Solutions Included

national5maths.co.uk

AH Mathematics

Practice Assessment One

Applications of Algebra & Calculus

Applications of Algebra and Calculus Assessment Standard 1.1

1 Expand $(2x + 3)^5$ using the Binomial theorem. **(3)**

Applications of Algebra and Calculus Assessment Standard 1.2

2 Complex numbers are defined as follows : $z_1 = p + i$ and $z_2 = 4 - 3i$.

Express the following in the form $a + ib$:

a) $z_1 z_2$

b) $\frac{z_1}{z_2}$ **(3)**

3 An arithmetic sequence is given by : 5, 17, 29, 41, ...

Find :

a) the 20^{th} term of the sequence. **(2)**

b) the sum of the first 20 terms. **(2)**

4 A geometric sequence is given by : 6, 30, 150, 750, ...

Find :

a) the 9^{th} term of the sequence. **(2)**

b) the sum of the first 9 terms. **(2)**

5 Find the first four terms of the Maclaurin series for $f(x) = e^{4x}$. **(3)**

Applications of Algebra and Calculus Assessment Standard 1.3

6 Evaluate $\sum_{k=1}^{14} (4k - 3)$. **(3)**

7 Use proof by induction to show that, for all $n \geq 1,\ n \in \mathbb{N}$

$$\sum_{r=1}^{n} 4r = 2n(n+1) \ . \qquad \textbf{(5)}$$

Applications of Algebra and Calculus Assessment Standard 1.4

8 $f(x) = \dfrac{x^2 - x + 3}{x - 2},\ x \in \mathbb{R} : x \neq 2$

For the graph $y = f(x)$:

a) Give the equation of the vertical asymptote. **(1)**

b) Show that there is a non-vertical asymptote and state the equation. **(2)**

9 Find the coordinates of the point of inflection on the graph of $f(x) = 4x^3 - 12x^2 + 5x$. **(4)**

10 Given that $f(x) = \sin(2x)$, sketch the graph of $|4f(x)|$ where $0 \leq x \leq \pi$. **(2)**

Applications of Algebra and Calculus Assessment Standard 1.5

11 A car begins travelling from rest along a straight road. Its velocity, $v(t)$ metres per second, is given by

$$v(t) = \frac{200t}{3t + 13} .$$

Find the acceleration of the car at 3 seconds. **(4)**

12 The area bounded by the curve $y = \sqrt{1 + \cos 2x}$ between $x = 0$

and $x = \dfrac{\pi}{2}$ is rotated 2π radians about the x – axis.

Determine the exact value of the volume of the solid formed. **(4)**

Applications of Algebra and Calculus (Practice 1) - Sol^{n}s

1) $(2x+3)^5$

$$= \sum_{r=0}^{5} \binom{5}{r} (2x)^{5-r} 3^r$$

$$= \binom{5}{0}(2x)^5 3^0 + \binom{5}{1}(2x)^4 3^1 + \binom{5}{2}(2x)^3 3^2$$

$$+ \binom{5}{3}(2x)^2 3^3 + \binom{5}{4}(2x)^1 3^4 + \binom{5}{5}(2x)^0 3^5$$

$$= 1 \cdot 32x^5 \cdot 1 + 5 \cdot 16x^4 \cdot 3 + 10 \cdot 8x^3 \cdot 9$$

$$+ 10 \cdot 4x^2 \cdot 27 + 5 \cdot 2x \cdot 81 + 1 \cdot 1 \cdot 243$$

$$\therefore \boxed{(2x+3)^5 = 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243}$$

2) $z_1 = p + i, \quad z_2 = 4 - 3i$

(a) $z_1 z_2 = (p+i)(4-3i)$

$= 4p + 4i - 3pi - 3i^2$

$= 4p + 4i - 3pi + 3$

$\therefore \boxed{z_1 z_2 = (4p+3) + (4-3p)i}$

(b) $\dfrac{z_1}{z_2} = \dfrac{p+i}{4-3i} \begin{matrix} \times (4+3i) \\ \times (4+3i) \end{matrix}$

$= \dfrac{(p+i)(4+3i)}{(4-3i)(4+3i)}$

$= \dfrac{4p + 4i + 3pi + 3i^2}{16 + 9}$

$= \dfrac{(4p-3) + (4+3p)i}{25}$

$\therefore \boxed{\dfrac{z_1}{z_2} = \dfrac{4p-3}{25} + \dfrac{4+3p}{25} i}$

3) $5, 17, 29, 41, \ldots$

(a) $a = 5, d = 12$, $u_n = a + (n-1)d$

$\therefore u_n = 5 + (n-1)12$

$\Rightarrow u_n = 12n - 7$

$\therefore u_{20} = 12.20 - 7$

$\Rightarrow u_{20} = 240 - 7$

$\Rightarrow \boxed{u_{20} = 233}$

(b) $S_n = \frac{n}{2}(2a + (n-1)d)$

$\therefore S_n = \frac{n}{2}(2(5) + (n-1)12)$

$\Rightarrow S_n = \frac{n}{2}(10 + 12n - 12)$

$\Rightarrow S_n = \frac{n}{2}(12n - 2)$

$\Rightarrow S_n = n(6n - 1)$

$\therefore S_{20} = 20(6.20 - 1)$

$\Rightarrow S_{20} = 20(120 - 1)$

$\Rightarrow \boxed{S_{20} = 2380}$

4) $6, 30, 150, 750, \ldots$

(a) $a = 6, \ r = 5$

$u_n = a \cdot r^{n-1}$

$\therefore u_n = 6 \cdot 5^{n-1}$

$\therefore u_9 = 6 \cdot 5^8$

$\Rightarrow \boxed{u_9 = 2\,343\,750}$

(b) $S_n = \dfrac{a(1-r^n)}{1-r}$

$\therefore S_n = \dfrac{6\cdot(1-5^n)}{1-5}$

$\Rightarrow S_n = \dfrac{6(5^n-1)}{4}$

$\therefore S_9 = \dfrac{6(5^9-1)}{4}$

$\Rightarrow \boxed{S_9 = 2\,929\,686}$

5) $f(x) = e^{4x}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\therefore e^{4x} = 1 + (4x) + \frac{(4x)^2}{2} + \frac{(4x)^3}{6} + \ldots$$

$$\Rightarrow e^{4x} = 1 + 4x + \frac{16x^2}{2} + \frac{64x^3}{6} + \ldots$$

$$\Rightarrow \boxed{e^{4x} = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \ldots}$$

6) $\sum_{k=1}^{14} (4k-3)$

$$= 4\left(\sum_{k=1}^{14} k\right) - 3\left(\sum_{k=1}^{14} 1\right)$$

$$= 4\left(\tfrac{1}{2} \cdot 14(14+1)\right) - 3.14$$

$$= 28.15 - 42$$

$$= \boxed{378}$$

7) $P(n): \sum_{r=1}^{n} 4r = 2n(n+1)$

Base Case $(n=1)$:

$$LHS = \sum_{r=1}^{1} 4r = 4.1 = 4$$

$$RHS = 2.1.2 = 4$$

So, as $LHS = RHS$, $P(1)$ is true.

Inductive Step:

Assume $P(k)$ is true, i.e., $\sum_{r=1}^{k} 4r = 2k(k+1)$.

RTP:

$$\sum_{r=1}^{k+1} 4r = 2(k+1)(k+2)$$

$$\sum_{r=1}^{k+1} 4r = \left(\sum_{r=1}^{k} 4r\right) + 4(k+1)$$

$$= 2k(k+1) + 4(k+1)$$

$$= 2(k+1)(k+2)$$

So, $P(k)$ true $\Rightarrow P(k+1)$ is true; together with $P(1)$ true, the PMI $\Rightarrow P(n)$ is true $\forall n \in \mathbb{N}$.

8) $$f(x) = \frac{x^2 - x + 3}{x - 2}$$

(a) $\boxed{\text{Vertical asymptote: } x = 2}$

(b)

$$\begin{array}{r|l} & x + 1 \\ \hline x - 2 & x^2 - x + 3 \\ & x^2 - 2x \\ \hline & \quad\; x + 3 \\ & \quad\; x - 2 \\ \hline & \qquad 5 \end{array}$$

$$\therefore \; f(x) = x + 1 + \frac{5}{x - 2}$$

$\therefore$ $\boxed{\text{Non-vertical asymptote: } y = x + 1}$

9) $f(x) = 4x^3 - 12x^2 + 5x$

$\therefore \; f'(x) = 12x^2 - 24x + 5$

$\therefore \; f''(x) = 24x - 24$

For P of I, $f''(x) = 0$; so,

$$24x - 24 = 0$$

$$\Rightarrow \quad \underline{x = 1}$$

For SPs, $f'(x) = 0$; so,

$$12x^2 - 24x + 5 = 0$$

$D = 336 > 0$. Hence, there are 2 SPs.

$$f'(1) = 12 - 24 + 5$$

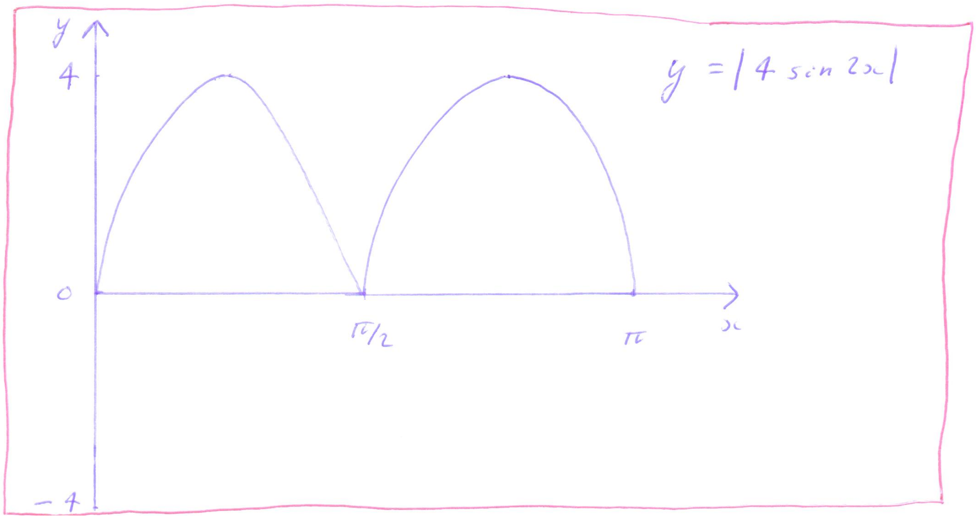
$$\Rightarrow \underline{f'(1) = -7 \neq 0}$$

So, <u>$x = 1$ is not a SP.</u>

$f(1) = 4 - 12 + 5 \Rightarrow \underline{f(1) = -3}$

$\therefore$ $\boxed{\text{P of I : } (1, -3)}$

10) $f(x) = \sin 2x$



11) $$V(t) = \frac{200t}{3t+13}$$

$$a(t) = v'(t) = \frac{200(3t+13) - 3(200t)}{(3t+13)^2}$$

$$\therefore\ a(t) = \frac{600t + 2600 - 600t}{(3t+13)^2}$$

$$\Rightarrow\ a(t) = \frac{2600}{(3t+13)^2}$$

$$\therefore\ a(3) = \frac{2600}{(9+13)^2}$$

$$\Rightarrow\ a(3) = \frac{2600}{484}$$

$$\Rightarrow\ \boxed{a(3) = \frac{650}{121}\ (\approx 5\cdot 37)\ ms^{-2}}$$

12) $V = \pi \int_0^{\pi/2} y^2 \, dx$

$\therefore V = \pi \int_0^{\pi/2} (1 + \cos 2x) \, dx$

$\Rightarrow V = \pi \left[x + \frac{1}{2} \sin 2x \right]_0^{\pi/2}$

$\Rightarrow V = \pi \left(\pi/2 + \frac{1}{2} \sin \pi \right) - \pi \left(0 + \frac{1}{2} \sin 0 \right)$

$\Rightarrow V = \frac{\pi^2}{2} + \frac{\pi}{2} . 0 - \pi . 0 - \frac{\pi}{2} . 0$

$\Rightarrow \boxed{V = \frac{\pi^2}{2}}$