Solutions Included

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AH Mathematics

Practice Assessment Three

Geometry, Proof & Systems of Equations

Geometry, Proof and Systems of Equations Assessment Standard 1.1

1 Solve the following system of equations using Gaussian elimination

$$\begin{aligned} 4x + y + 2z &= -13 \\ -x + 3z &= -10 \\ 3x + 4y + z &= 2 \end{aligned}$$ **(5)**

2 Given the matrices $A = \begin{pmatrix} 6 & 1 \\ -2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} p & 0 \\ 4 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -4 \\ -1 & q \end{pmatrix}$

where p and q are constants.

Find

a) $A - 2B + 5C$. **(2)**

b) BC. **(2)**

3 Given the matrices $D = \begin{pmatrix} 4 & -5 \\ 3 & p \end{pmatrix}$ and $E = \begin{pmatrix} q & 1 & -5 \\ 0 & 1 & 4 \\ 1 & -2 & 1 \end{pmatrix}$,

a) Find D^{-1}. **(2)**

b) Determine the value(s) of q for which E is singular. **(3)**

Geometry, Proof and Systems of Equations Assessment Standard 1.2

4 Given the vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, calculate $\mathbf{a} \times \mathbf{b}$. **(3)**

5 Find, in vector form, an equation for the line which passes through the points $(4, -1, 3)$ and $(5, 8, -2)$. **(2)**

6 Find, in Cartesian form, the equation of the plane which has normal vector $\begin{pmatrix} 6 \\ -4 \\ 7 \end{pmatrix}$ and passes through the point $(2, -2, 3)$. **(2)**

Geometry, Proof and Systems of Equations Assessment Standard 1.3

7 Given $z = -3 - 3\sqrt{3}\,i$

a) Find the modulus and argument of z using exact values.

b) Write z in polar form. **(3)**

8 Given $z = -2\left(\cos -\frac{2\pi}{3} + i\,\sin -\frac{2\pi}{3}\right)$

a) Write z in Cartesian form using exact values. **(2)**

b) Plot z on an Argand diagram. **(1)**

Geometry, Proof and Systems of Equations Assessment Standard 1.4

9 Use the Euclidean algorithm to obtain the greatest common divisor of 1 456 and 518. **(3)**

Geometry, Proof and Systems of Equations Assessment Standard 1.5

10 For any real numbers a and b, it is conjectured that

$$a > b + 7 \Rightarrow (a - 7)^2 > b^2$$

Use a counterexample to disprove this conjecture. **(3)**

11 Prove, by contradiction, that if $x - 2$ is irrational then x is irrational. **(4)**

12 Use direct proof to show that the product of any two square numbers is square. **(3)**

Geometry, Proof and Systems of Equations (Practice 3) - Sol'ns

1)
$$\begin{aligned} 4x + y + 2z &= -13 \\ -x \quad + 3z &= -10 \\ 3x + 4y + z &= 2 \end{aligned}$$

Augmented matrix is,

$$\left(\begin{array}{ccc|c} 4 & 1 & 2 & -13 \\ -1 & 0 & 3 & -10 \\ 3 & 4 & 1 & 2 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 + 3R_2]{R_1 \to R_1 + 4R_2} \left(\begin{array}{ccc|c} 0 & 1 & 14 & -53 \\ -1 & 0 & 3 & -10 \\ 0 & 4 & 10 & -28 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 4R_1} \left(\begin{array}{ccc|c} 0 & 1 & 14 & -53 \\ -1 & 0 & 3 & -10 \\ 0 & 0 & -46 & 184 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & 0 & 3 & -10 \\ 0 & 1 & 14 & -53 \\ 0 & 0 & -46 & 184 \end{array}\right)$$

$$\therefore \quad \begin{aligned} -x + 3z &= -10 \quad ① \\ y + 14z &= -53 \quad ② \\ -46z &= 184 \quad ③ \Rightarrow \underline{z = -4}\,; \end{aligned}$$

$② \Rightarrow y = -53 + 56 \Rightarrow \underline{y = 3}\,;\ ① \Rightarrow x = -12 + 10 \Rightarrow \underline{x = -2}$

$$\therefore \boxed{x = -2,\ y = 3,\ z = -4}$$

2) $A = \begin{pmatrix} 6 & 1 \\ -2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} p & 0 \\ 4 & -3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -4 \\ -1 & q \end{pmatrix}$

(a) $A - 2B + 5C = \begin{pmatrix} 6 & 1 \\ -2 & 0 \end{pmatrix} - 2\begin{pmatrix} p & 0 \\ 4 & -3 \end{pmatrix} + 5\begin{pmatrix} 1 & -4 \\ -1 & q \end{pmatrix}$

$\Rightarrow A - 2B + 5C = \begin{pmatrix} 6 & 1 \\ -2 & 0 \end{pmatrix} - \begin{pmatrix} 2p & 0 \\ 8 & -6 \end{pmatrix} + \begin{pmatrix} 5 & -20 \\ -5 & 5q \end{pmatrix}$

$\Rightarrow \boxed{A - 2B + 5C = \begin{pmatrix} 11-2p & -19 \\ -15 & 6+5q \end{pmatrix}}$

(b) $BC = \begin{pmatrix} p & 0 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ -1 & q \end{pmatrix}$

$\Rightarrow BC = \begin{pmatrix} p+0 & -4p+0 \\ 4+3 & -16-3q \end{pmatrix}$

$\Rightarrow \boxed{BC = \begin{pmatrix} p & -4p \\ 7 & -16-3q \end{pmatrix}}$

3) $D = \begin{pmatrix} 4 & -5 \\ 3 & p \end{pmatrix}$, $E = \begin{pmatrix} q & 1 & -5 \\ 0 & 1 & 4 \\ 1 & -2 & 1 \end{pmatrix}$

(a) $D^{-1} = \begin{pmatrix} 4 & -5 \\ 3 & p \end{pmatrix}^{-1}$

$\Rightarrow D^{-1} = \dfrac{1}{4p-(-5).3}\begin{pmatrix} p & 5 \\ -3 & 4 \end{pmatrix}$

$\Rightarrow \boxed{D^{-1} = \dfrac{1}{4p+15}\begin{pmatrix} p & 5 \\ -3 & 4 \end{pmatrix}}$

(b) $|E| = q\begin{vmatrix} 1 & 4 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 4 \\ 1 & 1 \end{vmatrix} + (-5)\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix}$

$\therefore |E| = q(1+8) - 1(-4) - 5(-1)$

$\Rightarrow |E| = 9q + 9$

E singular $\Rightarrow |E| = 0$. So,

$$9q + 9 = 0$$

$$\Rightarrow \boxed{q = -1}$$

4) $\underline{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

$\therefore \underline{a} \times \underline{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

$\Rightarrow \underline{a} \times \underline{b} = \begin{pmatrix} 1.2 - (-1)(-3) \\ -1.1 - 2.2 \\ 2(-3) - 1.1 \end{pmatrix}$

$\Rightarrow \boxed{\underline{a} \times \underline{b} = \begin{pmatrix} -1 \\ -5 \\ -7 \end{pmatrix}}$

5) $\underline{r} = \underline{a} + t\,\underline{d} \quad (\underline{d} = \underline{b} - \underline{a})$

$$\underline{a} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 5 \\ 8 \\ -2 \end{pmatrix}$$

$$\therefore \underline{d} = \begin{pmatrix} 5 \\ 8 \\ -2 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} \Rightarrow \underline{\underline{d} = \begin{pmatrix} 1 \\ 9 \\ -5 \end{pmatrix}}$$

$$\therefore \boxed{\underline{r} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 9 \\ -5 \end{pmatrix}}$$

6) $ax + by + cz = k$

$$\underline{n} = \begin{pmatrix} 6 \\ -4 \\ 7 \end{pmatrix}$$

$$\therefore 6x - 4y + 7z = k \quad (\pi)$$

$(2, -2, 3) \in \pi$, so,

$$6(2) - 4(-2) + 7(3) = k \Rightarrow \underline{k = 41}$$

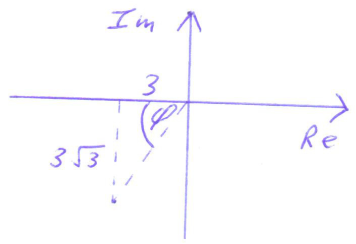
$$\therefore \boxed{6x - 4y + 7z = 41}$$

7) $z = -3 - 3\sqrt{3}\ i$

(a) $|z| = \sqrt{(-3)^2 + (-3\sqrt{3})^2}$

$\therefore |z| = \sqrt{9 + 9(3)} \Rightarrow |z| = \sqrt{36} \Rightarrow \boxed{|z| = 6}$

$\tan\varphi = \dfrac{3\sqrt{3}}{3}$



$\therefore \tan\varphi = \sqrt{3}$

$\Rightarrow \underline{\varphi = \dfrac{\pi}{3}}$

$\arg z = -(\pi - \varphi) \Rightarrow \arg z = -\left(\pi - \dfrac{\pi}{3}\right) \Rightarrow \boxed{\arg z = -\dfrac{2\pi}{3}}$

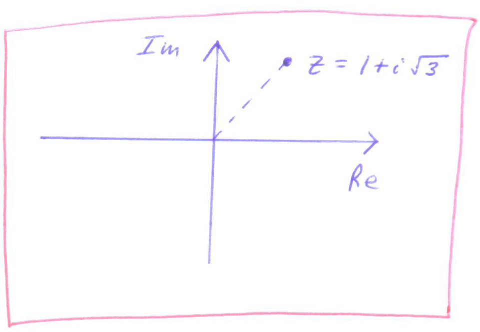
(b) $z = r(\cos\theta + i\sin\theta)$

$\therefore \boxed{z = 6\left(\cos\left(-\dfrac{2\pi}{3}\right) + i\sin\left(-\dfrac{2\pi}{3}\right)\right)}$

8) (a) $z = -2\left(\cos\left(-\dfrac{2\pi}{3}\right) + i\sin\left(-\dfrac{2\pi}{3}\right)\right)$

$\therefore z = -2\left(-\dfrac{1}{2} + i\left(-\dfrac{\sqrt{3}}{2}\right)\right) \Rightarrow \boxed{z = 1 + i\sqrt{3}}$

(b)



9) $1456 = 518 \cdot 2 + 420$

$518 = 420 \cdot 1 + 98$

$420 = 98 \cdot 4 + 28$

$98 = 28 \cdot 3 + 14$

$28 = 14 \cdot 2 + 0$

$\therefore \boxed{GCD(1456, 518) = 14}$

10) $a > b + 7 \Rightarrow (a-7)^2 > b^2$ (c)

Let $\boxed{a = 8,\ b = -9}$.

Then $b + 7 = -9 + 7 = -2$; so,

$a = 8 > -2 = b + 7$, i.e., $\underline{a > b + 7}$

$(a-7)^2 = (8-7)^2 = 1^2 = 1$

$b^2 = (-9)^2 = 81$

So, $(a-7)^2 \not> b^2$ as $1 < 81$.

Hence, $\boxed{\text{conjecture (c) is false}}$.

11) Assume $x-2$ is irrational and x is rational. So, $\exists\, p, q \in \mathbb{Z}$ $(q \neq 0)$ s.t.,

$$x = \frac{p}{q}$$

$$\therefore\ x - 2 = \frac{p}{q} - 2$$

$$\Rightarrow x - 2 = \frac{p-2q}{q} \quad (q \neq 0)$$

As $p, q \in \mathbb{Z}$, $p - 2q \in \mathbb{Z}$; so, $x - 2 \in \mathbb{Q}$.

But this contradicts $x-2$ being irrational.

So,

$$\boxed{x-2 \text{ irrational} \Rightarrow x \text{ irrational}}$$

12) Let $a = p^2$, $b = q^2$ $(p, q \in \mathbb{Z})$ be 2 square numbers. Then,

$$ab = p^2 q^2$$

$$\Rightarrow ab = (pq)^2$$

As $p, q \in \mathbb{Z}$, $pq \in \mathbb{Z}$; so, ab is square.

$$\therefore \boxed{a, b \text{ square} \Rightarrow ab \text{ square}}$$