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AH Mathematics

Practice Assessment Three

Applications
of
Algebra & Calculus

Applications of Algebra and Calculus Assessment Standard 1.1

1 Expand $(4x + 3)^5$ using the Binomial theorem. **(3)**

Applications of Algebra and Calculus Assessment Standard 1.2

2 Complex numbers are defined as follows : $z_1 = p + i$ and $z_2 = 3 - 5i$.

Express the following in the form $a + ib$:

a) $z_1 z_2$

b) $\dfrac{z_1}{z_2}$ **(3)**

3 An arithmetic sequence is given by : 7, 25, 43, 61, ...

Find :

a) the 16^{th} term of the sequence. **(2)**

b) the sum of the first 16 terms. **(2)**

4 A geometric sequence is given by : 4, 36, 324, 2 916, ...

Find :

a) the 5^{th} term of the sequence. **(2)**

b) the sum of the first 5 terms. **(2)**

5 Find the first four terms of the Maclaurin series for $f(x) = e^{6x}$. **(3)**

Applications of Algebra and Calculus Assessment Standard 1.3

6 Evaluate $\displaystyle\sum_{k=1}^{20} (6k - 5)$. **(3)**

7 Use proof by induction to show that, for all $n \geq 1,\ n \in \mathbb{N}$

$$\sum_{r=1}^{n} 10r = 5n(n+1)\ . \qquad \textbf{(5)}$$

Applications of Algebra and Calculus Assessment Standard 1.4

8 $f(x) = \dfrac{x^2 - 2x - 1}{x - 4},\ x \in \mathbb{R} : x \neq 4$

For the graph $y = f(x)$:

a) Give the equation of the vertical asymptote. **(1)**

b) Show that there is a non-vertical asymptote and state the equation. **(2)**

9 Find the coordinates of the point of inflection on the graph of $f(x) = 8x^3 + 48x^2 - 11x$. **(4)**

10 Given that $f(x) = \sin(2x)$, sketch the graph of $|6f(x)|$ where $0 \leq x \leq \pi$. **(2)**

Applications of Algebra and Calculus Assessment Standard 1.5

11 A car begins travelling from rest along a straight road. Its velocity, $v(t)$ metres per second, is given by

$$v(t) = \frac{400t}{7t + 15}.$$

Find the acceleration of the car at 5 seconds. **(4)**

12 The area bounded by the curve $y = \sqrt{1 + \cos 3x}$ between $x = 0$ and $x = \dfrac{\pi}{6}$ is rotated 2π radians about the x – axis.

Determine the exact value of the volume of the solid formed. **(4)**

Applications of Algebra and Calculus (Practice 3) - Sol^{n}s

1) $(4x+3)^5$

$$= \sum_{r=0}^{5} \binom{5}{r} (4x)^{5-r} 3^r$$

$$= \binom{5}{0}(4x)^5 3^0 + \binom{5}{1}(4x)^4 3^1 + \binom{5}{2}(4x)^3 3^2$$

$$+ \binom{5}{3}(4x)^2 3^3 + \binom{5}{4}(4x)^1 3^4 + \binom{5}{5}(4x)^0 3^5$$

$$= 1.1024x^5.1 + 5.256x^4.3 + 10.64x^3.9$$

$$+ 10.16x^2.27 + 5.4x.81 + 1.1.243$$

$$\therefore \boxed{(4x+3)^5 = 1024x^5 + 3840x^4 + 5760x^3 + 4320x^2 + 1620x + 243}$$

2) $z_1 = p + i$, $z_2 = 3 - 5i$

(a) $z_1 z_2 = (p+i)(3-5i)$

$= 3p + 3i - 5pi - 5i^2$

$= 3p + 3i - 5pi + 5$

$\therefore \boxed{z_1 z_2 = (3p+5) + (3-5p)i}$

(b) $\dfrac{z_1}{z_2} = \dfrac{p+i}{3-5i} \quad \dfrac{\times (3+5i)}{\times (3+5i)}$

$= \dfrac{(p+i)(3+5i)}{(3-5i)(3+5i)}$

$= \dfrac{3p + 3i + 5pi + 5i^2}{9 + 25}$

$= \dfrac{(3p-5) + (3+5p)i}{34}$

$\therefore \boxed{\dfrac{z_1}{z_2} = \dfrac{3p-5}{34} + \dfrac{3+5p}{34} i}$

3) $7, 25, 43, 61, \ldots$

(a) $a = 7,\ d = 18 \quad , \quad u_n = a + (n-1)d$

$\therefore\ u_n = 7 + (n-1)18$

$\Rightarrow\ \underline{u_n = 18n - 11}$

$\therefore\ u_{16} = 18.16 - 11$

$\Rightarrow\ \boxed{u_{16} = 277}$

(b) $S_n = \frac{n}{2}(2a + (n-1)d)$

$\therefore\ S_n = \frac{n}{2}(2(7) + (n-1)18)$

$\Rightarrow\ S_n = \frac{n}{2}(14 + 18n - 18)$

$\Rightarrow\ S_n = \frac{n}{2}(18n - 4)$

$\Rightarrow\ \underline{S_n = n(9n - 2)}$

$\therefore\ S_{16} = 16(9.16 - 2)$

$\Rightarrow\ S_{16} = 16(144 - 2)$

$\Rightarrow\ \boxed{S_{16} = 2272}$

4) $4, 36, 324, 2916, \ldots$

(a) $a = 4, \; r = 9$

$$u_n = a \cdot r^{n-1}$$

$$\therefore u_n = 4 \cdot 9^{n-1}$$

$$\therefore u_5 = 4 \cdot 9^4$$

$$\Rightarrow \boxed{u_5 = 26\,244}$$

(b) $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$\therefore S_n = \frac{4(9^n - 1)}{9 - 1}$$

$$\Rightarrow S_n = \frac{4(9^n - 1)}{8}$$

$$\therefore S_5 = \frac{(9^5 - 1)}{2}$$

$$\Rightarrow \boxed{S_5 = 29\,524}$$

5) $f(x) = e^{6x}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\therefore e^{6x} = 1 + (6x) + \frac{(6x)^2}{2} + \frac{(6x)^3}{6} + \dots$$

$$\Rightarrow e^{6x} = 1 + 6x + \frac{36x^2}{2} + \frac{216x^3}{6} + \dots$$

$$\Rightarrow \boxed{e^{6x} = 1 + 6x + 18x^2 + 36x^3 + \dots}$$

6) $\displaystyle\sum_{k=1}^{20} (6k - 5)$

$$= 6\left(\sum_{k=1}^{20} k\right) - 5\left(\sum_{k=1}^{20} 1\right)$$

$$= 6\left(\tfrac{1}{2} \cdot 20(20+1)\right) - 5 \cdot 20$$

$$= 60 \cdot 21 - 100$$

$$= \boxed{1160}$$

7) $P(n): \sum_{r=1}^{n} 10r = 5n(n+1)$

<u>Base Case ($n=1$):</u>

$$LHS = \sum_{r=1}^{1} 10r = 10.1 = 10$$

$$RHS = 5.1.2 = 10$$

So, as $LHS = RHS$, <u>$P(1)$ is true</u>.

<u>Inductive Step :</u>

Assume $P(k)$ is true, i.e., $\sum_{r=1}^{k} 10r = 5k(k+1)$.

<u>RTP:</u>

$$\sum_{r=1}^{k+1} 10r = 5(k+1)(k+2)$$

$$\sum_{r=1}^{k+1} 10r = \left(\sum_{r=1}^{k} 10r\right) + 10(k+1)$$

$$= 5k(k+1) + 10(k+1)$$

$$= 5(k+1)(k+2)$$

So, $P(k)$ true $\Rightarrow P(k+1)$ true ; together with $P(1)$ true, the PMI $\Rightarrow P(n)$ is true $\forall n \in \mathbb{N}$.

8) $f(x) = \dfrac{x^2 - 2x - 1}{x - 4}$

(a) $\boxed{\text{Vertical asymptote : } x = 4}$

(b)

$$\begin{array}{r}
x + 2 \\
x - 4 \overline{\smash{)}\, x^2 - 2x - 1} \\
\underline{x^2 - 4x} \\
2x - 1 \\
\underline{2x - 8} \\
7
\end{array}$$

$\therefore f(x) = x + 2 + \dfrac{7}{x - 4}$

$\therefore$ $\boxed{\text{Non-vertical asymptote : } y = x + 2}$

9) $f(x) = 8x^3 + 48x^2 - 11x$

$\therefore f'(x) = 24x^2 + 96x - 11$

$\therefore f''(x) = 48x + 96$

For P of I, $f''(x) = 0$; so,

$48x + 96 = 0$

$\Rightarrow \underline{x = -2}$

For SPs, $f'(x) = 0$; so,

$$24x^2 + 96x - 11 = 0$$

$D = 10\,272 > 0$. Hence, there are 2 SPs.

$$f'(-2) = 24(4) + 96(-2) - 11$$

$$\Rightarrow \underline{f'(-2) = -107 \neq 0}$$

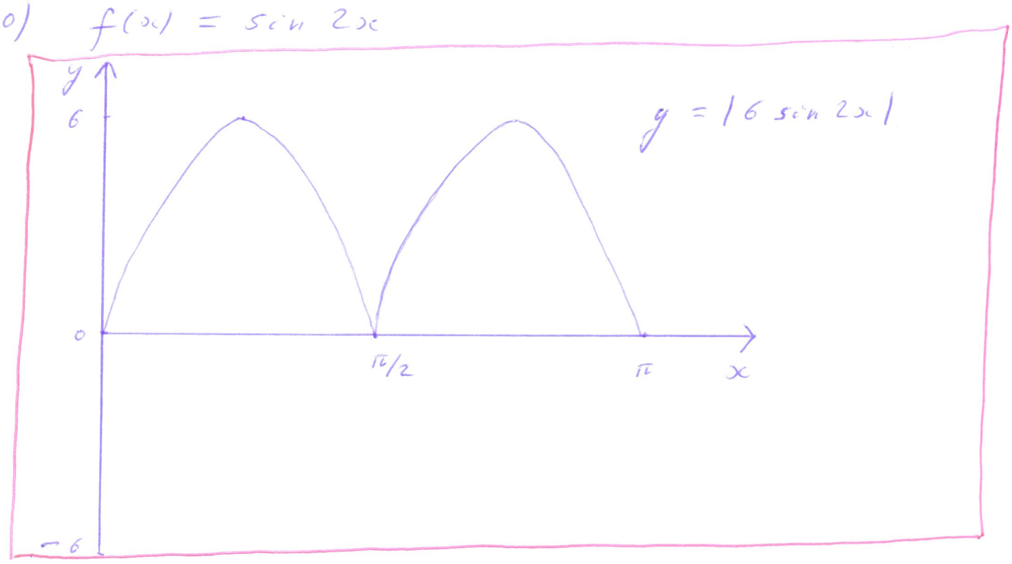
So, $\underline{x = -2 \text{ is not a SP.}}$

$$f(-2) = 8(-8) + 48(4) - 11(-2)$$

$$\Rightarrow \underline{f(-2) = 150}$$

$\therefore$ $\boxed{\text{P of I} : (-2, 150)}$

10) $f(x) = \sin 2x$



ii) $v(t) = \dfrac{400t}{7t+15}$

$$a(t) = v'(t) = \frac{400(7t+15) - 7(400t)}{(7t+15)^2}$$

$$\therefore \quad a(t) = \frac{2800t + 6000 - 2800t}{(7t+15)^2}$$

$$\Rightarrow \quad a(t) = \frac{6000}{(7t+15)^2}$$

$$\therefore \quad a(5) = \frac{6000}{(35+15)^2}$$

$$\Rightarrow \quad a(5) = \frac{6000}{2500}$$

$$\Rightarrow \quad \boxed{a(5) = \frac{12}{5} \quad (= 2 \cdot 4) \ ms^{-2}}$$

12) $V = \pi \int_0^{\pi/6} y^2 \, dx$

$\therefore V = \pi \int_0^{\pi/6} (1 + \cos 3x) \, dx$

$\Rightarrow V = \pi \left[x + \frac{1}{3} \sin 3x \right]_0^{\pi/6}$

$\Rightarrow V = \pi \left(\pi/6 + \frac{1}{3} \sin \frac{\pi}{2} \right) - \pi \left(0 + \frac{1}{3} \sin 0 \right)$

$\Rightarrow V = \frac{\pi^2}{6} + \frac{\pi}{3} . 1 - \pi . 0 - \frac{\pi}{3} . 0$

$\Rightarrow V = \frac{\pi^2}{6} + \frac{\pi}{3}$

$\Rightarrow \boxed{V = \frac{\pi}{6}(\pi + 2)}$