



**2009 Mathematics**

**Advanced Higher**

**Finalised Marking Instructions**

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## Solutions for AH maths 2009

1. (a)  $f(x) = (x + 1)(x - 2)^3$

$$f'(x) = (x - 2)^3 + 3(x + 1)(x - 2)^2 \quad \mathbf{1}$$

$$= (x - 2)^2((x - 2) + 3(x + 1))$$

$$= (x - 2)^2(4x + 1) \quad \mathbf{1}$$

$$= 0 \text{ when } x = 2 \text{ and when } x = -\frac{1}{4}. \quad \mathbf{1}$$

(b) *Method 1*

$$\frac{x^2}{y} + x = y - 5 \Rightarrow x^2 + xy = y^2 - 5y \quad \mathbf{1}$$

$$2x + x\frac{dy}{dx} + y = 2y\frac{dy}{dx} - 5\frac{dy}{dx} \quad \mathbf{2E1}$$

$$6 + 3\frac{dy}{dx} - 1 = -2\frac{dy}{dx} - 5\frac{dy}{dx}$$

$$5 = -10\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-1}{2} \quad \mathbf{1}$$

*Note:* a candidate may obtain  $\frac{dy}{dx} = \frac{2x+y}{2y-x-5}$  and then substitute.

*Method 2*

$$\frac{x^2}{y} + x = y - 5$$

$$\frac{2xy - x^2\frac{dy}{dx}}{y^2} + 1 = \frac{dy}{dx} \quad \mathbf{2E1}$$

$$\frac{-6 - 9\frac{dy}{dx}}{1} + 1 = \frac{dy}{dx}$$

$$-6 - 9\frac{dy}{dx} + 1 = \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{2} \quad \mathbf{2E1}$$

*Method 3*  $\frac{x^2}{y} + x = y - 5 \Rightarrow x^2\left(\frac{1}{y}\right) + x = y - 5$

$$2x\frac{1}{y} + x^2\left(-\frac{1}{y^2}\right)\frac{dy}{dx} + 1 = \frac{dy}{dx} \quad \mathbf{2E1}$$

$$-6 - 9\frac{dy}{dx} + 1 = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{2} \quad \mathbf{2E1}$$

*Note:* a candidate may obtain  $\frac{dy}{dx} = \frac{2xy+y^2}{y^2+x^2}$  (in 2 and 3) and then substitute.

2. (a)  $\det \begin{pmatrix} t+4 & 3t \\ 3 & 5 \end{pmatrix} = 5(t+4) - 9t$  1

$$= 20 - 4t$$

$$A^{-1} = \frac{1}{20 - 4t} \begin{pmatrix} 5 & -3t \\ -3 & t+4 \end{pmatrix}$$
 1,1

(b)  $20 - 4t = 0 \Rightarrow t = 5$  1

(c)  $\begin{pmatrix} t+4 & 3 \\ 3t & 5 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 6 & 5 \end{pmatrix} \Rightarrow t = 2$  1

3.  $e^y x^2 \frac{dy}{dx} = 1$

$$e^y \frac{dy}{dx} = x^{-2}$$

$$\int e^y dy = \int x^{-2} dx$$
 1

$$e^y = -x^{-1} + c$$
 1

$y = 0$  when  $x = 1$  so

$$1 = -1 + c \Rightarrow c = 2$$
 1

$$e^y = 2 - \frac{1}{x} \Rightarrow y = \ln \left( 2 - \frac{1}{x} \right)$$
 1

4. When  $n = 1$ , LHS =  $\frac{1}{1 \times 2} = \frac{1}{2}$ , RHS =  $1 - \frac{1}{2} = \frac{1}{2}$ . So true when  $n = 1$ . 1

Assume true for  $n = k$ ,  $\sum_{r=1}^k \frac{1}{r(r+1)} = 1 - \frac{1}{k+1}$ . 1

Consider  $n = k + 1$

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^k \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$$
 1

$$= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= 1 - \frac{k+2-1}{(k+1)(k+2)} = 1 - \frac{k+1}{(k+1)((k+1)+1)}$$
 1

$$= 1 - \frac{1}{((k+1)+1)}$$
 1

Thus, if true for  $n = k$ , statement is true for  $n = k + 1$ , and, since true for  $n = 1$ , true for all  $n \geq 1$ .

5. *Method 1*

$$\int_{\ln \frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$$

Let  $u = e^x - e^{-x}$ , then  $du = (e^x + e^{-x})dx$ . 1

When  $x = \ln \frac{3}{2}$ ,  $u = \frac{3}{2} - \frac{2}{3} = \frac{5}{6}$  and when  $x = \ln 2$ ,  $u = 2 - \frac{1}{2} = \frac{3}{2}$ . 1

$$\int_{\ln \frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \int_{5/6}^{3/2} \frac{du}{u}$$

$$= [\ln u]_{5/6}^{3/2} \quad 1$$

$$= \ln \frac{3}{2} - \ln \frac{5}{6} = \ln \frac{9}{5} \quad 1$$

*Method 2*

$$\int_{\ln \frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = [\ln(e^x - e^{-x})]_{\ln \frac{3}{2}}^{\ln 2} \quad 1,1$$

$$= \ln\left(2 - \frac{1}{2}\right) - \ln\left(\frac{3}{2} - \frac{2}{3}\right) \quad 1$$

$$= \ln \frac{3}{2} - \ln \frac{5}{6} = \ln \frac{9}{5} \quad 1$$

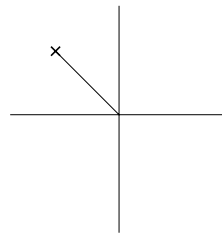
6.

$$\frac{(1 + 2i)^2}{7 - i} = \frac{1 + 4i - 4}{7 - i} \quad 1$$

$$= \frac{-3 + 4i}{7 - i} \times \frac{7 + i}{7 + i} \quad 1$$

$$= \frac{(-3 + 4i)(7 + i)}{50}$$

$$= -\frac{1}{2} + \frac{1}{2}i \quad 1$$



1

$$|z| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}\sqrt{2} \quad 1$$

$$\arg z = \tan^{-1} \frac{\frac{1}{2}}{-\frac{1}{2}} = \tan^{-1}(-1) = \frac{3\pi}{4} \text{ (or } 135^\circ\text{)}. \quad 1$$

7.

$$x = 2 \sin \theta \Rightarrow dx = 2 \cos \theta d\theta \quad \mathbf{1}$$

$$x = 0 \Rightarrow \theta = 0; x = \sqrt{2} \Rightarrow \sin \theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \quad \mathbf{1}$$

$$\begin{aligned} \int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}} dx &= \int_0^{\pi/4} \frac{4 \sin^2 \theta}{\sqrt{4-4 \sin^2 \theta}} (2 \cos \theta) d\theta \\ &= \int_0^{\pi/4} \frac{4 \sin^2 \theta}{2 \cos \theta} (2 \cos \theta) d\theta & \mathbf{1} \\ &= 2 \int_0^{\pi/4} (2 \sin^2 \theta) d\theta \\ &= 2 \int_0^{\pi/4} (1 - \cos 2\theta) d\theta & \mathbf{1} \\ &= 2 \left[ \theta - \frac{1}{2} \sin 2\theta \right]_0^{\pi/4} & \mathbf{1} \\ &= 2 \left\{ \left[ \frac{\pi}{4} - \frac{1}{2} \right] - 0 \right\} \\ &= \frac{\pi}{2} - 1 & \mathbf{1} \end{aligned}$$

8.

(a)  $(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \quad \mathbf{1}$

(b) Let  $x = -0.1$ , then  $\mathbf{1}$

$$\begin{aligned} 0.9^5 &= (1 + (-0.1))^5 \\ &= 1 - 0.5 + 0.1 - 0.01 + 0.0005 - 0.00001 & \mathbf{1} \\ &= 0.5 + 0.09 + 0.00049 \\ &= 0.59049 \end{aligned}$$

9.

$$\begin{aligned} \int_0^1 x \tan^{-1} x^2 dx &= \left[ \tan^{-1} x^2 \int x dx \right]_0^1 - \int_0^1 \frac{2x}{1+x^4} \frac{x^2}{2} dx & \mathbf{1,1} \\ &= \left[ \frac{1}{2} x^2 \tan^{-1} x^2 \right]_0^1 - \int_0^1 \frac{x^3}{1+x^4} dx \\ &= \left[ \frac{1}{2} x^2 \tan^{-1} x^2 \right]_0^1 - \left[ \frac{1}{4} \ln(1+x^4) \right]_0^1 & \mathbf{1} \\ &= \frac{1}{2} \tan^{-1} 1 - 0 - \left[ \frac{1}{4} \ln 2 - \frac{1}{4} \ln 1 \right] & \mathbf{1} \\ &= \frac{\pi}{8} - \frac{1}{4} \ln 2 & \mathbf{1,1} \end{aligned}$$

**10.**  $14654 = 11 \times 1326 + 68$  **1**  
 $1326 = 19 \times 68 + 34$   
 $68 = 2 \times 34$  **1**  
 $34 = 1326 - 19 \times 68$  **1**  
 $= 1326 - 19(14654 - 11 \times 1326)$   
 $= 210 \times 1326 - 19 \times 14654$  **1**

**11.** When  $x = 1, y = 1$ . **1**

$$y = x^{2x^2+1}$$

$$\Rightarrow \ln y = \ln(x^{2x^2+1})$$

$$= (2x^2 + 1) \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{2x^2 + 1}{x} + 4x \ln x$$
 **1,1**

Hence, when  $x = 1, y = 1$  and

$$\frac{dy}{dx} = 3 + 0 = 3.$$
 **1**

**12.**  $a_j = p^j \Rightarrow S_k = p + p^2 + \dots + p^k = \frac{p(p^k - 1)}{p - 1}$

$$S_n = \frac{p(p^n - 1)}{p - 1}$$
 **1**

$$S_{2n} = \frac{p(p^{2n} - 1)}{p - 1}$$
 **1**

$$\frac{p(p^{2n} - 1)}{p - 1} = \frac{65p(p^n - 1)}{p - 1}$$

$$(p^n + 1)(p^n - 1) = 65(p^n - 1)$$
 **1**

$$p^n + 1 = 65$$
 **1**

$$\Rightarrow p^n = 64$$

$$a_2 = p^2 \Rightarrow a_3 = p^3 \text{ but } a_3 = 2p \text{ so } p^3 = 2p$$

$$\Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2} \text{ since } p > 0.$$
 **1**

$$p^n = 64 = 2^6 = (\sqrt{2})^{12}$$

$$n = 12$$
 **1**

13.

$$f(x) = \frac{x^2 + 2x}{x^2 - 1} = \frac{x^2 + 2x}{(x - 1)(x + 1)}$$

Hence there are vertical asymptotes at  $x = -1$  and  $x = 1$ . 1

$$f(x) = \frac{x^2 + 2x}{x^2 - 1} = \frac{1 + \frac{2x}{x^2}}{1 - \frac{1}{x^2}} = \frac{1 + \frac{2}{x}}{1 - \frac{1}{x^2}}$$

$\rightarrow 1$  as  $x \rightarrow \infty$ .

So  $y = 1$  is a horizontal asymptote. 1

$$f(x) = \frac{x^2 + 2x}{x^2 - 1}$$

$$f'(x) = \frac{(2x + 2)(x^2 - 1) - (x^2 + 2x)2x}{(x^2 - 1)^2}$$

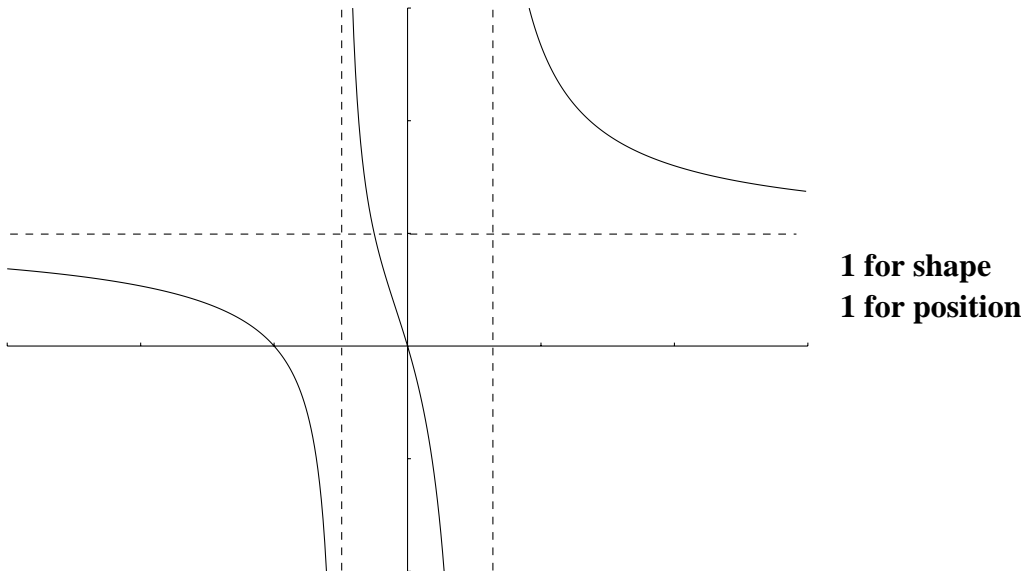
$$= \frac{2x^3 - 2x + 2x^2 - 2 - 2x^3 - 4x^2}{(x^2 - 1)^2} = \frac{-2(x^2 + x + 1)}{(x^2 - 1)^2}$$

$$= \frac{-2((x + \frac{1}{2})^2 + \frac{3}{4})}{(x^2 - 1)^2} < 0$$

Hence  $f(x)$  is a strictly decreasing function.

$$f(x) = \frac{x^2 + 2x}{x^2 - 1} = 0 \Rightarrow x = 0 \text{ or } x = -2$$

$$f(x) = \frac{x^2 + 2x}{x^2 - 1} = 1 \Rightarrow x^2 + 2x = x^2 - 1 \Rightarrow x = -\frac{1}{2}$$



Alternatively for the horizontal asymptote:

$$x^2 - 1 \left| \begin{array}{r} x^2 + 2x \\ x^2 \phantom{+ 2x} \\ \hline 2x + 1 \end{array} \right. \frac{1}{2x + 1} \Rightarrow f(x) = 1 + \frac{2x + 1}{x^2 - 1} \rightarrow 1 \text{ as } x \rightarrow \infty$$

14.

$$\frac{x^2 + 6x - 4}{(x + 2)^2(x - 4)} = \frac{A}{(x + 2)^2} + \frac{B}{x + 2} + \frac{C}{x - 4} \quad \mathbf{M1}$$

$$x^2 + 6x - 4 = A(x - 4) + B(x + 2)(x - 4) + C(x + 2)^2$$

Let  $x = -2$  then  $4 - 12 - 4 = -6A \Rightarrow A = 2.$  **1**

Let  $x = 4$  then  $16 + 24 - 4 = 36C \Rightarrow C = 1.$  **1**

Let  $x = 0$  then

$-4 = -4A - 8B + 4C \Rightarrow -4 = -8 - 8B + 4 \Rightarrow B = 0.$  **1**

Thus

$$\frac{x^2 + 6x - 4}{(x + 2)^2(x - 4)} = \frac{2}{(x + 2)^2} + \frac{1}{x - 4}.$$

Let  $f(x) = 2(x + 2)^{-2} + (x - 4)^{-1}$  then

$$f(x) = 2(x + 2)^{-2} + (x - 4)^{-1} \quad \Rightarrow \quad f(0) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \quad \mathbf{1}$$

$$f'(x) = -4(x + 2)^{-3} - (x - 4)^{-2} \quad \Rightarrow \quad f'(0) = -\frac{1}{2} - \frac{1}{16} = -\frac{9}{16} \quad \mathbf{1}$$

$$f''(x) = 12(x + 2)^{-4} + 2(x - 4)^{-3} \quad \Rightarrow \quad f''(0) = \frac{3}{4} - \frac{1}{32} = \frac{23}{32} \quad \mathbf{1}$$

Thus

$$\frac{x^2 + 6x - 4}{(x + 2)^2(x - 4)} = \frac{1}{4} - \frac{9x}{16} + \frac{23x^2}{64} + \dots \quad \mathbf{2E1}$$



15. (a)

$$(x + 1) \frac{dy}{dx} - 3y = (x + 1)^4$$

$$\frac{dy}{dx} - \frac{3}{x + 1}y = (x + 1)^3 \quad \mathbf{1}$$

Integrating factor:

$$\text{since } \int \frac{-3}{x + 1} dx = -3 \ln(x + 1). \quad \mathbf{1}$$

Hence the integrating factor is  $(x + 1)^{-3}$ .  $\mathbf{1}$

$$\frac{1}{(x + 1)^3} \frac{dy}{dx} - \frac{3}{(x + 1)^4}y = 1 \quad \mathbf{1}$$

$$\frac{d}{dx}((x + 1)^{-3}y) = 1$$

$$\frac{y}{(x + 1)^3} = \int 1 dx \quad \mathbf{1}$$

$$= x + c$$

$y = 16$  when  $x = 1$ , so  $2 = 1 + c \Rightarrow c = 1$ . Hence

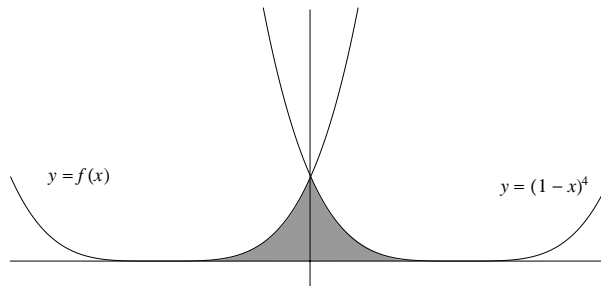
$$y = (x + 1)^4 \quad \mathbf{1}$$

(b)

$$(x + 1)^4 = (1 - x)^4$$

$$x + 1 = 1 - x \Rightarrow x = 0 \quad \mathbf{1}$$

or  $x + 1 = -1 + x$  which has no solutions.



$$\text{Area} = \int_{-1}^0 (x + 1)^4 dx + \int_0^1 (1 - x)^4 dx \quad \mathbf{M1}$$

$$= 2 \int_{-1}^0 (x + 1)^4 dx \quad \mathbf{1}$$

$$= \frac{2}{5} [(x + 1)^5]_{-1}^0 = \frac{2}{5} - 0 = \frac{2}{5} \quad \mathbf{1}$$

16. (a)

$$\begin{aligned}x + y - z &= 6 \\2x - 3y + 2z &= 2 \\-5x + 2y - 4z &= 1\end{aligned}$$

$$\begin{array}{ccc|ccc|ccc|c}1 & 1 & -1 & 6 & 1 & 1 & -1 & 6 & 1 & 1 & -1 & 6 \\2 & -3 & 2 & 2 & \Rightarrow & 0 & -5 & 4 & -10 & \Rightarrow & 0 & -5 & 4 & -10 \\-5 & 2 & -4 & 1 & & 0 & 7 & -9 & 31 & & 0 & 0 & -\frac{17}{5} & 17\end{array}$$

**1,1,1**

$$z = 17 \div \left(\frac{-17}{5}\right) = -5 \quad \mathbf{1}$$

$$-5y - 20 = -10 \Rightarrow y = -2$$

$$x - 2 + 5 = 6 \Rightarrow x = 3 \quad \mathbf{1}$$

(b) Let  $x = \lambda$ .

*Method 1*

In first plane:  $x + y - z = 6$ .

$$\lambda + (4\lambda - 14) - (5\lambda - 20) = 5\lambda - 5\lambda + 6 = 6. \quad \mathbf{1}$$

In the second plane:

$$2x - 3y + 2z = 2\lambda - 3(4\lambda - 14) + 2(5\lambda - 20) = 5\lambda - 5\lambda + 2 = 2. \quad \mathbf{1}$$

*Method 2*

$$y - z = 6 - \lambda \Rightarrow y = 6 + z - \lambda$$

$$-3y + 2z = 2 - 2\lambda \quad \mathbf{1}$$

$$(-18 - 3z + 3\lambda) + 2z = 2 - 2\lambda$$

$$-z = 20 - 5\lambda \Rightarrow z = 5\lambda - 20 \quad \mathbf{1}$$

$$\text{and } y = 4\lambda - 14$$

*Method 2*

$$x + y - z = 6 \quad (1)$$

$$2x - 3y + 2z = 2 \quad (2)$$

$$5x - z = 20 \quad (2) + 3(1)$$

$$4x - y = 14 \quad (2) + 2(1) \quad \mathbf{1}$$

$$y = 4x - 14$$

$$z = 5x - 20$$

$$x = \lambda, y = 4\lambda - 14, z = 5\lambda - 20 \quad \mathbf{1}$$

(c) Direction of  $L$  is  $\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ , direction of normal to the plane is  $-5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ . Letting  $\theta$  be the angle between these then

$$\cos \theta = \frac{-5 + 8 - 20}{\sqrt{42}\sqrt{45}} \quad \mathbf{1M,1}$$

$$= \frac{-17}{3\sqrt{210}}$$

This gives a value of  $113.0^\circ$  which leads to the angle  $113.0^\circ - 90^\circ = 23.0^\circ$ .

**1,1**