on *Page ten.*

Section B (Mathematics 3)

Marks

ONLY candidates doing the course Mathematics 1, 2 and 3 should attempt this Section.

Answer all the questions.

Answer these questions in a separate answer book, showing clearly the section chosen.

B1. Use the Euclidean algorithm to find integers x and y such that

$$149x + 139y = 1.$$ **4**

B2. Find the general solution of the following differential equation:

$$\frac{dy}{dx} + \frac{y}{x} = x, \qquad x > 0.$$ **4**

B3. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}.$$

Show that $AB = kI$ for some constant k, where I is the 3×3 identity matrix. Hence obtain (i) the inverse matrix A^{-1}, and (ii) the matrix A^2B. **4**

B4. Find the first four terms in the Maclaurin series for $(2 + x) \ln (2 + x)$. **4**

B5. Find the general solution of the following differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x - 1.$$ **5**

B6. Let L_1 and L_2 be the lines

$$L_1 : \quad x = 8 - 2t,\ y = -4 + 2t,\ z = 3 + t$$

$$L_2 : \quad \frac{x}{-2} = \frac{y+2}{-1} = \frac{z-9}{2}.$$

(*a*) (i) Show that L_1 and L_2 intersect and find their point of intersection. **4**

(ii) Verify that the acute angle between them is

$$\cos^{-1}\left(\tfrac{4}{9}\right).$$ **2**

(*b*) (i) Obtain an equation of the plane Π that is perpendicular to L_2 and passes through the point $(1, -4, 2)$. **3**

(ii) Find the coordinates of the point of intersection of the plane Π and the line L_1. **2**

[END OF SECTION B]

Section C (Statistics 1)

Marks

ONLY candidates doing the course Mathematics 1, 2 and Statistics 1 should attempt this Section.

Answer all the questions.

Answer these questions in a separate answer book, showing clearly the section chosen.

C1. A company has a wholesale department and a retail department. Of the accounts being audited in a particular year 60% are wholesale. Given that 2% of the wholesale accounts are in error and that 1% of the retail accounts are in error calculate the probability that an account, selected at random, is in error. **3**

Given that an account which has been checked is in error, calculate the probability that it was a retail account. **2**

C2. If X represents a temperature in degrees Celsius and Y the corresponding temperature in degrees Fahrenheit then $Y = 1{\cdot}8X + 32$. Given that X varies with a mean of 30 and standard deviation of 5, obtain the mean and standard deviation of Y. **4**

C3. Distinguish between stratified and cluster sampling and give an example of a situation where each might be used. **4**

C4. Interviews with a random sample of 400 adult residents in a city revealed that 320 were in favour of restricting a city centre area to pedestrian access only.

Obtain an approximate 95% confidence interval for the proportion of adult residents in favour of the restriction and state whether or not it supports a claim by an action group that the proportion in favour is at least 90%. **5**

C5. The heights of elderly women in a community are normally distributed with mean 160 cm and standard deviation 6 cm. The mean height of a sample of 36 elderly women with a major bone disease was found to be 157·5 cm. Test, at the 5% level of significance, the null hypothesis that the mean height in the population of women with the disease is 160 cm against the alternative hypothesis that it is less than 160 cm. **5**

Marks

C6. A legal company has two partners, Mr Smith and Mr Jones. The numbers of telephone calls received by the office switchboard for each, during the period 9.00 am to 9.15 am daily, have independent Poisson distributions with means 2 and 3 respectively.

(*a*) Obtain the probability that on a particular day, between 9.00 am and 9.15 am

(i) Mr Smith receives at least 3 calls, **2**

(ii) Mr Smith receives precisely 3 calls, and **2**

(iii) Mr Jones receives no calls. **1**

(*b*) State the daily mean of the **total** number of calls between 9.00 am and 9.15 am to the two partners and justify your answers. **2**

(*c*) Given that the sum of two independent Poisson distributions has a Poisson distribution, obtain the probability that the total number of calls between 9.00 am and 9.15 am to the two partners exceeds five. **2**

[END OF SECTION C]

Section D (Numerical Analysis 1)

ONLY candidates doing the course Mathematics 1, 2 and Numerical Analysis 1 should attempt this Section.

Answer all the questions.

Answer these questions in a separate answer book, showing clearly the section chosen.

Marks

D1. The polynomial p is the Taylor polynomial of degree three for a function f near $x = 1$. Express $p(1 + h)$ in the form

$$c_0 + c_1h + c_2h^2 + c_3h^3$$

for the function $f(x) = e^{1-2x}$. **3**

Estimate the value of $f(0{\cdot}98)$ using the *second* degree approximation, stating your answer to four decimal places. **2**

Write down the principal truncation error term for this second degree approximation and calculate its value. Hence state whether the second degree approximation is likely to be accurate to four decimal places. **2**

D2. The following data are available for function f

x	1	3	4
$f(x)$	1·7831	2·0226	1·9308

Use the Lagrange interpolation formula to estimate $f(2{\cdot}5)$ to three decimal places. **3**

D3. In the usual notation for forward differences of function values $f(x)$ tabulated at equally spaced values of x,

$$\Delta f_i = f_{i+1} - f_i$$

where $f_i = f(x_i)$ and $i = \ldots, -2, -1, 0, 1, 2, \ldots$.

Show that $\Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0$. **2**

If each value of f_i is subject to an error whose magnitude is less than or equal to ε, determine the magnitude of the maximum possible error in $\Delta^3 f_0$. **1**

D4. Values of a function f have been obtained as shown in the following table.

i	0	1	2	3	4	5
x_i	0·5	0·6	0·7	0·8	0·9	1·0
$f(x_i)$	−0·623	−0·152	0·455	1·222	2·176	3·340

Construct a difference table of third order for the data. **3**

Identify the value of $\Delta^3 f_1$ in this table. **1**

Using the Newton forward difference formula of degree three, and working to three decimal places, obtain a suitable approximation to $f(0{\cdot}63)$. **3**

D5. Derive Simpson's rule with two strips for evaluating an approximation to

$$\int_{-h}^{h} f(x)\,dx.$$ **4**

It is given that, for the integral

$$I = \int_0^1 x^2 \cos x\,dx,$$

$I_4 = 0{\cdot}2389685$ and $I_8 = 0{\cdot}2391235$ where I_n is the composite Simpson's rule estimate for this integral using n strips and seven decimal place arithmetic.

It is also given that for $f(x) = x^2 \cos x$,

$$f^{(\mathrm{iv})}(x) = (x^2 - 12)\cos x + 8x \sin x$$

and that $f^{(\mathrm{v})}(x)$ has no zero for $0 < x < 1$.

Obtain the maximum value of $\left|f^{(\mathrm{iv})}(x)\right|$ on the interval $[0,\ 1]$ and use this information to obtain an estimate of the maximum truncation error in I_8. Hence write down the value of I_8 to a suitable accuracy. **4**

Use Richardson extrapolation to obtain an improved estimate for I based on the given values of I_4 and I_8. **2**

By what approximate factor would you expect the truncation error in this answer to be reduced from that in I_8? Hence state this improved answer to an appropriate degree of accuracy. **2**

[END OF SECTION D]

Section E (Mechanics 1)

Marks

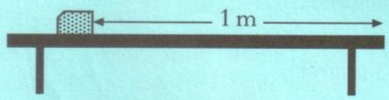
ONLY candidates doing the course Mathematics 1, 2 and Mechanics 1 should attempt this Section.

Answer all the questions.

Answer these questions in a separate answer book, showing clearly the section chosen.

Where appropriate, candidates should take the magnitude of the acceleration due to gravity as $9{\cdot}8\,\mathrm{m\,s^{-2}}$.

E1. A box on a horizontal table is projected with initial speed $0{\cdot}2\,\mathrm{m\,s^{-1}}$ towards the edge of a table, 1 metre away, as shown below.



Calculate the minimum value of the coefficient of friction between the box and the table to ensure that the box remains on the table. **4**

E2. A particle, initially projected from the origin, moves along the x-axis with velocity, measured in $\mathrm{m\,s^{-1}}$, given by

$$[2t(1-3t)+1]\,\mathbf{i},$$

where $\mathbf{i}$ is the unit vector in the positive direction of the x-axis and t is the time in seconds from the start of the motion.

Calculate

(*a*) the time when the acceleration is zero, and

(*b*) the acceleration when the particle returns to the origin. **5**

E3. John accelerates down a snowy hillside on his sledge, travelling in the direction of greatest slope. The hill is inclined at $30°$ to the horizontal, and the coefficient of friction between the sledge and the snow is μ.

(*a*) Show that the magnitude of the acceleration of John and the sledge is given by

$$\tfrac{1}{2}(1-\sqrt{3}\mu)g,$$

where $g\,\mathrm{m\,s^{-2}}$ is the magnitude of the acceleration due to gravity and $\mu < \frac{1}{\sqrt{3}}$. **4**

(*b*) On the lower part of the hill the snow becomes softer with $\mu > \frac{1}{\sqrt{3}}$.
Explain what happens to the sledge on the lower part of the hill. **2**

Marks

E4. Joan cycles due north at 20 kilometres per hour. She notices that the wind appears to come from the east. When she increases her speed to 30 kilometres per hour (still cycling due north), the wind appears to come from the north-east. Calculate the speed and direction of the wind. 6

E5. (*a*) A particle is projected from a point P with speed V m s^{-1} at an angle α to the horizontal. Derive an expression, in terms of V, α and g, for the range of the particle on the horizontal plane through P, where g m s^{-2} is the magnitude of the acceleration due to gravity. 4

(*b*) A gun fires a shell with speed V m s^{-1} from a point on a horizontal plane at a target on the plane distance D metres from the gun. The shell falls d metres short of the target when the angle of projection is 15°, and overshoots the target by d metres when projected at an angle of 30°.

(i) Show that d and D are related by the expression

$$d = \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)D,$$

and that

$$V^2 = \frac{4gD}{\sqrt{3}+1}.$$

5

(ii) Determine the possible angles of projection to ensure that the shell hits the target. 2

[*END OF SECTION E*]

[*END OF QUESTION PAPER*]