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AH Mathematics

Practice Assessment Two

Applications of Algebra & Calculus

Applications of Algebra and Calculus Assessment Standard 1.1

1 Expand $(3x + 2)^5$ using the Binomial theorem. **(3)**

Applications of Algebra and Calculus Assessment Standard 1.2

2 Complex numbers are defined as follows : $z_1 = p + i$ and $z_2 = 5 - 2i$.

Express the following in the form $a + ib$:

a) $z_1 z_2$

b) $\dfrac{z_1}{z_2}$ **(3)**

3 An arithmetic sequence is given by : 6, 20, 34, 48, ...

Find :

a) the 40^{th} term of the sequence. **(2)**

b) the sum of the first 40 terms. **(2)**

4 A geometric sequence is given by : 12, 84, 588, 4 116, ...

Find :

a) the 7^{th} term of the sequence. **(2)**

b) the sum of the first 7 terms. **(2)**

5 Find the first four terms of the Maclaurin series for $f(x) = e^{3x}$. **(3)**

Applications of Algebra and Calculus Assessment Standard 1.3

6 Evaluate $\displaystyle\sum_{k=1}^{12} (5k - 4)$. **(3)**

7 Use proof by induction to show that, for all $n \geq 1,\ n \in \mathbb{N}$

$$\sum_{r=1}^{n} 8r = 4n(n+1)\ .$$ **(5)**

Applications of Algebra and Calculus Assessment Standard 1.4

8 $f(x) = \dfrac{x^2 + 3x - 14}{x - 3},\ x \in \mathbb{R} : x \neq 3$

For the graph $y = f(x)$:

a) Give the equation of the vertical asymptote. **(1)**

b) Show that there is a non-vertical asymptote and state the equation. **(2)**

9 Find the coordinates of the point of inflection on the graph of $f(x) = 3x^3 - 18x^2 - 7x$. **(4)**

10 Given that $f(x) = \cos(2x)$, sketch the graph of $|5f(x)|$ where $0 \leq x \leq \pi$. **(2)**

Applications of Algebra and Calculus Assessment Standard 1.5

11 A car begins travelling from rest along a straight road. Its velocity, $v(t)$ metres per second, is given by

$$v(t) = \frac{300t}{4t + 9}.$$

Find the acceleration of the car at 2 seconds. **(4)**

12 The area bounded by the curve $y = \sqrt{1 + \sin 3x}$ between $x = 0$ and $x = \dfrac{\pi}{3}$ is rotated 2π radians about the x – axis.

Determine the exact value of the volume of the solid formed. **(4)**

Applications of Algebra and Calculus (Practice 2) - Sol^{n}s

1) $(3x+2)^5$

$$= \sum_{r=0}^{5} \binom{5}{r} (3x)^{5-r} 2^r$$

$$= \binom{5}{0}(3x)^5 2^0 + \binom{5}{1}(3x)^4 2^1 + \binom{5}{2}(3x)^3 2^2$$
$$+ \binom{5}{3}(3x)^2 2^3 + \binom{5}{4}(3x)^1 2^4 + \binom{5}{5}(3x)^0 2^5$$

$$= 1.243x^5.1 + 5.81x^4.2 + 10.27x^3.4$$
$$+ 10.9x^2.8 + 5.3x.16 + 1.1.32$$

$$\therefore \boxed{(3x+2)^5 = 243x^5 + 810x^4 + 1080x^3 + 720x^2 + 240x + 32}$$

2) $z_1 = p + i, \quad z_2 = 5 - 2i$

(a) $z_1 z_2 = (p+i)(5-2i)$

$= 5p + 5i - 2pi - 2i^2$

$= 5p + 5i - 2pi + 2$

$\therefore \boxed{z_1 z_2 = (5p+2) + (5-2p)i}$

(b) $\dfrac{z_1}{z_2} = \dfrac{p+i}{5-2i} \quad \dfrac{\times (5+2i)}{\times (5+2i)}$

$= \dfrac{(p+i)(5+2i)}{(5-2i)(5+2i)}$

$= \dfrac{5p + 5i + 2pi + 2i^2}{25+4}$

$= \dfrac{(5p-2) + (5+2p)i}{29}$

$\therefore \boxed{\dfrac{z_1}{z_2} = \dfrac{5p-2}{29} + \dfrac{5+2p}{29} i}$

3) $6, 20, 34, 48, \ldots$

(a) $a = 6$, $d = 14$, $u_n = a + (n-1)d$

$\therefore \quad u_n = 6 + (n-1)14$

$\Rightarrow \quad \underline{u_n = 14n - 8}$

$\therefore \quad u_{40} = 14 . 40 - 8$

$\Rightarrow \quad u_{40} = 560 - 8$

$\Rightarrow \quad \boxed{u_{40} = 552}$

(b) $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

$\therefore \quad S_n = \frac{n}{2}\left(2(6) + (n-1)14\right)$

$\Rightarrow \quad S_n = \frac{n}{2}\left(12 + 14n - 14\right)$

$\Rightarrow \quad S_n = \frac{n}{2}\left(14n - 2\right)$

$\Rightarrow \quad \underline{S_n = n(7n - 1)}$

$\therefore \quad S_{40} = 40(7 . 40 - 1)$

$\Rightarrow \quad S_{40} = 40(280 - 1)$

$\Rightarrow \quad \boxed{S_{40} = 11\,160}$

4) $12, 84, 588, 4116, \ldots$

(a) $a = 12, \; r = 7$

$u_n = a \cdot r^{n-1}$

$\therefore \; u_n = 12 \cdot 7^{n-1}$

$\therefore \; u_7 = 12 \cdot 7^6$

$\Rightarrow \boxed{u_7 = 1\,411\,788}$

(b) $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$\therefore \; S_n = \dfrac{12(7^n - 1)}{7 - 1}$

$\Rightarrow S_n = \dfrac{12(7^n - 1)}{6}$

$\therefore \; S_7 = 2(7^7 - 1)$

$\Rightarrow \boxed{S_7 = 1\,647\,084}$

5) $f(x) = e^{3x}$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\therefore\ e^{3x} = 1 + 3x + \frac{(3x)^2}{2} + \frac{(3x)^3}{6} + \dots$$

$$\Rightarrow\ e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{6} + \dots$$

$$\Rightarrow\ \boxed{e^{3x} = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \dots}$$

6) $\sum_{k=1}^{12} (5k - 4)$

$$= 5\left(\sum_{k=1}^{12} k\right) - 4\left(\sum_{k=1}^{12} 1\right)$$

$$= 5\left(\tfrac{1}{2} . 12 . (12+1)\right) - 4 . 12$$

$$= 30 . 13 - 48$$

$$= \boxed{342}$$

7) $P(n): \sum_{r=1}^{n} 8r = 4n(n+1)$

Base Case ($n=1$):

$$LHS = \sum_{r=1}^{1} 8r = 8.1 = 8$$

$$RHS = 4.1.2 = 8$$

So, as LHS = RHS, P(1) is true.

Inductive Step:

Assume P(k) is true, i.e., $\sum_{r=1}^{k} 8r = 4k(k+1)$.

RTP:

$$\left[\sum_{r=1}^{k+1} 8r = 4(k+1)(k+2) \right]$$

$$\sum_{r=1}^{k+1} 8r = \left(\sum_{r=1}^{k} 8r \right) + 8(k+1)$$

$$= 4k(k+1) + 8(k+1)$$

$$= 4(k+1)(k+2)$$

$$\boxed{\text{So, } P(k) \text{ true} \Rightarrow P(k+1) \text{ true; together with } P(1) \text{ true, the PMI} \Rightarrow P(n) \text{ is true } \forall n \in \mathbb{N}.}$$

8) $$f(x) = \frac{x^2+3x-14}{x-3}$$

(a) $\boxed{\text{Vertical asymptote : } x = 3}$

(b)

$$\begin{array}{r|l} & x + 6 \\ \hline x - 3 & x^2 + 3x - 14 \\ & x^2 - 3x \\ \hline & \quad 6x - 14 \\ & \quad 6x - 18 \\ \hline & \qquad\quad 4 \end{array}$$

$$\therefore \; f(x) = x + 6 + \frac{4}{x-3}$$

$\therefore$ $\boxed{\text{Non-vertical asymptote : } y = x + 6}$

9) $$f(x) = 3x^3 - 18x^2 - 7x$$

$$\therefore \; f'(x) = 9x^2 - 36x - 7$$

$$\therefore \; f''(x) = 18x - 36$$

For P of I, $f''(x) = 0$; so,

$$18x - 36 = 0$$

$$\Rightarrow \quad x = 2$$

For SPs, $f'(x) = 0$; so,

$$9x^2 - 36x - 7 = 0$$

$D = 1548 > 0$. Hence, there are 2 SPs.

$$f'(2) = -43 \neq 0$$

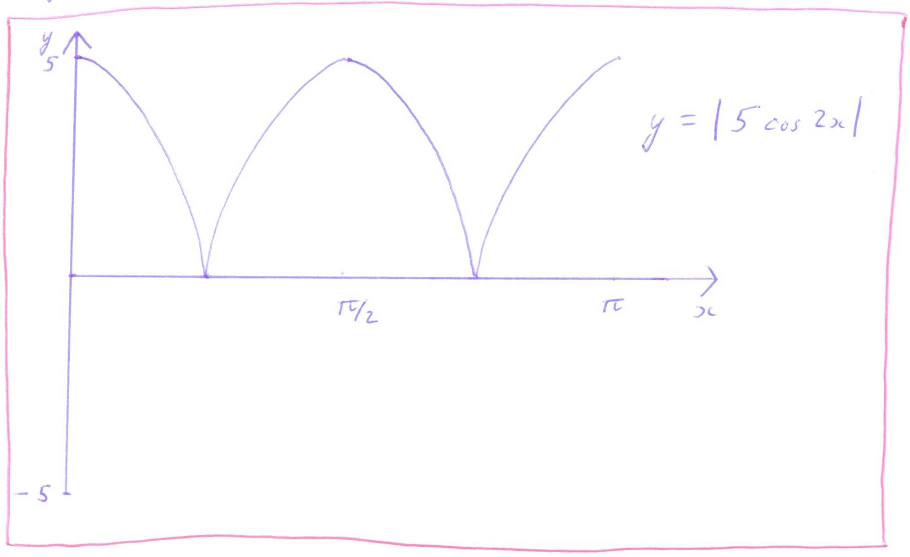
So, $x = 2$ is not a SP.

$$f(2) = 3(8) - 18(4) - 7(2)$$

$$\Rightarrow f(2) = -62$$

$\therefore$ $\boxed{\text{P of I : } (2, -62)}$

10) $f(x) = \cos 2x$



11) $$v(t) = \frac{300t}{4t+9}$$

$$a(t) = v'(t) = \frac{300(4t+9) - 4(300t)}{(4t+9)^2}$$

$$\therefore\ a(t) = \frac{1200t + 2700 - 1200t}{(4t+9)^2}$$

$$\Rightarrow\ a(t) = \frac{2700}{(4t+9)^2}$$

$$\therefore\ a(2) = \frac{2700}{(8+9)^2}$$

$$\Rightarrow\ \boxed{a(2) = \frac{2700}{289}\ (\simeq 9.34)\ ms^{-2}}$$

12) $V = \pi \int_0^{\pi/3} y^2 \, dx$

$\therefore V = \pi \int_0^{\pi/3} (1 + \sin 3x) \, dx$

$\Rightarrow V = \pi \left[x - \frac{1}{3} \cos 3x \right]_0^{\pi/3}$

$\Rightarrow V = \pi \left(\pi/3 - \frac{1}{3} \cos \pi \right) - \pi \left(0 - \frac{1}{3} \cos 0 \right)$

$\Rightarrow V = \frac{\pi^2}{3} - \frac{\pi}{3}(-1) - \pi . 0 + \frac{\pi}{3} . 1$

$\Rightarrow V = \frac{\pi^2}{3} + \frac{2\pi}{3}$

$\Rightarrow \boxed{V = \frac{\pi}{3}(\pi + 2)}$