Solutions Included

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AH Mathematics

Practice Assessment Two

Geometry, Proof & Systems of Equations

Geometry, Proof and Systems of Equations Assessment Standard 1.1

1 Solve the following system of equations using Gaussian elimination

$$\begin{aligned} 3x + y + 5z &= -26 \\ -x \quad\quad + 2z &= -2 \\ 4x + 3y + z &= -16 \end{aligned}$$ **(5)**

2 Given the matrices $A = \begin{pmatrix} 2 & 4 \\ -3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} p & 0 \\ 1 & -4 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & -7 \\ -2 & q \end{pmatrix}$

where p and q are constants.

Find

a) $A + 4B - 3C$. **(2)**

b) BC. **(2)**

3 Given the matrices $D = \begin{pmatrix} 5 & -2 \\ 1 & p \end{pmatrix}$ and $E = \begin{pmatrix} q & 1 & -4 \\ 0 & 1 & 3 \\ 1 & -4 & 1 \end{pmatrix}$,

a) Find D^{-1}. **(2)**

b) Determine the value(s) of q for which E is singular. **(3)**

Geometry, Proof and Systems of Equations Assessment Standard 1.2

4 Given the vectors $\mathbf{a} = 4\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, calculate $\mathbf{a} \times \mathbf{b}$. **(3)**

5 Find, in vector form, an equation for the line which passes through the points $(1, -4, 3)$ and $(6, 2, -4)$. **(2)**

6 Find, in Cartesian form, the equation of the plane which has normal vector $\begin{pmatrix} 5 \\ -3 \\ 11 \end{pmatrix}$ and passes through the point $(2, -4, 1)$. **(2)**

Geometry, Proof and Systems of Equations Assessment Standard 1.3

7 Given $z = -5 + 5\sqrt{3}\, i$

a) Find the modulus and argument of z using exact values.

b) Write z in polar form. **(3)**

8 Given $z = -4\left(\cos \dfrac{2\pi}{3} + i \sin \dfrac{2\pi}{3}\right)$

a) Write z in Cartesian form using exact values. **(2)**

b) Plot z on an Argand diagram. **(1)**

Geometry, Proof and Systems of Equations Assessment Standard 1.4

9 Use the Euclidean algorithm to obtain the greatest common divisor of 1 104 and 608. **(3)**

Geometry, Proof and Systems of Equations Assessment Standard 1.5

10 For any real numbers a and b, it is conjectured that

$$a > b + 5 \Rightarrow (a - 5)^2 > b^2$$

Use a counterexample to disprove this conjecture. **(3)**

11 Prove, by contradiction, that if $x - 6$ is irrational then x is irrational. **(4)**

12 Use direct proof to show that the product of any two even numbers is even. **(3)**

Geometry, Proof and Systems of Equations (Practice 2) - Sol^{n}s

1)

$$3x + y + 5z = -26$$
$$-x + 2z = -2$$
$$4x + 3y + z = -16$$

Augmented matrix is,

$$\left(\begin{array}{ccc|c} 3 & 1 & 5 & -26 \\ -1 & 0 & 2 & -2 \\ 4 & 3 & 1 & -16 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 + 4R_2]{R_1 \to R_1 + 3R_2} \left(\begin{array}{ccc|c} 0 & 1 & 11 & -32 \\ -1 & 0 & 2 & -2 \\ 0 & 3 & 9 & -24 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 3R_1} \left(\begin{array}{ccc|c} 0 & 1 & 11 & -32 \\ -1 & 0 & 2 & -2 \\ 0 & 0 & -24 & 72 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & 0 & 2 & -2 \\ 0 & 1 & 11 & -32 \\ 0 & 0 & -24 & 72 \end{array}\right)$$

$\therefore$ $-x + 2z = -2$ ①

$y + 11z = -32$ ②

$-24z = 72$ ③ $\Rightarrow \underline{z = -3}$;

② $\Rightarrow y = -32 - 11(-3) \Rightarrow \underline{y = 1}$; ① $\Rightarrow x = 2 - 6 \Rightarrow \underline{x = -4}$

$\therefore$ $\boxed{x = -4, y = 1, z = -3}$

2) $A = \begin{pmatrix} 2 & 4 \\ -3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} p & 0 \\ 1 & -4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & -7 \\ -2 & q \end{pmatrix}$

(a) $A + 4B - 3C = \begin{pmatrix} 2 & 4 \\ -3 & 0 \end{pmatrix} + 4\begin{pmatrix} p & 0 \\ 1 & -4 \end{pmatrix} - 3\begin{pmatrix} 2 & -7 \\ -2 & q \end{pmatrix}$

$\Rightarrow A + 4B - 3C = \begin{pmatrix} 2 & 4 \\ -3 & 0 \end{pmatrix} + \begin{pmatrix} 4p & 0 \\ 4 & -16 \end{pmatrix} - \begin{pmatrix} 6 & -21 \\ -6 & 3q \end{pmatrix}$

$\Rightarrow \boxed{A + 4B - 3C = \begin{pmatrix} 4p-4 & 25 \\ 7 & -16-3q \end{pmatrix}}$

(b) $BC = \begin{pmatrix} p & 0 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 2 & -7 \\ -2 & q \end{pmatrix}$

$\Rightarrow BC = \begin{pmatrix} 2p-0 & -7p+0 \\ 2+8 & -7-4q \end{pmatrix}$

$\Rightarrow \boxed{BC = \begin{pmatrix} 2p & -7p \\ 10 & -7-4q \end{pmatrix}}$

3) $D = \begin{pmatrix} 5 & -2 \\ 1 & p \end{pmatrix}$, $E = \begin{pmatrix} q & 1 & -4 \\ 0 & 1 & 3 \\ 1 & -4 & 1 \end{pmatrix}$

(a) $D^{-1} = \begin{pmatrix} 5 & -2 \\ 1 & p \end{pmatrix}^{-1}$

$\Rightarrow D^{-1} = \dfrac{1}{5p-(-2).1}\begin{pmatrix} p & 2 \\ -1 & 5 \end{pmatrix}$

$\Rightarrow \boxed{D^{-1} = \dfrac{1}{5p+2}\begin{pmatrix} p & 2 \\ -1 & 5 \end{pmatrix}}$

(b) $|E| = q\begin{vmatrix} 1 & 3 \\ -4 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} + (-4)\begin{vmatrix} 0 & 1 \\ 1 & -4 \end{vmatrix}$

$\therefore |E| = q(1+12) - (-3) - 4(-1)$

$\Rightarrow |E| = 13q + 7$

E singular $\Rightarrow |E| = 0$. So,

$$13q + 7 = 0$$

$$\Rightarrow \boxed{q = -\frac{7}{13}}$$

4) $\underline{a} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$\therefore \underline{a} \times \underline{b} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$\Rightarrow \underline{a} \times \underline{b} = \begin{pmatrix} 1.3 - (-1)(-1) \\ -1.2 - 4.3 \\ 4.(-1) - 1.2 \end{pmatrix}$

$$\Rightarrow \boxed{\underline{a} \times \underline{b} = \begin{pmatrix} 2 \\ -14 \\ -6 \end{pmatrix}}$$

5) $\underline{r} = \underline{a} + t\,\underline{d} \quad (\underline{d} = \underline{b} - \underline{a})$

$$\underline{a} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix}$$

$$\therefore \underline{d} = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \Rightarrow \underline{\underline{d} = \begin{pmatrix} 5 \\ 6 \\ -7 \end{pmatrix}}$$

$$\therefore \boxed{\underline{r} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 6 \\ -7 \end{pmatrix}}$$

6) $ax + by + cz = k$

$$\underline{n} = \begin{pmatrix} 5 \\ -3 \\ 11 \end{pmatrix}$$

$$\therefore 5x - 3y + 11z = k \qquad (\pi)$$

$(2, -4, 1) \in \pi$, so,

$$5(2) - 3(-4) + 11(1) = k \Rightarrow \underline{k = 33}$$

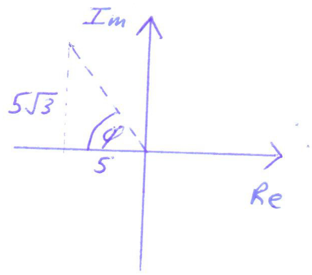
$$\therefore \boxed{5x - 3y + 11z = 33}$$

7) $z = -5 + 5\sqrt{3}\, i$

(a) $|z| = \sqrt{(-5)^2 + (5\sqrt{3})^2}$

$\therefore |z| = \sqrt{25 + 25(3)} \Rightarrow |z| = \sqrt{100} \Rightarrow \boxed{|z| = 10}$

$\tan\varphi = \dfrac{5\sqrt{3}}{5}$



$\therefore \tan\varphi = \sqrt{3}$

$\Rightarrow \varphi = \dfrac{\pi}{3}$

$\arg z = \pi - \varphi \Rightarrow \arg z = \pi - \dfrac{\pi}{3} \Rightarrow \boxed{\arg z = \dfrac{2\pi}{3}}$

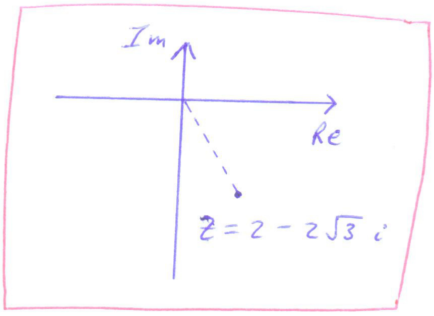
(b) $z = r(\cos\theta + i\sin\theta)$

$\therefore \boxed{z = 10\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)}$

8) (a) $z = -4\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

$\therefore z = -4\left(-\dfrac{1}{2} + i\left(\dfrac{\sqrt{3}}{2}\right)\right) \Rightarrow \boxed{z = 2 - 2\sqrt{3}\, i}$

(b)



9) $$1104 = 608 . 1 + 496$$
$$608 = 496 . 1 + 112$$
$$496 = 112 . 4 + 48$$
$$112 = 48 . 2 + 16$$
$$48 = 16 . 3$$

$\therefore$ $\boxed{GCD(1104, 608) = 16}$

10) $a > b + 5 \not\Rightarrow (a-5)^2 > b^2$ (C)

Let $\boxed{a = 6, b = -7}$.

Then $b + 5 = -7 + 5 = -2$; so,

$a = 6 > -2 = b + 5$, i.e., $\underline{a > b + 5}$

$$(a-5)^2 = (6-5)^2 = 1^2 = 1$$
$$b^2 = (-7)^2 = 49$$

So, $(a-5)^2 \not> b^2$ as $1 < 25$.

Hence, $\boxed{\text{conjecture (c) is false}}$

11) Assume $x-6$ is irrational and x is rational.

So, $\exists\ p, q \in \mathbb{Z}$ $(q \neq 0)$ s.t.,

$$x = \frac{p}{q}$$

$$\therefore\ x - 6 = \frac{p}{q} - 6$$

$$\Rightarrow\ x - 6 = \frac{p - 6q}{q} \quad (q \neq 0)$$

As $p, q \in \mathbb{Z}$, $p - 6q \in \mathbb{Z}$; so, $x - 6 \in \mathbb{Q}$.

But this contradicts $x-6$ being irrational.

So,

$$\boxed{x - 6 \text{ irrational} \Rightarrow x \text{ irrational}}$$

12) Let $a = 2m$, $b = 2n$ $(m, n \in \mathbb{N})$

be 2 even numbers. Then,

$$ab = (2m)(2n)$$

$$\Rightarrow ab = 4mn$$

$$\Rightarrow ab = 2(2mn)$$

As $m, n \in \mathbb{N}$, $2mn \in \mathbb{N}$; so, ab is even.

$$\therefore\ \boxed{a, b \text{ even} \Rightarrow ab \text{ even}}$$