Solutions Included

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AH Mathematics

Practice Assessment One

Geometry, Proof & Systems of Equations

Geometry, Proof and Systems of Equations Assessment Standard 1.1

1 Solve the following system of equations using Gaussian elimination

$$\begin{aligned} 3x + y + 2z &= -9 \\ -x + 4z &= -1 \\ 2x + 4y + z &= 1 \end{aligned} \qquad \textbf{(5)}$$

2 Given the matrices $A = \begin{pmatrix} 4 & 2 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} p & 0 \\ 3 & -5 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & -1 \\ -4 & q \end{pmatrix}$

where p and q are constants.

Find

a) $3A - 2B + C$. **(2)**

b) BC. **(2)**

3 Given the matrices $D = \begin{pmatrix} 2 & -6 \\ 4 & p \end{pmatrix}$ and $E = \begin{pmatrix} q & 1 & -5 \\ 0 & 1 & 2 \\ 1 & -3 & 1 \end{pmatrix}$,

a) Find D^{-1}. **(2)**

b) Determine the value(s) of q for which E is singular. **(3)**

Geometry, Proof and Systems of Equations Assessment Standard 1.2

4 Given the vectors $\mathbf{a} = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$,
calculate $\mathbf{a} \times \mathbf{b}$. **(3)**

5 Find, in vector form, an equation for the line which passes through the points $(2, -3, 4)$ and $(1, 7, -3)$. **(2)**

6 Find, in Cartesian form, the equation of the plane which has normal vector $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ and passes through the point $(5, -2, 6)$. **(2)**

Geometry, Proof and Systems of Equations Assessment Standard 1.3

7 Given $z = 4\sqrt{3} - 4i$

a) Find the modulus and argument of z using exact values.

b) Write z in polar form. **(3)**

8 Given $z = 4\left(\cos -\frac{\pi}{3} + i \sin -\frac{\pi}{3}\right)$

a) Write z in Cartesian form using exact values. **(2)**

b) Plot z on an Argand diagram. **(1)**

Geometry, Proof and Systems of Equations Assessment Standard 1.4

9 Use the Euclidean algorithm to obtain the greatest common divisor of 1 440 and 1 005. **(3)**

Geometry, Proof and Systems of Equations Assessment Standard 1.5

10 For any real numbers a and b, it is conjectured that

$$a > b + 3 \Rightarrow (a - 3)^2 > b^2$$

Use a counterexample to disprove this conjecture. **(3)**

11 Prove, by contradiction, that if $x + 3$ is irrational then x is irrational. **(4)**

12 Use direct proof to show that the product of any two odd numbers is odd. **(3)**

Geometry, Proof and Systems of Equations (Practice 1) - Sol^{n}s

1)

$$\begin{aligned} 3x + y + 2z &= -9 \\ -x \quad\quad + 4z &= -1 \\ 2x + 4y + z &= 1 \end{aligned}$$

Augmented matrix is,

$$\left(\begin{array}{ccc|c} 3 & 1 & 2 & -9 \\ -1 & 0 & 4 & -1 \\ 2 & 4 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 + 2R_2]{R_1 \to R_1 + 3R_2} \left(\begin{array}{ccc|c} 0 & 1 & 14 & -12 \\ -1 & 0 & 4 & -1 \\ 0 & 4 & 9 & -1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 4R_1} \left(\begin{array}{ccc|c} 0 & 1 & 14 & -12 \\ -1 & 0 & 4 & -1 \\ 0 & 0 & -47 & 47 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & 0 & 4 & -1 \\ 0 & 1 & 14 & -12 \\ 0 & 0 & -47 & 47 \end{array}\right)$$

$$\therefore \quad \begin{aligned} -x + 4z &= -1 \quad (1) \\ y + 14z &= -12 \quad (2) \\ -47z &= 47 \quad (3) \Rightarrow \underline{z = -1} \,; \end{aligned}$$

$(2) \Rightarrow y = -12 - 14(-1) \Rightarrow \underline{y = 2}$; $(1) \Rightarrow x = -4 + 1 \Rightarrow \underline{x = -3}$

$$\therefore \boxed{x = -3, \; y = 2, \; z = -1}$$

2) $A = \begin{pmatrix} 4 & 2 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} p & 0 \\ 3 & -5 \end{pmatrix}$, $C = \begin{pmatrix} 3 & -1 \\ -4 & q \end{pmatrix}$.

(a) $3A - 2B + C = 3\begin{pmatrix} 4 & 2 \\ -1 & 0 \end{pmatrix} - 2\begin{pmatrix} p & 0 \\ 3 & -5 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ -4 & q \end{pmatrix}$

$\Rightarrow 3A - 2B + C = \begin{pmatrix} 12 & 6 \\ -3 & 0 \end{pmatrix} - \begin{pmatrix} 2p & 0 \\ 6 & -10 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ -4 & q \end{pmatrix}$

$\Rightarrow \boxed{3A - 2B + C = \begin{pmatrix} 15 - 2p & 5 \\ -13 & 10 + q \end{pmatrix}}$

(b) $BC = \begin{pmatrix} p & 0 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -4 & q \end{pmatrix}$

$\Rightarrow BC = \begin{pmatrix} 3p - 0 & -p + 0 \\ 9 + 20 & -3 - 5q \end{pmatrix}$

$\Rightarrow \boxed{BC = \begin{pmatrix} 3p & -p \\ 29 & -3 - 5q \end{pmatrix}}$

3) $D = \begin{pmatrix} 2 & -6 \\ 4 & p \end{pmatrix}$, $E = \begin{pmatrix} 9 & 1 & -5 \\ 0 & 1 & 2 \\ 1 & -3 & 1 \end{pmatrix}$

(a) $D^{-1} = \begin{pmatrix} 2 & -6 \\ 4 & p \end{pmatrix}^{-1}$

$\Rightarrow D^{-1} = \frac{1}{2p - 4(-6)} \begin{pmatrix} p & 6 \\ -4 & 2 \end{pmatrix}$

$\Rightarrow \boxed{D^{-1} = \frac{1}{2p + 24} \begin{pmatrix} p & 6 \\ -4 & 2 \end{pmatrix}}$

(b) $|E| = q\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + (-5)\begin{vmatrix} 0 & 1 \\ 1 & -3 \end{vmatrix}$

$\therefore |E| = q(1+6) - (-2) - 5(-1)$

$\Rightarrow |E| = 7q + 2 + 5$

$\Rightarrow |E| = 7q + 7$

E singular $\Rightarrow |E| = 0$. So,

$$7q + 7 = 1$$

$$\Rightarrow \boxed{q = -1}$$

4) $\underline{a} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}, \ \underline{b} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

$\therefore \underline{a} \times \underline{b} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

$\Rightarrow \underline{a} \times \underline{b} = \begin{pmatrix} 1 \cdot 1 - (-1)(-2) \\ -1 \cdot 3 - 5 \cdot 1 \\ 5 \cdot (-2) - 1 \cdot 3 \end{pmatrix}$

$\Rightarrow \boxed{\underline{a} \times \underline{b} = \begin{pmatrix} -1 \\ -8 \\ -13 \end{pmatrix}}$

5) $\underline{r} = \underline{a} + t\,\underline{d} \quad (\underline{d} = \underline{b} - \underline{a})$

$\underline{a} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \; ; \; \underline{b} = \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$

$\therefore \underline{d} = \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \Rightarrow \underline{\underline{d} = \begin{pmatrix} -1 \\ 10 \\ -7 \end{pmatrix}}$

$\therefore \boxed{\underline{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + t\begin{pmatrix} -1 \\ 10 \\ -7 \end{pmatrix}}$

6) $ax + by + cz = k$

$\underline{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$

$\therefore 3x - y + 4z = k \qquad (\pi)$

$(5, -2, 6) \in \pi$, so,

$3(5) - (-2) + 4(6) = k \Rightarrow \underline{k = 41}$

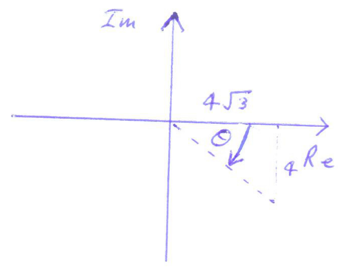
$\therefore \boxed{3x - y + 4z = 41}$

7) $z = 4\sqrt{3} - 4i$

(a) $|z| = \sqrt{(4\sqrt{3})^2 + (-4)^2}$

$\therefore |z| = \sqrt{16(3) + 16} \Rightarrow |z| = \sqrt{64} \Rightarrow \boxed{|z| = 8}$

$\tan\theta = -\frac{4}{4\sqrt{3}}$



$\therefore \tan\theta = -\frac{1}{\sqrt{3}}$

$\Rightarrow \underline{RA = \frac{\pi}{6}}$

$(\theta = \arg z)$

$\therefore \theta = \boxed{\arg z = -\frac{\pi}{6}}$

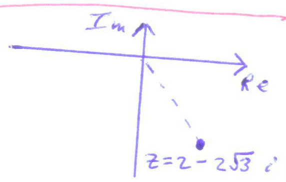
(b) $z = r(\cos\theta + i\sin\theta)$

$\therefore \boxed{z = 8\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)}$

8) (a) $z = 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

$\therefore z = 4\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right) \Rightarrow \boxed{z = 2 - 2\sqrt{3}\,i}$

(b)



9)

$$1440 = 1005 \cdot 1 + 435$$

$$1005 = 435 \cdot 2 + 135$$

$$435 = 135 \cdot 3 + 30$$

$$135 = 30 \cdot 4 + 15$$

$$30 = 15 \cdot 2 + 0$$

$\therefore$ $\boxed{GCD(1440, 1005) = 15}$

10)

$$a > b + 3 \Rightarrow (a-3)^2 > b^2 \quad (C)$$

Let $\boxed{a = 4,\ b = -5}$.

Then $b + 3 = -5 + 3 = -2$; so,

$a = 4 > -2 = b + 3$, i.e., $\underline{a > b + 3}$

$$(a-3)^2 = (4-3)^2 = 1^2 = 1$$

$$b^2 = (-5)^2 = 25$$

So, $\underline{(a-3)^2 \not> b^2}$ as $1 < 25$.

Hence, $\boxed{\text{conjecture (C) is false}}$

11) Assume $x+3$ is irrational and x is rational.

So, $\exists$ $p, q \in \mathbb{Z}$ $(q \neq 0)$ s.t.,

$$x = \frac{p}{q}$$

$$\therefore \quad x+3 = \frac{p}{q} + 3$$

$$\Rightarrow \quad x+3 = \frac{p+3q}{q} \quad (q \neq 0)$$

As $p, q \in \mathbb{Z}$, $p+3q \in \mathbb{Z}$; so, $x+3 \in \mathbb{Q}$.

But this contradicts $x+3$ being irrational.

So,

$$\boxed{x+3 \text{ irrational} \Rightarrow x \text{ irrational}}$$

12) Let $a = 2m+1$, $b = 2n+1$ $(m, n \in \mathbb{N})$ be 2 odd numbers. Then,

$$ab = (2m+1)(2n+1)$$

$$\Rightarrow \quad ab = 4mn + 2m + 2n + 1$$

$$\Rightarrow \quad ab = 2(2mn + m + n) + 1$$

As $m, n \in \mathbb{N}$, $2mn + m + n \in \mathbb{N}$; so ab is odd.

$$\therefore \quad \boxed{a, b \text{ odd} \Rightarrow ab \text{ odd}}$$